ON THE DUAL CANONICAL AND KAZHDAN-LUSZTIG BASES AND 3412, 4231-AVOIDING PERMUTATIONS

MARK SKANDERA

Abstract. Using Du's characterization of the dual canonical basis of the coordinate ring $\mathcal{O}(GL(n,\mathbb{C}))$, we express all elements of this basis in terms of immanants. We then give a new factorization of permutations w avoiding the patterns 3412 and 4231, which in turn yields a factorization of the corresponding Kazhdan-Lusztig basis elements $C'_w(q)$ of the Hecke algebra $H_n(q)$. Using the immanant and factorization results, we show that for every totally nonnegative immanant $\mathrm{Imm}_f(x)$ and its expansion $\sum d_w \mathrm{Imm}_w(x)$ with respect to the basis of Kazhdan-Lusztig immanants, the coefficient d_w must be nonnegative when w avoids the patterns 3412 and 4231.

1. Introduction

Studying methods of solving of the quantum Yang-Baxter equation, Drinfeld [10] and Jimbo [19] introduced a quantization $U_q(\mathfrak{sl}(n,\mathbb{C}))$ of the universal enveloping algebra $U(\mathfrak{sl}(n,\mathbb{C}))$. An explosion of mathematical research soon led to a quantization $\mathcal{O}_q(SL(n,\mathbb{C}))$ of the coordinate ring $\mathcal{O}(SL(n,\mathbb{C}))$, related by Hopf algebra duality to $U_q(\mathfrak{sl}(n,\mathbb{C}))$, and to a development of the representation theory of these algebras now known as quantum groups. In particular, Kashiwara [21] and Lusztig [25], [26] introduced a modification $\dot{U}$ of $U_q(\mathfrak{sl}(n,\mathbb{C}))$ and a *canonical* (or *crystal*) *basis* of this algebra which has many interesting representation theoretic properties. A corresponding dual basis of $\mathcal{O}_q(SL(n,\mathbb{C}))$ is known as the *dual canonical basis*, and may be viewed as the projection of a certain basis of a quantization $\mathbb{C}_q[x_{1,1},\dots,x_{n,n}]$ of the polynomial ring $\mathbb{C}[x_{1,1},\dots,x_{n,n}]$. This latter basis is also called the *dual canonical basis*. (See [11].) An elementary description of canonical and dual canonical bases has been rather elusive, especially in the nonquantum ($q=1$) setting.

Results of Lusztig [27] imply that when we specialize at $q=1$, the elements of the dual canonical basis of $\mathbb{C}[x_{1,1},\dots,x_{n,n}]$ are *totally nonnegative* (TNN) polynomials in the following sense. We define a matrix with real entries to be *totally nonnegative* (TNN) if each of its minors is nonnegative. (See e.g. [14].) We define a polynomial $p(x) \in \mathbb{C}[x_{1,1},\dots,x_{n,n}]$ to be *totally nonnegative* (TNN) if for each $n \times n$ TNN matrix

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$A = (a_{i,j})$, we have

$$p(A) \underset{\text{def}}{=} p(a_{1,1}, \ldots, a_{n,n}) \geq 0.$$

While it is not true that every TNN polynomial belongs to the dual canonical cone, we will show that certain coordinates of a TNN polynomial with respect to the dual canonical basis must be nonnegative. Our criterion involves avoidance of the patterns 3412 and 4231 in permutations and thus links total nonnegativity to smoothness in Schubert varieties.

In Section 2 we review Du's formulation of the dual canonical basis and express its elements in terms of functions called immanants. In Sections 3 and 4 we state factorization theorems for 3412-avoiding, 4231-avoiding permutations and for the corresponding Kazhdan-Lusztig basis elements. In Section 5 we combine immanant and factorization results to prove that for each TNN homogeneous element $p(x)$ of the coordinate ring $\mathcal{O}(SL(n, \mathbb{C}))$, certain coordinates of $p(x)$ with respect to the dual canonical basis must be nonnegative.

2. Kazhdan-Lusztig immanants and the dual canonical basis

The dual canonical bases of $\mathcal{O}(SL(n, \mathbb{C}))$ and $\mathcal{O}(GL(n, \mathbb{C}))$ may be obtained easily from a basis of the polynomial ring $\mathbb{C}[x] = \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$. We will call this basis also *the dual canonical basis*.

We will find it convenient to express monomials in $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ in terms of permutations in the symmetric group S_r according to the following conventions. Define a left action of S_n on n-letter words by letting the adjacent transposition s_i swap the letters in positions i and $i+1$. We call the word $w \circ (1 \cdots n)$ the *one-line notation* of w. Thus, the one-line notation of $s_1 s_2$ in S_3 is $s_1(s_2(123)) = s_1(132) = 312$. Denote the one-line notations of w and of w^{-1} by $w_1 \cdots w_n$ and $w_1^{-1} \cdots w_n^{-1}$, respectively. If a factorization of w into adjacent transpositions $w = s_{i_1} \cdots s_{i_\ell}$ is as short as possible, we call the number $\ell = \ell(w)$ the *length* of w. It is well known that $\ell(w)$ is equal to the number of pairs (w_i, w_j) satisfying $w_i > w_j$ and $i < j$.

Before explicitly describing the dual canonical basis of $\mathbb{C}[x]$, let us look at a multigrading of this ring in terms of multisets. $\mathbb{C}[x]$ has a traditional grading by degree,

$$\mathbb{C}[x] = \bigoplus_{r \geq 0} \mathcal{A}_r,$$

where $\mathcal{A}_r$ is the complex span of degree-r monomials. We may refine this grading by defining a multigrading of $\mathcal{A}_r$ indexed by pairs of r-element multisets. Let $\mathcal{M}(n, r)$ be the set of r-element multisets of $[n] \underset{\text{def}}{=} \{1, \ldots, n\}$. Then we have

$$\mathcal{A}_r = \bigoplus_{M, M' \in \mathcal{M}(n,r)} \mathcal{A}_r(M, M'), \tag{2.1}$$

where $\mathcal{A}_r(M, M')$ is the homogeneous component of multidegree (M, M'), i.e. the complex span of monomials in which the multiset of row indices is M and the multiset of column indices is M'. For example, the polynomial $x_{1,1}x_{2,1}^2x_{3,3} - x_{1,1}x_{2,1}x_{2,3}x_{3,1}$ belongs to the component $\mathcal{A}_3(1223, 1113)$ of $\mathbb{C}[x_{1,1}, \dots, x_{3,3}]$.

Closely related to the multigrading (2.1) are *generalized submatrices* of x. Given two r-element multisets $M = m_1 \cdots m_r$, $M' = m'_1 \cdots m'_r$ of $[n]$ (written as weakly increasing words), define the (M, M') *generalized submatrix* of x to be the $r \times r$ matrix

$$x_{M,M'} = \begin{bmatrix} x_{m_1,m'_1} & x_{m_1,m'_2} & \cdots & x_{m_1,m'_r} \\ x_{m_2,m'_1} & x_{m_2,m'_2} & \cdots & x_{m_2,m'_r} \\ \vdots & \vdots & & \vdots \\ x_{m_r,m'_1} & x_{m_r,m'_2} & \cdots & x_{m_r,m'_r} \end{bmatrix}.$$

Letting $y = x_{M,M'}$, we see that for every permutation $w \in S_r$ having one-line notation $w_1 \cdots w_r$, the monomial

$$y_{1,w_1} \cdots y_{r,w_r} = x_{m_1,m'_{w_1}} \cdots x_{m_r,m'_{w_r}} \tag{2.2}$$

belongs to $\mathcal{A}_r(M, M')$. In particular, we may express the polynomial in the preceding paragraph in terms of the matrix $y = x_{1223,1113}$ as $y_{1,1}y_{2,2}y_{3,3}y_{4,4} - y_{1,1}y_{2,2}y_{3,4}y_{4,3}$.

