PERMANENTAL INEQUALITIES FOR CERTAIN TOTALLY NONNEGATIVE MATRICES

Sihong Pan, Mark Skandera and Jiayuan Wang

Lehigh University

Outline

(1) Totally nonnegative matrices and polynomials
(2) Classical and recent inequalities
(3) Inequalities for special matrices
(4) Open problems

Polynomials in matrix entries

Define $x = (x_{i,j})_{i,j=1}^{n}$ and $\mathbb{R}[x] := \mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$.

Evaluate $f(x) \in \mathbb{R}[x]$ at $A = (a_{i,j}) \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ by

$$f(A) := f(a_{1,1}, a_{1,2}, \ldots, a_{n,n}).$$

Ex:

$$\det(x) = \sum_{w\in\mathfrak{S}_n} \epsilon(w) x_{1,w_1} \cdots x_{n,w_n}, \quad \mathrm{per}(x) = \sum_{w\in\mathfrak{S}_n} x_{1,w_1} \cdots x_{n,w_n},$$

$$\det(A) = \sum_{w\in\mathfrak{S}_n} \epsilon(w) a_{1,w_1} \cdots a_{n,w_n}, \quad \mathrm{per}(A) = \sum_{w\in\mathfrak{S}_n} a_{1,w_1} \cdots a_{n,w_n},$$

where $\mathfrak{S}_n$ = permutations $w = w_1 \cdots w_n$ of $1 \cdots n$.

Totally nonnegative matrices and polynomials

Given $n \times n$ matrix $A = (a_{i,j})$, subsets $I, J \subseteq [1, n] := \{1, \ldots, n\}$, define (I, J)-*submatrix* $A_{I,J} = (a_{i,j})_{i \in I, j \in J}$.

Call A a *totally nonnegative (TNN) matrix* if $\det(A_{I,J}) \geq 0$ for all $I, J \subseteq [1, n]$, $|I| = |J|$.

Ex: $$A = \begin{bmatrix} 8 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 8 \end{bmatrix}; \quad \det(A_{12,13}) = \det \begin{bmatrix} 8 & 1 \\ 2 & 2 \end{bmatrix} = 14.$$

Call $f(x)$ a *totally nonnegative (TNN) polynomial* if $f(A) \geq 0$ whenever A is a TNN matrix.

Ex: $\det(x), \quad x_{i,j}, \quad \mathrm{per}(x), \quad \det(x_{12,12})x_{3,3} + x_{1,1}\mathrm{per}(x_{23,23})$.

Polynomials and inequalities

Ex: (More interesting, FGJ '01)

$$\det(x_{13,13})\det(x_{24,24}) - \det(x_{12,12})\det(x_{34,34})$$

is a TNN polynomial.

In general, $g(x) - f(x)$ is a TNN polynomial if and only if

$$f(A) \leq g(A)$$

for all TNN matrices A.

Motivation: To solve certain integrable systems in physics,

(1) Use inverse scattering method (nonlinear Fourier transform).
(2) This requires representations of quantum group $\mathcal{O}_q(\mathrm{SL}_n(\mathbb{C}))$.
(3) Can construct these with *dual canonical basis* of $\mathcal{O}_q(\mathrm{SL}_n(\mathbb{C}))$.
(4) This complicated basis is related to TNN polynomials.

Determinantal vs. permanental inequalities

Thm: (K '53, obv) For all A TNN,

$$\det(A) \leq a_{1,1} \cdots a_{n,n}, \qquad \mathrm{per}(A) \geq a_{1,1} \cdots a_{n,n}.$$

Thm: (F '67, obv) For all A TNN, $I \subseteq [n]$, $\overline{I} := [n] \smallsetminus I$,

$$\det(A) \leq \det(A_{I,I}) \det(A_{\overline{I},\overline{I}}), \qquad \mathrm{per}(A) \geq \mathrm{per}(A_{I,I})\mathrm{per}(A_{\overline{I},\overline{I}}).$$

