ON A NONNEGATIVITY PROPERTY OF THE DUAL CANONICAL BASIS

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Outline

(1) Quantum groups
(2) Immanants and the dual canonical basis
(3) Total nonnegativity and the dual canonical cone
(4) Planar networks
(5) Factorization of Kazhdan-Lusztig basis elements

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Quantum groups

Hopf algebras	bases
$U(\mathfrak{sl}_n\mathbb{C}), U_q(\mathfrak{sl}_n\mathbb{C})$	canonical
$\mathcal{O}(SL_n\mathbb{C}), \mathcal{O}_q(SL_n\mathbb{C})$	dual canonical

$$x = (x_{1,1}, \ldots, x_{n,n})$$

$$\begin{aligned} \text{projections} : \mathbb{C}_q\langle x\rangle &\to \mathbb{C}_q\langle x\rangle/(\det_q(x) - 1) \cong \mathcal{O}_q(SL_n\mathbb{C}) \\ \mathbb{C}[x] &\to \mathbb{C}[x]/(\det(x) - 1) \cong \mathcal{O}(SL_n\mathbb{C}) \\ \text{DCB} &\mapsto \text{DCB} \end{aligned}$$

Nonnegativity properties of DCB of $\mathbb{C}[x]$ have interpretations in representation theory, symmetric functions, other areas.

Immanants in dual canonical basis of $\mathbb{C}_q\langle x\rangle$, $\mathbb{C}[x]$

Kazhdan-Lusztig immanants $\{\mathrm{Imm}_v(x;q) \,|\, v \in S_n\}$ in $\mathbb{C}_q\langle x\rangle$:

$$\mathrm{Imm}_v(x;q) = \sum_{w \geq v} \frac{\epsilon_{v,w}}{q_{v,w}} Q_{v,w}(q) x_{1,w(1)} \cdots x_{n,w(n)},$$

$$\epsilon_{v,w} = (-1)^{\ell(w)-\ell(v)},$$

$$q_{v,w} = (q^{1/2})^{\ell(w)-\ell(v)},$$

$$Q_{v,w}(q) = P_{w_0 w, w_0 v}(q).$$

Nonquantum ($q = 1$) analogs in $\mathbb{C}[x]$:

$$\mathrm{Imm}_v(x) = \sum_{w \geq v} \epsilon_{v,w} Q_{v,w}(1) x_{1,w(1)} \cdots x_{n,w(n)}.$$

The dual canonical basis of $\mathbb{C}[x_{1,1}, \ldots, x_{3,3}]$

K-L immanants of generalized submatrices of $\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}$:

$$\mathrm{Imm}_{4132} \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \end{bmatrix}, \quad \mathrm{Imm}_{321} \begin{bmatrix} x_{2,1} & x_{2,2} & x_{2,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{bmatrix},$$

$$\mathrm{Imm}_{12} \begin{bmatrix} x_{1,1} & x_{1,3} \\ x_{2,1} & x_{2,3} \end{bmatrix}, \quad \mathrm{Imm}_{43215} \begin{bmatrix} x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} \\ x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}.$$

Total nonnegativity

Call a matrix *totally nonnegative* (TNN) if each of its minors is nonnegative.

Call a polynomial $p(x) \in \mathbb{R}[x_{1,1}, \ldots, x_{n,n}]$ *totally nonnegative* if $p(a_{1,1}, \ldots, a_{n,n}) \geq 0$ for all TNN matrices A.

Examples:

- (L) dual canonical basis elements
- (JS) irreducible character immanants
- (F-G-J, R-S) certain $\Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x)$

Theorem: (H, D, R-S, S) These all belong to the dual canonical cone.

Main Theorem

Suppose $p(x) = \sum_{w \in S_n} d_w \mathrm{Imm}_w(x)$ is TNN.

Q: For which $w \in S_n$, if any, must we have $d_w \geq 0$?

Thm: (S) If w avoids patterns 3412, 4231, then $d_w \geq 0$.

Ex: $\mathrm{Imm}_{4123}(x) - \mathrm{Imm}_{2341}(x)$ is not TNN.

Ex: (S-Z) This polynomial is TNN, but does not belong to the dual canonical cone:

$$\mathrm{Imm}_{1432}(x) + \mathrm{Imm}_{3214}(x) - \mathrm{Imm}_{3412}(x).$$

Planar network theorem

Thm: (S) For w avoiding 3412, 4231, there is a planar network G whose (TNN) path matrix A satisfies

$$\mathrm{Imm}_v(A) = \delta_{v,w}.$$

$$G(43152) = \qquad , \qquad A(43152) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Q: Is $\mathrm{Imm}_{14523}(x) - 2\mathrm{Imm}_{43152}(x)$ TNN?

A: No. $\mathrm{Imm}_{14523}(A) - 2\mathrm{Imm}_{43152}(A) = -2$.

Reversal factorization theorem

Reversal: In S_9, $s_{[3,6]} = 126543789$.