The multigrading (2.1) is also closely related to parabolic subgroups of S_r as follows. To an r-element multiset M, associate a subset $\iota(M)$ of the generators $\{s_1, \dots, s_{r-1}\}$ of S_r by

$$\iota(M) = \{s_j \,|\, m_j = m_{j+1}\}.$$

Let $I = \iota(M)$, $J = \iota(M')$ be the subsets of generators of $W = S_r$ corresponding to multisets M, M', and denote the corresponding parabolic subgroups by W_I, W_J. Letting W_I and W_J act by left and right multiplication on all $r \times r$ matrices (restricting the defining representation of S_r to the parabolic subgroups), we see that $x_{M,M'}$ is fixed by this action.

The dual canonical basis of $\mathbb{C}[x_{1,1}, \dots, x_{n,n}]$ consists of homogeneous elements with respect to the multigrading (2.1). Du [11, Sec. 2] stated a new formula for the elements of this basis and justified it [12] by combining earlier results of Dipper and James [8] and Grojnowski and Lusztig [15]. This formula relies upon alternating sums [11, Sec. 1] of (inverse) Kazhdan-Lusztig polynomials,

$$\widetilde{Q}_{u,w}(q) = \sum_{\substack{v \in W_I w W_J \\ u \leq v \leq w}} (-1)^{\ell(w)-\ell(v)} P_{w_0 v, w_0 u}(q),$$

where u and w are maximal representatives of cosets in $W_I \backslash W / W_J$, $\ell(w)$ is the length of w, and $\leq$ is the Bruhat order on S_r. (See [18].) These polynomials $\widetilde{Q}_{u,w}(q)$ are generalizations of Deodhar's q-parabolic Kazhdan-Lusztig polynomials [7], for when $I = \emptyset$ we have

$$\widetilde{Q}_{u,w}(q) = \widetilde{P}^J_{w_0ww_0^J, w_0uw_0^J}(q),$$

where w_0 and w_0^J are the longest elements of W and W_J, respectively.

We will express the dual canonical basis in terms of generalized submatrices and Kazhdan-Lusztig immanants $\{\mathrm{Imm}_u(x) \,|\, u \in S_n\}$ introduced in [30],

$$\mathrm{Imm}_u(x) = \sum_{w \geq u} (-1)^{\ell(w)-\ell(u)} P_{w_0w,w_0u}(1) x_{1,w_1} \cdots x_{n,w_n}. \tag{2.3}$$

Theorem 2.1. *Let M, M' be r-element multisets of $[n]$ and define $W = S_r$. The nonzero polynomials in the set $\{\mathrm{Imm}_v(x_{M,M'}) \,|\, v \in W\}$ are the dual canonical basis of $\mathcal{A}_r(M, M')$. In particular, the permutations v corresponding to nonzero polynomials are maximal length representatives of double cosets in $W_{\iota(M)} \backslash W / W_{\iota(M')}$.*

Proof. Let $I = \iota(M)$, $J = \iota(M')$. By [11, Lem. 2.2], (see also [12, Sec. 4], [13, Sec. 3]) the dual canonical basis elements of $\mathcal{A}_r(M, M')$ are in bijective correspondence with cosets in $W_{\iota(M)} \backslash W / W_{\iota(M')}$, and each has the form

$$Z_u = \sum_{z \geq u} (-1)^{\ell(z')-\ell(u')} \widetilde{Q}_{u',z'}(1) \prod_{i,j=1}^{n} x_{i,j}^{\alpha(z,i,j)},$$

where u, z are minimal representatives of double cosets in $W_I \backslash W / W_J$, u', z' are the respective maximal coset representatives, and

$$\alpha(z, i, j) = |\{k \,|\, m_k = i, m'_{z_k} = j\}|.$$

For any pair (u, z) of minimal coset representatives and the corresponding pair (u', z') of maximal coset representatives, we have that $u \leq z$ if and only if $u' \leq z'$. (See e.g. [9, Lem. 2.2], [28, Prop. 31].) We may therefore rewrite Du's description by summing over only maximal coset representatives,

$$Z_u = \sum_{z' \geq u'} (-1)^{\ell(z')-\ell(u')} \widetilde{Q}_{u',z'}(1) \prod_{i,j=1}^{n} x_{i,j}^{\alpha(z,i,j)}.$$

Let $y = x_{M,M'}$. Then for any function $f : S_r \to \mathbb{C}$ we have

$$\mathrm{Imm}_f(y) = \sum_{w \in S_r} f(w) y_{1,w_1} \cdots y_{r,w_r}.$$

Since each permutation v in the double coset $W_I w W_J$ satisfies

$$y_{1,v_1} \cdots y_{r,v_r} = y_{1,w_1} \cdots y_{r,w_r},$$

we may sum over these double cosets,

$$\mathrm{Imm}_f(y) = \sum_{D \in W_I \backslash W / W_J} \Big(\sum_{v \in D} f(v) \Big) y_{1,w_1} \cdots y_{r,w_r}, \tag{2.4}$$

where w is any representative of the double coset D. Note that $y_{i,w_i} = x_{j,k}$ if $m_i = j$ and $m'_{w_i} = k$. Thus the exponent of $x_{i,j}$ in $y_{1,w_1} \cdots y_{r,w_r}$ is equal to $\alpha(w,i,j)$,

$$y_{1,w_1} \cdots y_{r,w_r} = \prod_{i,j=1}^{n} x_{i,j}^{\alpha(w,i,j)}.$$

Now for any $u \in S_r$ consider the function $f_u : v \mapsto (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1)$ and the corresponding Kazhdan-Lusztig immanant $\mathrm{Imm}_u(y) = \mathrm{Imm}_{f_u}(y)$ of y. If u is not maximal in $W_I u W_J$, then we have $su > u$ for some transposition s in I, or we have $us > u$ for some transposition s in J. (See, e.g., [5, Thm. 1.2].) By [30, Cor. 6.4] either of these conditions implies that $\mathrm{Imm}_u(y) = 0$. Thus for any maximal coset representative u' we have by (2.4) that

$$\mathrm{Imm}_{u'}(y) = \sum_{D \in W_I \backslash W / W_J} \Big(\sum_{\substack{v \in D \\ v \geq u'}} (-1)^{\ell(v)-\ell(u')} P_{w_0 v, w_0 u'}(1) \Big) \prod_{i,j=1}^{n} x_{i,j}^{\alpha(w',i,j)},$$

where w' is the maximal representative of D. We include the inequality $v \geq u'$ because the number $P_{w_0 v, w_0 u'}(1)$ is zero otherwise. For each coset D and its maximal representative w', the inner sum is equal to

$$(-1)^{\ell(w')-\ell(u')} \sum_{\substack{v \in W_I w' W_J \\ u' \leq v \leq w'}} (-1)^{\ell(w')-\ell(v)} P_{w_0 v, w_0 u'}(1) = \begin{cases} (-1)^{\ell(w')-\ell(u')} \widetilde{Q}_{u',w'}(1), & \text{if } w' \geq u', \\ 0 & \text{otherwise,} \end{cases}$$

and we have

$$\mathrm{Imm}_{u'}(y) = \sum_{\substack{W_I w' W_J \\ w' \geq u'}} (-1)^{\ell(w')-\ell(u')} \widetilde{Q}_{u',w'}(1) \prod_{i,j=1}^{n} x_{i,j}^{\alpha(w,i,j)} = Z_u,$$

as desired. □

Quantizing the Kazhdan-Lusztig immanants by

$$\mathrm{Imm}_v(x;q) = \sum_{w \geq v} (-q^{-1/2})^{\ell(w)-\ell(v)} \widetilde{Q}_{v,w}(q) x_{1,w_1} \cdots x_{n,w_n},$$

one constructs the dual canonical basis of quantum $\mathcal{A}_r(M,M')$ by taking all of the polynomials $(q^{1/2})^{\ell(w_0^J)-\ell(w_0^I)} \mathrm{Imm}_v(x_{M,M'};q)$, where $I = \iota(M)$, $J = \iota(M')$, v is a maximal length coset representative in $W_I \backslash W / W_J$, and w_0^I, w_0^J are the maximal length elements of W_I, W_J. (See [3], [11], [13] for other descriptions of this basis.)