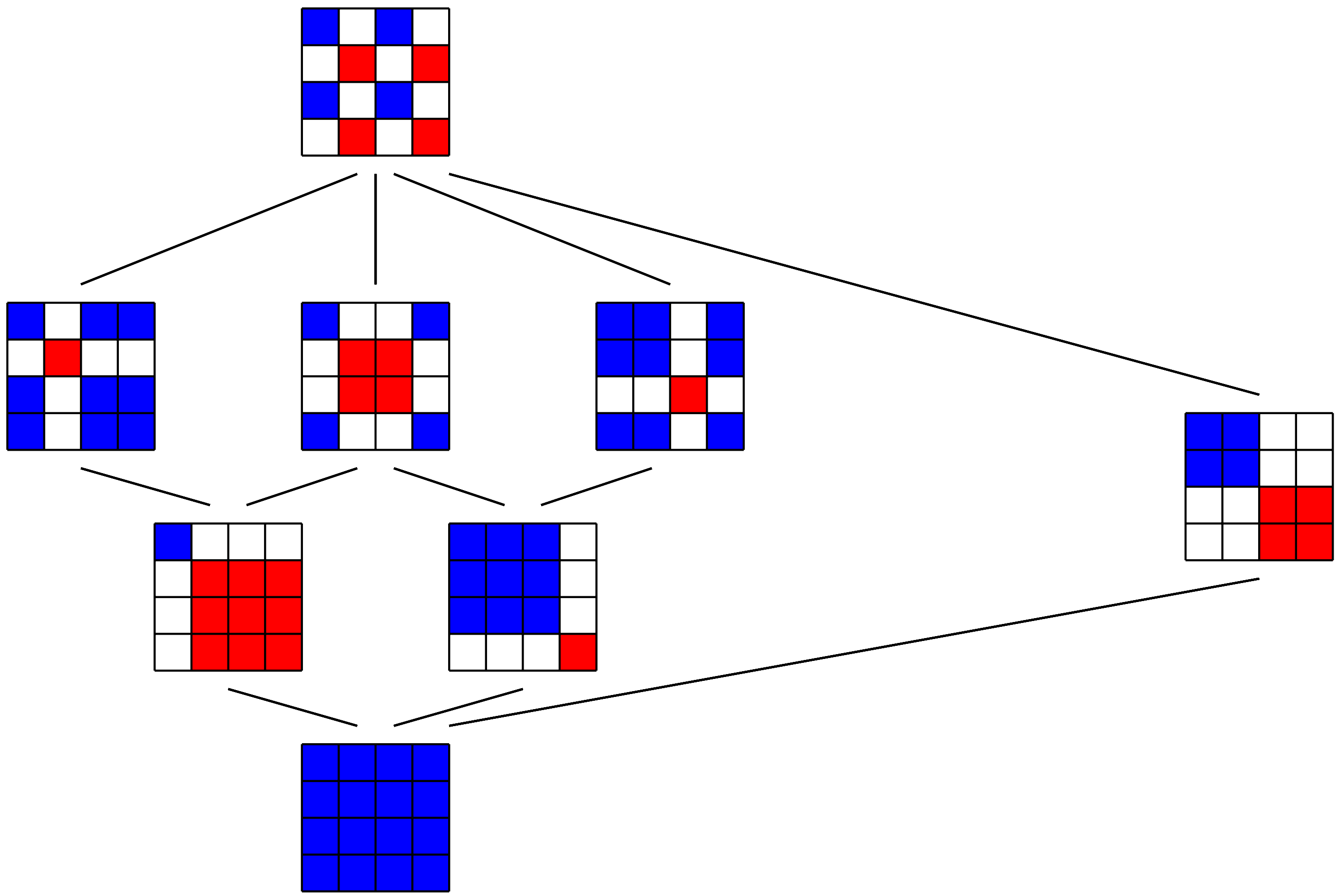
Fact: FGJ ('01) characterized all $I, J \subseteq [1, n]$ for which we have

$$\det(A_{I,I}) \det(A_{\overline{I},\overline{I}}) \leq \det(A_{J,J}) \det(A_{\overline{J},\overline{J}})$$

for all TNN A.

Determinantal inequalities in 4×4 matrices

Products of complementary principal minors are related by



.

A conjecture

Quest: Can we turn this poset upside down to characterize all inequalities of the form

$$\mathrm{per}(A_{I,I})\mathrm{per}(A_{\overline{I},\overline{I}}) \geq \mathrm{per}(A_{J,J})\mathrm{per}(A_{\overline{J},\overline{J}})?$$

Ans: No, but this seems to work for some pairs of products.