Thm: (S) For w avoiding 3412, 4231, we have

$$w = s_{I_1} \cdots s_{I_k},$$
$$C'_w = g(\sqrt{q})C'_{s_{I_1}} \cdots C'_{s_{I_k}}.$$

$\{C'_w \,|\, w \in S_n\}$ = Kazhdan-Lusztig basis of $H_n(q)$.
In terms of natural basis $\{T_w \,|\, w \in S_n\}$,

$$C'_w(q) = q^{-\ell(w)/2} \sum_{v \leq w} P_{v,w}(q) T_v.$$

Fact: $C'_{s_{[i,j]}}(q) = q^{-\ell(s_{[i,j]})/2} \sum_{v \in S_{[i,j]}} T_v.$

Examples of Kazhdan-Lusztig basis elements

$$C'_{s_i} = C'_{s_{[i,i+1]}} = q^{-1/2}(1 + T_{s_i}),$$
$$C'_{s_{[1,3]}} = q^{-3/2}(1 + T_{s_1} + T_{s_2} + T_{s_1 s_2} + T_{s_2 s_1} + T_{s_1 s_2 s_1}).$$

$$621354 = s_{[1,3]} s_{[3,4]} s_{[4,6]},$$
$$C'_{621354} = C'_{s_{[1,3]}} C'_{s_{[3,4]}} C'_{s_{[4,6]}},$$

$$654213 = s_{[1,5]} s_{[3,5]} s_{[3,6]},$$
$$C'_{654213} = \frac{q^3}{3_q!^2} C'_{s_{[1,5]}} C'_{s_{[3,5]}} C'_{s_{[3,6]}},$$
$$= \frac{q^{3/2}}{3_q!} C'_{s_{[1,5]}} C'_{s_{[3,6]}}.$$

Kazhdan-Lusztig factorization questions

Q: (Open) For which w does C'_w factor as

$$C'_w = g(\sqrt{q})C'_{I_1} \cdots C'_{I_k}?$$

A: (B-W) For w avoiding 321, hexagon patterns.

A: (S) For w avoiding $3412, 4231$.

These results agree for w avoiding $321, 3412$.

Q: Are these the best possible results?

A: No. 4231 doesn't avoid 321 or 4231, but

$$C'_{4231} = C'_{s_{[1,2]}} C'_{s_{[2,4]}} C'_{s_{[1,2]}}.$$

The Hecke algebra $H_n(q)$

Generators over $\mathbb{C}[q^{1/2}, q^{-1/2}]$: $T_{s_1}, \ldots, T_{s_{n-1}}$.

Relations:

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e && \text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} && \text{for } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} && \text{for } |i-j| \geq 2. \end{aligned}$$

Natural basis: $\{T_w \,|\, w \in S_n\}$,

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}, \qquad (w = s_{i_1} \cdots s_{i_\ell} \text{ reduced}).$$

Bar involution:

$$\overline{q} = q^{-1}, \qquad \overline{T_w} = (T_{w^{-1}})^{-1}.$$

The Kazhdan-Lusztig basis

Thm: (K-L '79) There is a unique basis $\{C'_w \mid w \in S_n\}$ of $H_n(q)$ which satisfies

(1) $\overline{C'_w} = C'_w$ for all w,
(2) $C'_w = q^{-\ell(w)/2} \sum_{v \leq w} P_{v,w}(q) T_v$,

where

(a) $P_{w,w}(q) = 1$ for all w,
(b) $P_{v,w}(q) \in \mathbb{Z}[q]$ for all v, w,
(c) $\deg P_{v,w}(q) \leq (\ell(w) - \ell(v) - 1)/2$ for all $v < w$.

Call this the (signless) *Kazhdan-Lusztig basis.*

Quantum polynomial ring

Define $\mathcal{A}(n;q) \cong \mathbb{C}[q^{1/2}, q^{-1/2}]\langle x_{1,1}, \ldots, x_{n,n}\rangle$, modulo

$$\begin{aligned}
x_{i,\ell}x_{j,k} &= x_{j,k}x_{i,\ell} && \text{if } i<j,\ k<\ell,\\
x_{i,\ell}x_{i,k} &= q^{1/2}x_{i,k}x_{i,\ell} && \text{if } k<\ell,\\
x_{j,k}x_{i,k} &= q^{1/2}x_{i,k}x_{j,k} && \text{if } i<j,\\
x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{1/2}-q^{-1/2})x_{i,\ell}x_{j,k} && \text{if } i<j,\ k<\ell.
\end{aligned}$$

A central element is the quantum determinant,

$$\det_q(x) = \sum_{w\in S_n} (-q^{-1/2})^{\ell(w)} x_{1,w(1)} \cdots x_{n,w(n)}.$$

We have $\mathcal{O}_q(SL_n\mathbb{C}) \cong \mathcal{A}(n;q)/(\det_q(x)-1)$.