Letting $\mathbf{B}$ be the dual canonical basis of $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$, we have the following well-known formulas for the dual canonical bases of the coordinate rings of $GL(n,\mathbb{C})$ and $SL(n,\mathbb{C})$. The dual canonical basis of

$$\mathcal{O}(GL(n,\mathbb{C})) \cong \mathbb{C}[x_{1,1},\ldots,x_{n,n},t]/(\det(x)t-1)$$

is obtained by dividing elements of $\mathbf{B}$ by powers of the determinant,

$$\bigcup_{r\geq 0}\ \bigcup_{M,M'\in\mathcal{M}(n,r)} \{\mathrm{Imm}_w(x)\det(x)^{-k}\,|\,k\geq 0; w \text{ maximal in } W_{\iota(M)}\backslash S_r/W_{\iota(M')}\}.$$

The dual canonical basis of

$$\mathcal{O}(SL(n,\mathbb{C})) \cong \mathbb{C}[x_{1,1},\ldots,x_{n,n}]/(\det(x)-1)$$

is obtained by projecting $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ onto $\mathcal{O}(SL(n,\mathbb{C}))$, or by setting $\det(x)=1$ in $\mathcal{O}(GL(n,\mathbb{C}))$.

3. Factorization of 3412-avoiding, 4231-avoiding permutations

The factorization of Kazhdan-Lusztig basis elements in Section 4 and the nonnegativity results in Section 5 depend upon a strategic factorization of permutations whose one-line notations avoid certain patterns.

Let v be a permutation v in S_k with one-line notation $v_1\cdots v_k$. Given a word $u=u_1\cdots u_k$ on a totally ordered alphabet, we will say that u *matches the pattern* v if the letters of u appear in the same relative order as those of v. We will also say that u_1 *matches the* v_1, etc. For example, a word $u_1u_2u_3$ matches the pattern 312, with u_1 matching the 3, u_2 matching the 1, and u_3 matching the 2, if we have $u_2<u_3<u_1$. Given a permutation w in S_n having one-line notation $w_1\cdots w_n$, we will say for any indices $i_1<\cdots<i_k$ in $[n]$ that the word $w_{i_1}\cdots w_{i_k}$ is a *subword* of w. We will say that w *avoids the pattern* v if no subword of w matches the pattern v. We will also call such a permutation v*-avoiding.* In particular we will be interested in permutations which avoid the patterns 3412 and 4231.

A somewhat customary map $\oplus : S_n\times S_m \to S_{n+m}$ may be defined in terms of adjacent transpositions by

$$s_{i_1}\cdots s_{i_\ell}\oplus s_{j_1}\cdots s_{j_k} \underset{\mathrm{def}}{=} s_{i_1}\cdots s_{i_\ell}s_{j_1+n}\cdots s_{j_k+n},$$

or equivalently, in terms of one-line notation by

$$u_1\cdots u_n\oplus v_1\cdots v_m \underset{\mathrm{def}}{=} u_1\cdots u_n\cdot(v_1+n)\cdots(v_m+n).$$

We define a permutation w to be $\oplus$*-indecomposable* if it can not be expressed as $w=u\oplus v$. It is easy to see the following properties of the map $\oplus$.

Observation 3.1. *Let u and v be permutations in S_m and S_n.*

(1) u and v avoid the patterns 3412 *and* 4231 *if and only if $u \oplus v$ does.*
(2) $u' \leq u$ and $v' \leq v$ if and only if $u' \oplus v' \leq u \oplus v$.
(3) $\ell(u) + \ell(v) = \ell(u \oplus v)$.

We will factor each 3412-avoiding, 4231-avoiding permutation as a product of *reversals*, permutations $s_{[i,j]}$ having a one-line notation of the form

$$1 \cdots (i-1) \cdot j(j-1) \cdots i \cdot (j+1) \cdots n,$$

for some interval $[i, j]$. Call a sequence $(s_{I_1}, \dots, s_{I_p})$ of reversals a *reversal factorization* of w if $w = s_{I_1} \cdots s_{I_p}$. Note that every permutation has a reversal factorization. Indeed, each adjacent transposition s_i is the reversal $s_{[i,i+1]}$. We define $s_{[i,i]}$ to be the identity permutation, and we denote the left and right endpoints of an interval I by $\lambda(I)$ and $\rho(I)$, respectively.

Let $w \in S_n$ be $\oplus$-indecomposable. Call a reversal factorization $(s_{I_1}, \dots, s_{I_p})$ of w an *indecomposable zig-zag factorization* if there exist a positive integer r and a sequence of nonnegative integers

$$j_1, \dots, j_r, k_1, \dots, k_r, \tag{3.1}$$

all odd except possibly for j_1 and k_r which may also be zero, such that $I_1, \dots, I_p$ may be labeled as

$$\begin{aligned} A_0, B_{1,1}, \dots, B_{1,j_1}, A_1, C_{1,1}, \dots, C_{1,k_1}, D_1,& \\ \dots, B_{i,1}, \dots, B_{i,j_i}, A_i, C_{i,1}, \dots, C_{i,k_i}, D_i, \dots,& \\ B_{r,1}, \dots, B_{r,j_r}, A_r, C_{r,1}, \dots, C_{r,k_r}, D_r,& \end{aligned} \tag{3.2}$$

with labels $A_0, B_{1,1}, \dots, B_{1,j_1}$ unused if $j_1 = 0$ and labels $A_r, C_{r,1}, \dots, C_{r,k_r}$ unused if $k_r = 0$, and with the endpoints of the intervals satisfying the following conditions:

(1) For each i satisfying $j_i \neq 0$ we have
$$\begin{aligned} \lambda(A_{i-1}) < \lambda(B_{i,1}) = \lambda(B_{i,2}) < \lambda(B_{i,3}) = \cdots < \lambda(B_{i,j_i}) = \lambda(D_i), \\ \rho(A_{i-1}) = \rho(B_{i,1}) < \rho(B_{i,2}) = \rho(B_{i,3}) < \cdots = \rho(B_{i,j_i}) < \rho(D_i). \end{aligned}$$
(2) For each i satisfying $k_i \neq 0$ we have
$$\begin{aligned} \lambda(A_i) = \lambda(C_{i,1}) > \lambda(C_{i,2}) = \lambda(C_{i,3}) > \cdots = \lambda(C_{i,k_i}) > \lambda(D_i), \\ \rho(A_i) > \rho(C_{i,1}) = \rho(C_{i,2}) > \rho(C_{i,3}) = \cdots > \rho(C_{i,k_i}) = \rho(D_i). \end{aligned}$$
(3) For $i = 1, \dots, r$ we have $\rho(B_{i,j_i}) < \lambda(C_{i,k_i})$.
(4) For $i = 1, \dots, r-1$ we have $\rho(C_{i,1}) < \rho(B_{i+1,1})$.

By the above conditions, we have that the number p in an indecomposable zig-zag factorization $(s_{I_1}, \dots, s_{I_p})$ is odd, and that the interval I_p is labeled D_r. When $p \geq 3$,

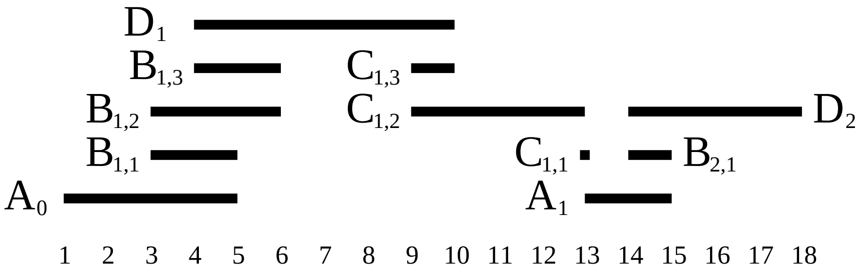

FIGURE 3.1. Intervals corresponding to an indecomposable zig-zag factorization.

then at least one of the integers (3.1) is positive, and the interval I_1 is labeled A_0 if $j_1 > 0$ and A_1 otherwise. Furthermore, the above conditions imply that the intervals

$$\{A_i \,|\, 0 \leq i \leq r\} \cup \{B_{i,j} \,|\, 1 \leq i \leq r, j \text{ even}\} \cup \{C_{i,j} \,|\, 1 \leq i \leq r, j \text{ even}\} \cup \{D_i \,|\, 1 \leq i \leq r\},$$

i.e. $\{I_1, I_3, I_5, \ldots, I_p\}$, have cardinality at least two (of course, excluding unused labels as listed after (3.2).) The assumption that w is $\oplus$-indecomposable implies that we have $I_1 \cup \cdots \cup I_p = [n]$. For convenience, we also will define the empty reversal factorization of the identity element of S_1 to be an indecomposable zig-zag factorization.