Fact: (FGJ, '01) Define

$$O = O(n) := [1,n] \smallsetminus 2\mathbb{Z}, \qquad E = E(n) := [1,n] \cap 2\mathbb{Z}.$$

For all A TNN and $1 \leq h \leq n$, we have

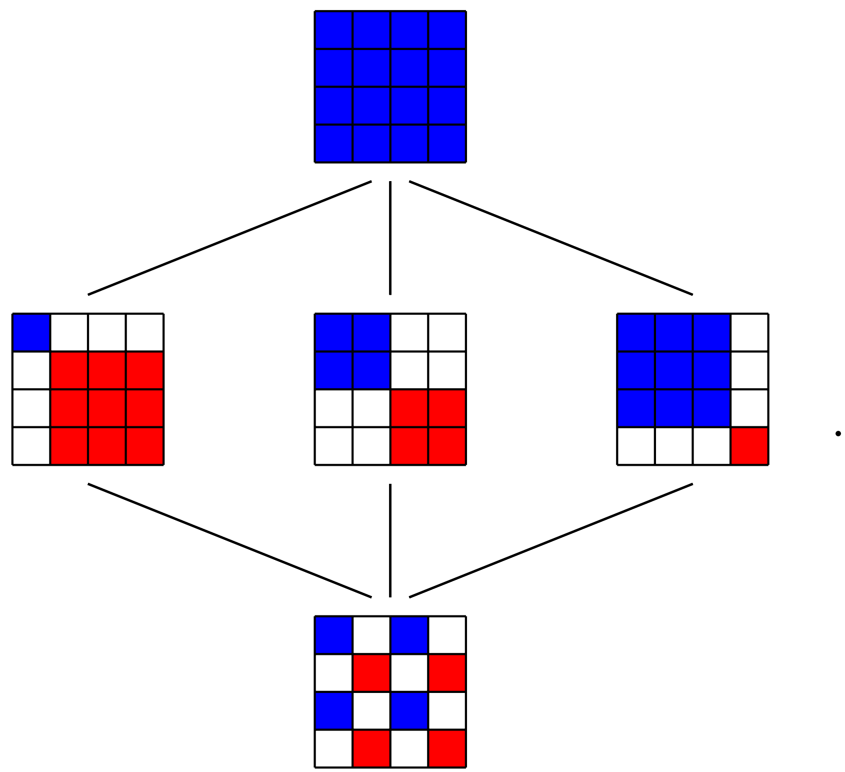
$$\det(A_{[1,h],[1,h]}) \det(A_{[h+1,n],[h+1,n]}) \leq \det(A_{O,O}) \det(A_{E,E}).$$

Conj: For all A TNN and $1 \leq h \leq n$, we have

$$\mathrm{per}(A_{[1,h],[1,h]})\mathrm{per}(A_{[h+1,n],[h+1,n]}) \geq \mathrm{per}(A_{O,O})\mathrm{per}(A_{E,E}).$$

Permanental inequalities in 4×4 matrices

Some complementary permanent products are related by



But permanents are harder to study than determinants.

Special TNN matrices

Let $\mathcal{A}_n$ be the set of $n \times n$, 0-1 matrices $A = (a_{i,j})$ satisfying

$$a_{i,i} = 1, \qquad a_{i,j} \geq a_{i,j+1}, \qquad a_{i,j} \leq a_{i+1,j}.$$

Ex: $(n = 3)$ $\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}, \begin{bmatrix}1&1&0\\1&1&0\\1&1&1\end{bmatrix}, \begin{bmatrix}1&0&0\\1&1&1\\1&1&1\end{bmatrix}, \begin{bmatrix}1&1&0\\1&1&1\\1&1&1\end{bmatrix}, \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}.$

Fact: The matrices in $\mathcal{A}_n$ are TNN. Some facts about *all* TNN matrices A are proved first for $A \in \mathcal{A}_n$.

Ex: Let χ be an irreducible $\mathfrak{S}_n$-character.
For A TNN, we know only that $\mathrm{Imm}_\chi(A) \geq 0$.
For $A \in \mathcal{A}_n$, a visual representation of A gives value of $\mathrm{Imm}_\chi(A)$.