If a permutation w decomposes as $w = w^{(1)} \oplus \cdots \oplus w^{(m)}$ and has a reversal factorization W which is a concatenation $W_1 \cdots W_m$ of indecomposable zig-zag factorizations corresponding to $w^{(1)}, \ldots, w^{(m)}$, then we will call W a *zig-zag factorization* of w.

Figure 3.1 shows intervals, labeled as in (3.2), corresponding to the indecomposable zig-zag factorization

$$(s_{[1,5]}, s_{[3,5]}, s_{[3,6]}, s_{[4,6]}, s_{[13,15]}, s_{[13,13]}, s_{[9,13]}, s_{[9,10]}, s_{[4,10]}, s_{[14,15]}, s_{[14,18]}) \tag{3.3}$$

of the permutation in S_{18} having one-line notation

$$(10, 9, 8, 2, 1, 3, 7, 6, 13, 12, 11, 5, 18, 17, 4, 16, 15, 14). \tag{3.4}$$

In addition to illuminating the choice of the term *zig-zag*, Figure 3.1 suggests partially ordering the intervals $(I_1, \ldots, I_p)$ used in a zig-zag factorization by defining $I_i < I_j$ if $i < j$ and $I_i \cap I_j \neq \emptyset$, and by taking the transitive closure of this relation. For an indecomposable zig-zag factorization, all comparabilities in this poset are given by

$$\begin{aligned} A_{i-1} < B_{i,1} < \cdots < B_{i,j_1} < D_i, \\ A_i < C_{i,1} < \cdots < C_{i,k_1} < D_i, \end{aligned} \tag{3.5}$$

for $i = 1, \ldots, r$. We will return to this poset in Section 5.

Note that an indecomposable zig-zag factorization of w^{-1} is obtained by listing the intervals (3.2) in the order

$$\begin{aligned}(3.6)\quad & D_1, B_{1,j_1}, \dots, B_{1,1}, A_0, C_{1,k_1}, \dots, C_{1,1}, D_2, B_{2,j_2}, \dots, B_{2,1}, A_1, \\ & \dots, C_{i,k_i}, \dots, C_{i,1}, D_{i+1}, B_{i+1,j_{i+1}}, \dots, B_{i+1,1}, A_i, \dots, \\ & C_{r-1,k_{r-1}}, \dots, C_{r-1,1}, D_r, B_{r,j_r}, \dots, B_{r,1}, A_{r-1}, C_{r,k_r}, \dots, C_{r,1}, A_r.\end{aligned}$$

From a zig-zag factorization of w, one can easily identify certain decreasing subwords of the one-line notations of w and w^{-1}.

Observation 3.2. *Let w have a zig-zag factorization $(s_{I_1}, \dots, s_{I_p})$. For numbers $a < b$, if the earliest interval in this factorization to contain a is also the earliest to contain b, then we have $w_a > \cdots > w_b$. If the latest interval to contain a is also the latest to contain b, then we have $w_a^{-1} > \cdots > w_b^{-1}$.*

Moreover, knowledge that a permutation has a zig-zag factorization allows us to use decreasing subwords of the one-line notations of w and w^{-1} to construct the zig-zag factorization.

Observation 3.3. *Suppose that w has an indecomposable zig-zag factorization with intervals labeled as in (3.2), and let ℓ, m be the greatest indices for which the one-line notations of w and w^{-1} satisfy*

$$\begin{aligned} w_1 &> \cdots > w_\ell, \\ w_1^{-1} &> \cdots > w_m^{-1}. \end{aligned} \tag{3.7}$$

Then we have the following.

(1) *$\ell = m$ if and only if $w = s_{[1,n]}$.*
(2) *$\ell < m$ if and only if $j_1 = 0$, $D_1 = [1, m]$, and $C_{1,k_1} = [\ell + 1, m]$.*
(3) *$\ell > m$ if and only if $A_0 = [1, \ell]$ and $B_{1,1} = [m + 1, \ell]$.*

These two observations imply that the zig-zag factorization of a permutation is unique if it exists, and that we may construct it with the following algorithm.

(1) Initialize the current factorization to be the empty sequence ( ).
(2) Let $t - 1$ be the greatest index for which the one-line notations of w and w^{-1} satisfy
$$w_1 \cdots w_{t-1} = w_1^{-1} \cdots w_{t-1}^{-1} = 1 \cdots (t - 1).$$
(3) If $t - 1 = n$, output the current factorization sequence and stop.
(4) If the longest initial decreasing subwords of $w_t \cdots w_n$ and $w_t^{-1} \cdots w_n^{-1}$ are equally long, do
    (a) Let ℓ be the greatest index satisfying $w_t > \cdots > w_\ell$, $w_t^{-1} > \cdots > w_\ell^{-1}$ and replace w by $s_{[t,\ell]}w$.
    (b) Append $s_{[t,\ell]}$ to the current factorization sequence.

Otherwise, if the longest initial decreasing subword of $w_t \cdots w_n$ is longer than the longest inital decreasing subword of $w_t^{-1} \cdots w_n^{-1}$, do

(a) While the greatest indices ℓ, m satisfying $w_t > \cdots > w_\ell$, $w_t^{-1} > \cdots > w_m^{-1}$ satisfy $\ell > m$, do
   (i) Append the reversals $(s_{[t,\ell]},\ s_{[m+1,\ell]})$ to a temporary factorization sequence.
   (ii) Replace t by $m+1$ and w by $s_{[t,\ell]} s_{[m+1,\ell]} w$.

(b) Append the temporary factorization sequence to the current factorization sequence.

Otherwise if the longest initial decreasing subword of $w_t \cdots w_n$ is shorter than the longest inital decreasing subword of $w_t^{-1} \cdots w_n^{-1}$, do

(a) While the greatest indices ℓ, m satisfying $w_t > \cdots > w_\ell$, $w_t^{-1} > \cdots > w_m^{-1}$ satisfy $\ell < m$, do
   (i) Prepend the reversals $(s_{[\ell+1,m]},\ s_{[t,m]})$ to a temporary factorization sequence.
   (ii) Replace t by $\ell+1$ and w by $w s_{[t,m]} s_{[\ell+1,m]}$.

(b) Append the temporary factorization sequence to the current factorization sequence.

(5) Return to (2) with the updated permutation w.

Assuming that a permutation w has an indecomposable zig-zag factorization or that it avoids the patterns 3412 and 4231 allows us to restate Equation (3.7) more precisely.

Lemma 3.4. *Let w be a permutation in S_n and let a, b be the greatest indices for which the one-line notations of w and w^{-1} satisfy*

$$\begin{aligned} w_1 > \cdots > w_b, \\ w_1^{-1} > \cdots > w_a^{-1}. \end{aligned} \tag{3.8}$$

Assume that $a \leq b$. If w avoids the patterns 3412 and 4231 or has a zig-zag factorization, then we have

$$w_{b+1-i} = i, \qquad i = 1, \ldots, a. \tag{3.9}$$

Furthermore, we have $w_j = a+1$ for some $j \geq b+1$, unless $a = b = n$.

Proof. If $b = n$ then we also have $a = n$ and the conclusions of the lemma are clearly true. Assume therefore that $b < n$.