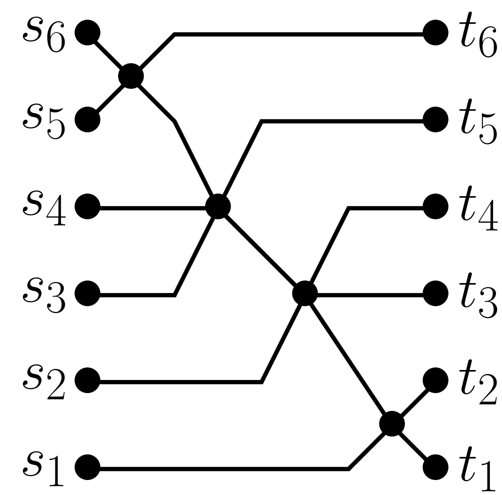
Star networks and path matrices

Matrix $A \in \mathcal{A}_n$ corresponds to network $F = F(A)$ with boundary vertices $s_1, \ldots, s_n$, $t_1, \ldots, t_n$, edges $\rightarrow$ and

$$a_{i,j} = \begin{cases} 1 & \text{if there is a path from } s_i \text{ to } t_j, \\ 0 & \text{otherwise.} \end{cases}$$

Ex :

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$



.

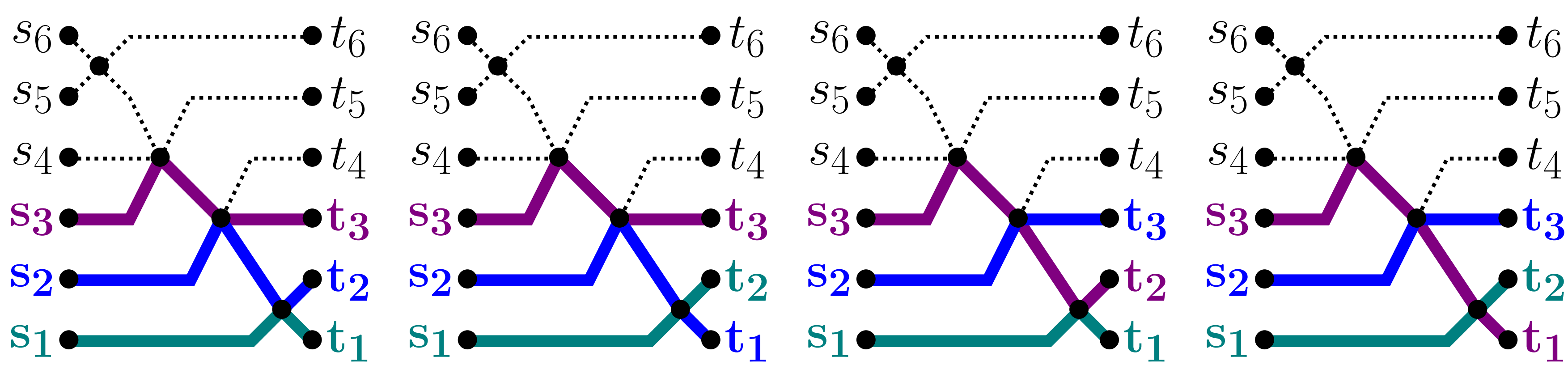
Call A the *path matrix* of $F(A)$.

Permanents of path matrices

Fact: $\mathrm{per}(A) = \#$ path families from $\{s_1, \ldots, s_n\}$ to $\{t_1, \ldots, t_n\}$, (not necessarily respectively).

Fact: $\mathrm{per}(A_{I,I}) = \#$ path families from $\{s_i \mid i \in I\}$ to $\{t_i \mid i \in I\}$.