Suppose first that w avoids the patterns 3412 and 4231 and we do not have the equalities (3.9). Let c be the greatest integer for which $w_{b+1-c} \neq c$, $c \leq a$. If $w_{b+1-c} < c$, then we have $w_b < 1$, a contradiction. We therefore have $w_{b+1-c} > c$, implying $w_{b+1-a} > a$ and $c = a$ by our choice of c. Since a and b are the greatest indices satisfying (3.8), we have that $w_b < w_{b+1}$, and that $a(a+1)$ is a subword of $w_{b+2} \cdots w_n$. If $w_1 > w_{b+1}$, then the subword $w_1 w_b w_{b+1} a$ of $w_1 \cdots w_n$ matches the pattern 4231, a contradiction. If $w_1 < w_{b+1}$, then the subword $w_1 w_{b+1} a(a+1)$ matches the pattern 3412, another contradiction. We therefore have (3.9).

Next suppose that w has a zig-zag factorization $(s_{I_1}, \ldots, s_{I_p})$. By Observation 3.3 we have $I_1 = [1, b]$, $I_2 = [a+1, b]$, and by definition we have $\lambda(I_i) \geq a+1$ for $i = 3, \ldots, p$. Thus the one-line notation of the product $s_{I_2} \cdots s_{I_p}$ begins with $1 \cdots a$. Applying $s_{I_1} = s_{[1,b]}$ on the left reverses the letters $1, \ldots, a$ and puts them in positions $b+1-a, \ldots, b$. Again we have (3.9). Thus,

$$w_1 > \cdots > w_{b-a} > w_{b-a+1} = a. \tag{3.10}$$

Now suppose that the letter $a+1$ appears in the subword $w_1 \cdots w_b$ of w. By (3.10) we must have $w_{b-a} = a+1$, which contradicts the maximality of a in (3.8). Since $b < n$, we conclude that $a+1$ appears in the word $w_{b+1} \cdots w_n$. $\square$

Using this lemma, we can now prove the equivalence of zig-zag factorization and avoidance of the patterns 3412 and 4231.

Theorem 3.5. *A permutation $u \in S_n$ avoids the patterns* 3412 *and* 4231 *if and only if it has a zig-zag factorization.*

Proof. By Observation 3.1 and the definition of zig-zag factorization, it suffices to prove that an $\oplus$-indecomposable permutation u avoids the patterns 3412 and 4231 if and only if it has an indecomposable zig-zag factorization.

It is easy to verify the statement for permutations in S_1, S_2, S_3. Using induction on n, assume the statement to be true also for permutations in $S_4, \ldots, S_{n-1}$. If neither u nor u^{-1} satisfies the conditions of w in (3.8) and (3.9) then u does not avoid the patterns 3412 and 4231 and u does not have a zig-zag factorization. Assume therefore that u satisfies these conditions and define $v \in S_n$ by $u = s_{[1,b]} s_{[a+1,b]} v$, where a and b are the maximal indices satisfying (3.8). Then the one-line notation of u is

$$u_1 \cdots u_n = u_1 \cdots u_{b-a} \cdot a \cdots 21 \cdot u_{b+1} \cdots u_n = v_{a+1} \cdots v_b \cdot a \cdots 21 \cdot v_{b+1} \cdots v_n,$$

and the one-line notation of v is

$$v_1 \cdots v_n = 12 \cdots a \cdot v_{a+1} \cdots v_b \cdot v_{b+1} \cdots v_n = 12 \cdots a \cdot u_1 \cdots u_{b-a} \cdot u_{b+1} \cdots u_n.$$

First we claim that u avoids the patterns 3412 and 4231 if and only if v does. Assume that some subword $v_i v_j v_k v_\ell$ of v matches the pattern 3412 or 4231. Then we must have $i \geq a+1$, since otherwise all letters to the right of position i in v would be greater than v_i. But then $v_i v_j v_k v_\ell$ is also a subword of u.

Now assume that a subword $u_i u_j u_k u_\ell$ of u matches the pattern 3412 or 4231. If at least one of the letters $u_i u_j u_k u_\ell$ belongs to the range $[1, a]$, then one such letter must match the 1 in 3412 or 4231. But then a subword of the decreasing word

$u_1 \cdots u_{b-a}$ must match 34 or 423, which is impossible. Thus $u_i u_j u_k u_\ell$ is a subword of $u_1 \cdots u_{b-a} \cdot u_{b+1} \cdots u_1 = v_{a+1} \cdots v_n$.

Next we claim that u has an indecomposable zig-zag factorization if and only if v has a zig-zag factorization. Assume that u has an indecomposable zig-zag factorization $(s_{I_1}, \ldots, s_{I_r})$. By Observation 3.3 we have $I_1 = [1, b]$, $I_2 = [a+1, b]$. Thus $(s_{I_3}, \ldots, s_{I_r})$ is a zig-zag factorization of v.

Now assume that v has a zig-zag factorization $(s_{J_1}, \ldots s_{J_r})$. Let c and d be the greatest indices for which we have

$$\begin{gathered} v_{a+1} > \cdots > v_c \\ v_{a+1}^{-1} > \cdots > v_d^{-1}. \end{gathered}$$

Since $u_1 \cdots u_{b-a}$ is a decreasing word, we have $c \geq b$. If $d = c$ then by Observation 3.3, v decomposes as $v = 1 \cdots a \oplus c(c-1) \cdots (a+1) \oplus v'$ for some permutation $v' \in S_{n-c}$, and u decomposes as $u = s_{[1,b]} s_{[a+1,b]} s_{[a+1,c]} \oplus v'$, contradicting the $\oplus$-indecomposability of u. Suppose therefore that $d > c$. Then for some i we have $J_i = [c+1, d]$, $J_{i+1} = [a+1, d]$, and

$$\lambda(J_1), \ldots, \lambda(J_{i-1}) > c + 1 \geq b + 1.$$

It follows that $(s_{[1,b]}, s_{[a+1,b]}, s_{J_1}, \ldots, s_{J_r})$ is an indecomposable zig-zag factorization of u. Now suppose that $d < c$. Then $J_1 = [a+1, c]$, $J_2 = [d+1, c]$. If $c = b$, then $v_a^{-1} = a$ and $v_{a+1}^{-1} = b$ by Lemma 3.4. This implies that $u_a^{-1} = b - a + 1$ and $u_{a+1}^{-1} = b - a$, contradicting our choice of a to be the minimal index satisfying $u_1^{-1} > \cdots > u_a^{-1}$. If $c > b$, then $(s_{[1,b]}, s_{[a+1,b]}, s_{J_1}, \ldots, s_{J_r})$ is an indecomposable zig-zag factorization of u. □

The number of inversions in a permutation possessing an indecomposable zig-zag permutation is given by the following formula.

Proposition 3.6. *Let w avoid the patterns 3412 and 4231 and have an indecomposable zig-zag factorization $(s_{I_1}, \ldots, s_{I_p})$ with intervals labeled as in (3.2). Then we have*

$$\begin{aligned} \ell(w) &= \binom{|A_0|}{2} + \sum_{i=1}^{r} \left(\binom{|A_i|}{2} + \binom{|D_i|}{2} + \sum_{j=1}^{j_i} (-1)^j \binom{|B_{i,j}|}{2} + \sum_{k=1}^{k_i} (-1)^k \binom{|C_{i,k}|}{2} \right) \\ &= \sum_{j=1}^{p} (-1)^{j+1} \binom{|I_j|}{2}. \end{aligned}$$

Proof. Suppose first that w is the reversal $s_{[1,n]}$. Then the sequence (3.1) is $(j_1, k_1) = (0, 0)$ with $r = 1$, and the interval $[1, n]$ is labeled D_1. The above formula therefore is equal to $\binom{|D_1|}{2} = \binom{n}{2} = \ell(s_{[1,n]})$.

Now suppose that $w \neq s_{[1,n]}$. Assume that the interval I_1 is labeled A_0. (If this is not the case, we can apply the following argument to w^{-1}, since $\ell(w^{-1}) = \ell(w)$.) Assume by induction that the claimed formula holds for permutations in $S_1, \ldots, S_{n-1}$, and define a permutation v by $w = s_{A_0} s_{B_{1,1}} v$. Then v is the direct sum of the identity permutation in $S_{\lambda(B_{1,1})-1}$ with a 3412-avoiding, 4231-avoiding permutation in $S_{n-\lambda(B_{1,1})}$, and we have that

$$\ell(v) = \sum_{i=1}^{r} \left(\binom{|A_i|}{2} + \binom{|D_i|}{2} + \sum_{j=2}^{j_i} (-1)^j \binom{|B_{i,j}|}{2} + \sum_{k=1}^{k_i} (-1)^k \binom{|C_{i,k}|}{2} \right).$$

Since the one-notation of v begins with $1 \cdots (\lambda(B_{1,1})-1)$ and is followed by a decreasing sequence of $|B_{1,1}|$ letters, the permutation $s_{B_{1,1}} v$ has $\binom{|B_{1,1}|}{2}$ fewer inversions than v. Since the one-line notation of $s_{B_{1,1}} v$ begins with an increasing sequence of $|A_0|$ letters, w has $\binom{|A_0|}{2}$ more inversions than this, and we have the desired formula. □

From the zig-zag factorization of a permutation, one may construct a reduced expression as follows.

Corollary 3.7. *Let w avoid the patterns 3412 and 4231 and have an indecomposable zig-zag factorization with intervals labeled as in (3.2). Choose reduced expressions*

$$\begin{gathered} \{W(A_i) \,|\, 0 \leq i \leq r\}, \\ \{W(B_{i,j}) \,|\, 1 \leq i \leq r,\ j = 2, 4, \ldots, j_i - 1\}, \\ \{W(C_{i,k}) \,|\, 1 \leq i \leq r,\ k = 2, 4, \ldots, k_i - 1\}, \\ \{W(D_i) \,|\, 1 \leq i \leq r\} \end{gathered}$$

for the permutations

$$\begin{gathered} \{s_{A_0} s_{B_{1,1}}\} \cup \{s_{A_i} s_{C_{i,1}} s_{B_{i+1,1}} \,|\, 0 < i < r\} \cup \{s_{A_r} s_{C_{r,1}}\}, \\ \{s_{B_{i,j}} s_{B_{i,j+1}} \,|\, 1 \leq i \leq r,\ j = 2, 4, \ldots, j_i - 1\}, \\ \{s_{C_{i,k}} s_{B_{i,j+1}} \,|\, 1 \leq i \leq r,\ k = 2, 4, \ldots, k_i - 1\}, \\ \{s_{D_i} \,|\, 1 \leq i \leq r\}. \end{gathered}$$

Then a reduced expression for w in terms of adjacent transpositions is given by the concatenation of the reduced expressions

$$\begin{aligned} &W(A_0), W(B_{1,2}), \ldots, W(B_{1,j_1-1}), W(A_1), W(C_{1,2}), \ldots, W(C_{1,k_1-1}), W(D_1), \\ &\ldots, W(B_{i,2}), \ldots, W(B_{i,j_i-1}), W(A_i), W(C_{i,2}), \ldots, W(C_{i,k_i-1}), W(D_i), \ldots, \\ &W(B_{r,2}), \ldots, W(B_{r,j_r-1}), W(A_r), W(C_{r,1}), \ldots, W(C_{r,k_r}), W(D_r). \end{aligned} \tag{3.11}$$

Proof. Multiplying the reduced expressions in (3.11), we clearly obtain w. By Proposition 3.6, the length of this expression for w is as short as possible and the expression is therefore reduced. □

Furthermore, for each 3412-avoiding, 4231-avoiding permutation w, we have the following description of permutations $v \leq w$.

Proposition 3.8. *Let $(s_{I_1}, \dots, s_{I_p})$ be a zig-zag factorization of $w \in S_n$. Then $u \leq w$ if and only if there exist permutations $t_{I_1}, \dots, t_{I_p}$ satisfying $u = t_{I_1} \cdots t_{I_p}$ and $t_{I_j} \leq s_{I_j}$ for $j = 1, \dots, p$.*

Proof. By Observation 3.1, we may assume that w is $\oplus$-indecomposable. Label the intervals $I_1, \dots, I_p$ as in (3.2). Assume that I_1 is labeled A_0. (If I_1 is labeled D_1, the claim is trivial; if it is labeled A_1, we may replace w by w^{-1} in the claim and use the factorization (3.6) for w^{-1}.)

Suppose first that $u \leq w$. Choosing a reduced expression W for w as in (3.11), we can find a reduced expression U for u which is a subexpression of W. It is then easy to express U as a product of the desired form.

Now suppose that we have $u = t_{I_1} \cdots t_{I_p}$, with $t_{I_j} \leq s_{I_j}$ for $j = 1, \dots, p$. Combining the inequalities

$$\begin{aligned}
t_{A_0} t_{B_{1,1}} &\leq s_{A_0}, && \\
t_{A_i} t_{C_{i,1}} t_{B_{i+1,1}} &\leq s_{A_i}, && 1 \leq i < r, \\
t_{A_r} t_{C_{r,1}} &\leq s_{A_r}, && \\
t_{B_{i,j}} t_{B_{i+1,j}} &\leq s_{B_{i,j}}, && 1 \leq i \leq r, \quad j = 2, 4, \dots, j_i - 1, \\
t_{C_{i,k}} t_{C_{i+1,k}} &\leq s_{C_{i,k}}, && 1 \leq i \leq r, \quad k = 2, 4, \dots, k_i - 1, \\
t_{D_i} &\leq s_{D_i}, && 1 \leq i \leq r.
\end{aligned}$$

with a reduced expression of the form (3.11) for w, we have that $u \leq w$. □

A second description of permutations less than or equal to a 3412-avoiding, 4231-avoiding permutation is as follows.

Lemma 3.9. *Let $w \in S_n$ avoid the patterns 3412 and 4231 and have an indecomposable zig-zag factorization $(s_{I_1}, \dots, s_{I_p})$. Then each $v \leq w$ factors uniquely as $v = uy$ with $\ell(v) = \ell(u) + \ell(y)$ and*

$$\begin{aligned}
&u \leq s_{I_1}, \\
&y \leq zy, \qquad \text{for all } z \leq s_{I_1}, \\
&y \leq s_{I_3} \cdots s_{I_p}.
\end{aligned} \tag{3.12}$$

Furthermore, this factorization defines a bijection between permutations $v \leq w$ and pairs (u, y) satisfying (3.12).

Proof. Let S_{I_1} be the parabolic subgroup of S_n corresponding to the interval I_1, and let W^{I_1} be the set of minimum length representatives of cosets of the form $S_{I_1} y$. It is well known that elements y of W^{I_1} are characterized by the condition that $y \leq zy$

for all $z \in S_{I_1}$, that every permutation $v \in S_n$ factors uniquely as $v = uy$ with $u \in S_{I_1}$, $y \in W^{I_1}$, $\ell(v) = \ell(u) + \ell(y)$, and that this factorization defines a bijection $\phi : S_n \to S_{I_1} \times W^{I_1}$.

To see that ϕ satisfies $\phi(\{v \,|\, v \leq w\}) \subset S_{I_1} \times \{y \in W^{I_1} \,|\, y \leq s_{I_3} \cdots s_{I_p}\}$, choose a permutation $v \leq w$. By Proposition 3.8, we can factor v as $t_{I_1} t_{I_3} \cdots t_{I_p}$ with $t_{I_j} \leq s_{I_j}$ for j odd and with $\ell(v) = \ell(t_{I_1}) + \ell(t_{I_3}) + \cdots + \ell(t_{I_p})$. Furthermore, it is easy to choose $t_{I_1}, \ldots, t_{I_p}$ so that the first $|I_1|$ letters in the one-line notation of $t_{I_3} \cdots t_{I_p}$ are increasing. Thus we have $u = t_{I_1} \in S_{I_1}$ and $y = t_{I_3} \cdots t_{I_p} \in W^{I_1}$. The containment $\phi^{-1}(S_{I_1} \times \{y \in W^{I_1} \,|\, y \leq s_{I_3} \cdots s_{I_p}\}) \subset \{v \,|\, v \leq w\}$ follows immediately from Proposition 3.8. □

4. A Factorization of Kazhdan-Lusztig basis elements

While each nonnegative linear combination of dual canonical basis elements is a totally nonnegative polynomial, the converse of this statement is false. Intimately related to this fact is a vector space duality between the homogeneous component $\mathcal{A}_n([n],[n])$ of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ and the group algebra $\mathbb{C}[S_n]$ defined by

$$\langle x_{1,u(1)} \cdots x_{n,u(n)}, v \rangle = \delta_{u,v}.$$

In particular, Kazhdan and Lusztig [22] defined a basis $\{C'_w(q) \,|\, w \in S_n\}$ of the Hecke algebra $H_n(q)$ by

$$C'_v(q) = q^{-\ell(v)/2} \sum_{u \leq v} P_{u,v}(q) T_u,$$

where $\{P_{u,v}(q) \,|\, u, v \in S_n\}$ are the *Kazhdan-Lusztig polynomials*, which we used earlier the definition (2.3) of Kazhdan-Lusztig immanants. These polynomials have no known elementary formula.