Ex: $\mathrm{per}(A_{[1,3],[1,3]}) = \mathrm{per} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 4$ path families,



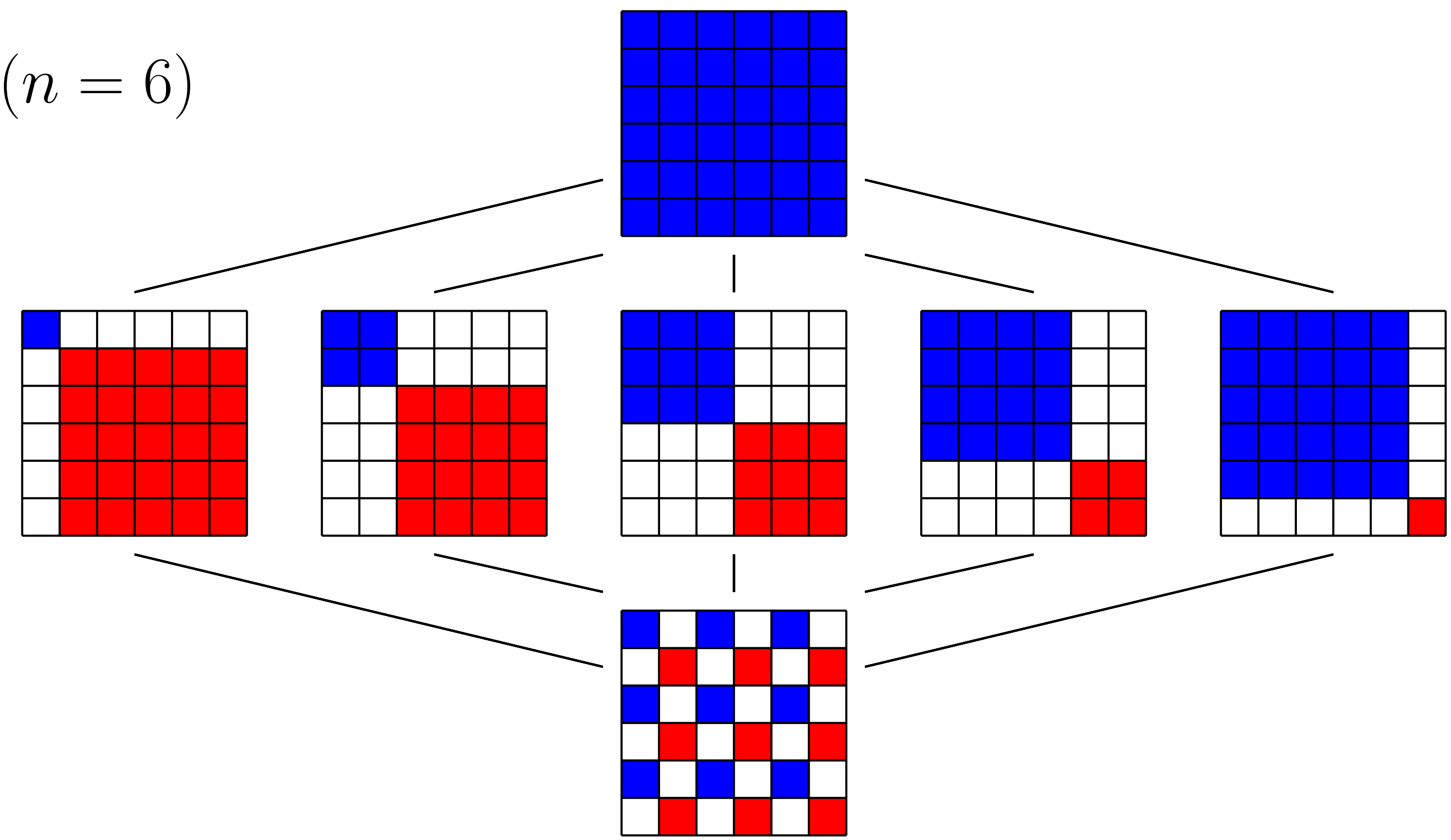
.

Main result

Thm: (PSW, '25) For all $A \in \mathcal{A}_n$ and $1 \le h \le n$, we have

$$\mathrm{per}(A_{[1,h],[1,h]})\mathrm{per}(A_{[h+1,n],[h+1,n]}) \ge \mathrm{per}(A_{O,O})\mathrm{per}(A_{E,E}).$$

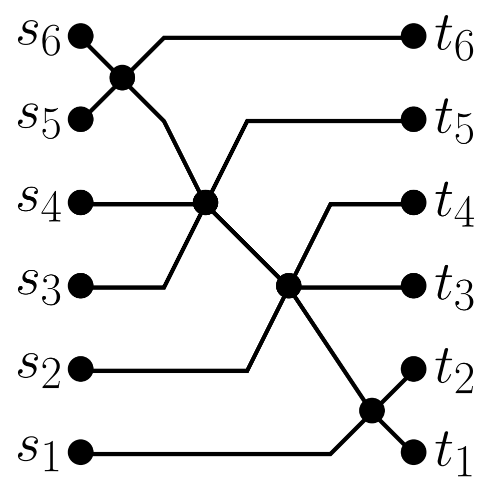
$(n = 6)$



.