Specializing the Kazhdan-Lusztig basis at $q = 1$, we obtain a basis of $\mathbb{C}[S_n]$. Dual to this is the basis of Kazhdan-Lusztig immanants,

$$\langle \mathrm{Imm}_u(x), C'_v(1) \rangle = \delta_{u,v}.$$

Since no elementary formula is known for the Kazhdan-Lusztig polynomials, it is not surprising that also no elementary formula is known for the Kazhdan-Lusztig basis of the Hecke algebra or for the Kazhdan-Lusztig immanants. Nevertheless, we can deduce certain properties of the Kazhdan-Lusztig immanants by studying Kazhdan-Lusztig basis elements corresponding to reversals and to 3412-avoiding, 4231-avoiding permutations.

For each adjacent transposition s_i in S_n, the Kazhdan-Lusztig basis element $C'_{s_i}(q)$ has the form $q^{-1/2}(T_e + T_{s_i})$. The following result of Billey and Warrington [2] uses pattern avoidance to characterize basis elements which factor as products of these.

Theorem 4.1. *Let $s_{i_1} \cdots s_{i_\ell}$ be a reduced expression for w. Then we have*

$$C'_w(q) = C'_{s_{i_1}}(q) \cdots C'_{s_{i_\ell}}(q)$$

if and only if w avoids the patterns 321, 56781234, 46781235, 56718234, 46718235.

To state an analogous factorization result in Theorem 4.3, we generalize from adjacent transpositions to reversals, for which we have

$$C'_{s_{[i,j]}}(q) = (q^{-1/2})^{\binom{j-i+1}{2}} \sum_{v \leq s_{[i,j]}} T_v. \tag{4.1}$$

More generally still, results linking pattern avoidance, smoothness in Schubert varieties and the Kazhdan-Lusztig basis [4], [6], [23] imply that basis elements corresponding to 3412-avoiding, 4231-avoiding permutations have the form

$$C'_w(q) = q^{-\ell(w)/2} \sum_{v \leq w} T_v.$$

That is, the Kazhdan-Lusztig polynomial $P_{v,w}(q)$ is identically 1 when w avoids the patterns 3412 and 4231. (See also [1, Ch. 6].) While the above expression is itself rather simple, we will show in Theorem 4.3 that it factors as a product of basis elements of the form (4.1). This factorization, closely related to the zig-zag factorization defined in Section 3, will help describe the dual cone of total nonnegativity, to be defined in Section 5.

For any nonnegative integer k, define the polynomials k_q and $k_q!$ by

$$k_q = \begin{cases} 1 + q + \cdots + q^{k-1} & \text{if } k > 0, \\ 0 & \text{if } k = 0, \end{cases}$$

$$k_q! = \begin{cases} 1_q 2_q \cdots k_q & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Specializing the polynomials at $q = 1$, we obtain k and $k!$ respectively. Since $k_q!$ is a generating function for permutations in S_k by length, i.e. $k_q! = \sum_{w \in S_k} q^{\ell(w)}$, the Hecke algebra identity $T_v C'_{s_{[i,j]}}(q) = q^{\ell(v)} C'_{s_{[i,j]}}(q)$ for $v \leq s_{[i,j]}$ implies that we have

$$(C'_{s_{[i,j]}}(q))^2 = (q^{-1/2})^{\binom{j-i+1}{2}} (j-i+1)_q! C'_{s_{[i,j]}}(q). \tag{4.2}$$

The author is grateful to K. Peterson for pointing out this identity.

Proposition 4.2. *Let $w \in S_n$ avoid the patterns* 3412 *and* 4231 *and have an indecomposable zig-zag factorization $(s_{[1,m]}, s_{[j+1,m]}, s_{I_3}, \ldots, s_{I_p})$. Let e be the identity element of S_j, and define $w' \in S_{n-j}$ by $e \oplus w' = s_{I_3} \cdots s_{I_p}$. Then we have*

$$C'_{s_{[1,m]}}(q) C'_{e \oplus w'}(q) = (m-j)_q! (q^{-1/2})^{\binom{m-j}{2}} C'_w(q). \tag{4.3}$$

Proof. By Proposition 3.6, we have $\ell(w) = \ell(e \oplus w') + \binom{m}{2} - \binom{m-j}{2}$. Thus, since w avoids the patterns 3412 and 4231, the right-hand side of Equation (4.3) is equal to

$$(m-j)_q!(q^{-1/2})^{\binom{m}{2}+\ell(w')} \sum_{v \le w} T_v. \tag{4.4}$$

Since $s_{[1,m]}$ and w' avoid the patterns 3412 and 4231, the left-hand side is equal to

$$(q^{-1/2})^{\binom{m}{2}+\ell(w')} \sum_{u \le s_{[1,m]}} T_u \sum_{v \le e \oplus w} T_v. \tag{4.5}$$

For indices i, i', let $S_{[i,i']}$ be the parabolic subgroup of S_n generated by $s_i, \ldots, s_{i'-1}$. Define $X \subset S_{[1,m]}$ to be the set of minimum length representatives of cosets $uS_{[j+1,m]}$ in $S_{[1,m]}$. Then each permutation $u \in S_{[1,m]}$ factors uniquely as $u = xz$ with $x \in X$ and $z \in S_{[j+1,m]}$. Using Proposition 3.8 and the definition of an indecomposable zig-zag factorization, it is straightforward to show that the group $S_{[j+1,m]}$ acts by left multiplication on the set $\{v \,|\, v \le e \oplus w'\}$. Let $Y \subset S_{[j+1,m]}$ be the set of minimum length representatives of the resulting orbits. Then Y is also the set of minimum length representatives in the orbits resulting from the action of $S_{[1,m]}$ on $\{v \,|\, v \le w\}$.

By Lemma 3.9 and Equation (4.2) we then have

$$\begin{aligned}
\sum_{u \le s_{[1,m]}} T_u \sum_{v \le e \oplus w'} T_v &= \sum_{x \in X} T_x \Big(\sum_{z \in S_{[j+1,m]}} T_z \Big)^2 \sum_{y \in Y} T_y \\
&= (m-j)_q! \sum_{\substack{x \in X \\ z \in S_{[j+1,m]} \\ y \in Y}} T_x T_z T_y \\
&= (m-j)_q! \sum_{\substack{u \in S_{[1,m]} \\ y \in Y}} T_u T_y \\
&= (m-j)_q! \sum_{v \le w} T_v.
\end{aligned} \tag{4.6}$$

By (4.4) and (4.5) this gives the desired equality (4.3). $\square$

The factorization result for permutations given in Theorem 3.5 now translates into the following factorization result for Kazhdan-Lusztig basis elements. Given a sequence of intervals $(I_1, \ldots, I_m)$, define the Hecke algebra element

$$\Phi(I_1, \ldots, I_m; q) = C'_{s_{I_1}}(q) \cdots C'_{s_{I_m}}(q).$$

Theorem 4.3. *Let w avoid the patterns 3412 and 4231 and have an indecomposable zig-zag factorization $(s_{I_1}, \ldots, s_{I_p})$ with intervals labeled as in (3.2). Then $C'_w(q)$*

factors as

$$C'_w(q) = \frac{(q^{1/2})^{\binom{|I_2|}{2}+\binom{|I_4|}{2}+\cdots+\binom{|I_{p-1}|}{2}}}{|I_2|_q!|I_4|_q!\cdots|I_{p-1}|_q!}\Phi(I_1, I_3, \ldots, I_p; q). \tag{4.7}$$

Proof. Assume that Equation (4.7) holds for permutations in $S_1, \ldots, S_{n-1}$.