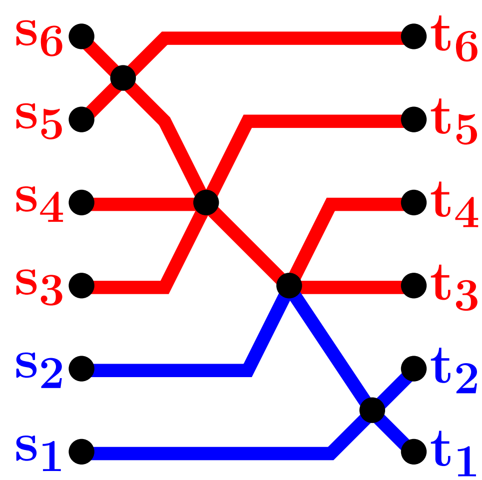
Ex: Recall the path matrix and star network

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$



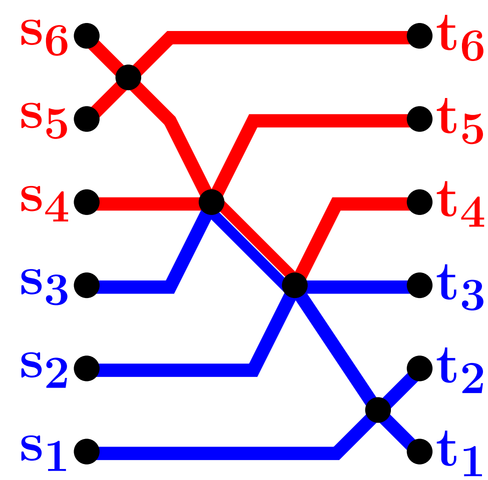
$\mathrm{per}(A_{12,12})\mathrm{per}(A_{3456,3456}) = 2 \cdot 12 = 24$ path families:

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 1 & 1 & 0 & 0 \\ 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix},$$



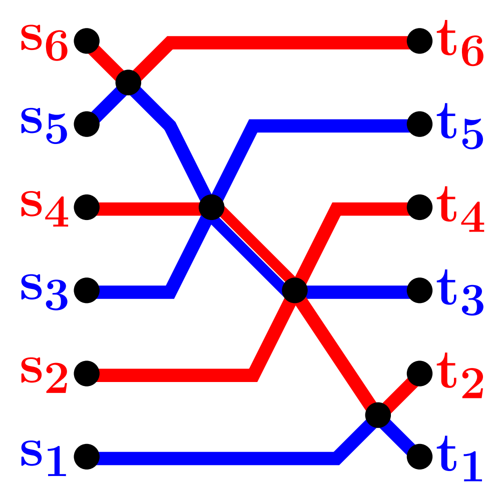
$\mathrm{per}(A_{123,123})\mathrm{per}(A_{456,456}) = 4 \cdot 4 = 16$ path families:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$



$\mathrm{per}(A_{135,135})\mathrm{per}(A_{246,246}) = 2 \cdot 2 = 4$ path families:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$



Partial results and open problems

Fact: (PSW, '25) For all $2n \times 2n$ TNN matrices A, $n \leq 6$ we have

$$\mathrm{per}(A_{[1,n],[1,n]})\mathrm{per}(A_{[n+1,2n],[n+1,2n]}) \geq \mathrm{per}(A_{O,O})\mathrm{per}(A_{E,E}).$$

Prob: Show that this holds for all n.

Prob: Characterize all inequalities of the form

$$\mathrm{per}(A_{I,I})\mathrm{per}(A_{\overline{I},\overline{I}}) \geq \mathrm{per}(A_{J,J})\mathrm{per}(A_{\overline{J},\overline{J}}).$$

Prob: For each n, find the extreme rays of the cone of TNN polynomials which are homogeneous of degree n.