If the interval I_1 is labeled as A_0, define $v = s_{I_3}s_{I_4}\cdots s_{I_p}$. Then $v = e \oplus w'$ for some 3412-avoiding, 4231-avoiding permutation w' in $S_{n'}$, $n' < n$. Using the induction hypothesis and Proposition 3.6 we then have

$$C'_v(q) = \frac{(q^{1/2})^{\binom{|I_4|}{2}+\binom{|I_6|}{2}+\cdots+\binom{|I_{p-1}|}{2}}}{|I_4|_q!|I_6|_q!\cdots|I_{p-1}|_q!}\Phi(I_3, I_5, \ldots, I_p; q). \tag{4.8}$$

By Proposition 4.2 we may multiply both sides of (4.8) by

$$\frac{q^{\frac{1}{2}\binom{|I_2|}{2}}}{|I_2|_q!}C'_{s_{I_1}}(q)$$

to obtain Equation (4.7).

If the interval I_1 is labeled as A_1, then apply the above argument to the zig-zag factorization (3.6) of w^{-1} to factor $C'_{w^{-1}}(q)$ as in Equation (4.7) and reverse the order of the factors to obtain a factorization of $C'_w(q)$. Finally, if the interval I_1 is labeled as D_1, then the result follows from Equation (4.1). □

Since $C'_u(q)C'_v(q) = C'_{u\oplus v}(q)$, we may factor $C'_w(q)$ for any 3412-avoiding, 4231-avoiding permutation w as a product of expressions of the form (4.7). On the other hand, we may use Equation (4.2) to obtain the following alternative to the factorization (4.7),

$$C'_w(q) = \frac{q^{\binom{|I_2|}{2}+\binom{|I_4|}{2}+\cdots+\binom{|I_{p-1}|}{2}}}{(|I_2|_q!|I_4|_q!\cdots|I_{p-1}|_q!)^2}\Phi(I_1, I_2, \ldots, I_p; q).$$

To illustrate Theorem 4.3, let $w \in S_{18}$ be the permutation having one-line notation (3.4). Using the zig-zag factorization (3.3) of w, we may factor $C'_w(q)$ as

$$\begin{aligned}
&\frac{q^{\frac{1}{2}\left(\binom{3}{2}+\binom{3}{2}+\binom{1}{2}+\binom{2}{2}+\binom{2}{2}\right)}}{3_q!3_q!0_q!1_q!1_q!}C'_{s_{[1,5]}}(q)C'_{s_{[3,6]}}(q)C'_{s_{[13,15]}}(q)C'_{s_{[9,13]}}(q)C'_{s_{[4,10]}}(q)C'_{s_{[14,18]}}(q)\\
&\qquad = \frac{q^4}{3_q!^2}C'_{s_{[1,5]}}(q)C'_{s_{[3,6]}}(q)C'_{s_{[13,15]}}(q)C'_{s_{[9,13]}}(q)C'_{s_{[4,10]}}(q)C'_{s_{[14,18]}}(q).
\end{aligned}$$

Note that in Figure 3.1 it is easy to distinguish between the intervals which contribute Kazhdan-Lusztig basis elements to the above expression and the intervals which contribute q-factorials and powers of q. In terms of the poset defined in (3.5), the latter intervals are equal to intersections of the intervals which they cover and which cover them.

The factorizations in Theorems 4.1 and 4.3 are related by more than their appearance. The following result shows that the two factorizations are identical for permutations which avoid all seven of the forbidden patterns, equivalently, for permutations which avoid the two patterns 321 and 3412.

Proposition 4.4. *Let w avoid the patterns* 321 *and* 3412*. Then the factorizations of $C'_w(q)$ in Theorems 4.1 and 4.3 agree.*

Proof. Since the operation $\oplus$ preserves avoidance of both patterns, we may assume w to have an indecomposable zig-zag factorization $(s_{I_1}, \ldots, s_{I_p})$. By Observation 3.2, we have

$$|I_m| = \begin{cases} 2 & m \text{ odd}, \\ 1 & m \text{ even}. \end{cases}$$

By Proposition 3.6, $\ell(w)$ is equal to the number of these intervals having cardinality 2. Thus the zig-zag factorization of w (ignoring reversals of the form $s_{[i,i]}$) is a reduced expression for w, and the factorization

$$C'_w(q) = C'_{s_{I_1}}(q) C'_{s_{I_3}}(q) \cdots C'_{s_{I_p}}(q)$$

given by Theorem 4.3 agrees with that given by Theorem 4.1. □

It is easy to see that neither Theorem 4.1 nor Theorem 4.3 generalizes the other, e.g. by using them to factor $C'_{4312}(q)$ and $C'_{3412}(q)$. On the other hand, they must be special cases of a more general fact since we have

$$C'_{4231}(q) = C'_{s_{[1,2]}}(q) C'_{s_{[2,4]}}(q) C'_{s_{[1,2]}}(q),$$

while 4231 avoids neither 321 nor 4231. It would be interesting to precisely state this generalization.

Question 4.5. *For which permutations w does $C'_w(q)$ factor as $g(\sqrt{q})\Phi(I_1, \ldots, I_m; q)$ for some intervals $I_1, \ldots, I_m$ and some rational function g?*

5. The dual cone of total nonnegativity

In [30, Sec. 7], cones of TNN and SNN elements of $\mathrm{span}_{\mathbb{C}}\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in S_n\}$ were defined. Virtually all of the known TNN and SNN polynomials belong to these cones. (See [30, Sec. 1].) Generalizing these definitions a bit, we will define the following cones of functions on $n \times n$ matrices. Define the *dual canonical cone*, the *dual cone of total nonnegativity*, and the *dual cone of Schur nonnegativity*, which we will denote by $\check{C}_{\mathbf{B}}$, $\check{C}_{\mathrm{TNN}}$, and $\check{C}_{\mathrm{SNN}}$, respectively, to be the cones whose extreme rays are homogeneous elements of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ belonging to $\mathbf{B}$, having the TNN property, and having the SNN property, respectively. Our use of the term *dual* refers to the relationship of this point of view to that of Stembridge [32], who defined the *cone of total nonnegativity* to be the smallest cone in $\mathbb{C}[S_n]$ containing all of elements of the form $\sum_{w \in S_n} a_{1,w(1)} \cdots a_{n,w(n)} w$, where $A = (a_{i,j})$ is a totally nonnegative matrix.

Using this terminology, we have the following.

Corollary 5.1. *The dual canonical cone is contained in the intersection of the dual cones of total nonnegativity and Schur nonnegativity.*

Proof. By Theorem 2.1, the dual canonical basis elements are Kazhdan-Lusztig immanants of generalized submatrices of $x = (x_{i,j})_{i,j=1}^n$. By [16], [30], [29], these are TNN and SNN. ☐

The author and A. Zelevinsky have verified that the containment of $\check{C}_{\mathbf{B}}$ in $\check{C}_{\mathrm{TNN}}$ is strict. In particular, the homogeneous element

$$\mathrm{Imm}_{3214}(x) + \mathrm{Imm}_{1432}(x) - \mathrm{Imm}_{3412}(x) \tag{5.1}$$

belongs to $\check{C}_{\mathrm{TNN}} \smallsetminus \check{C}_{\mathbf{B}}$. Moreover we have used cluster algebras and Maple to show that this element is equal to a subtraction-free rational expression in matrix minors. Thus the cone $\check{C}_{SFR}$ of functions which have this subtraction-free rational function property must also properly contain $\check{C}_{\mathbf{B}}$. On the other hand, the element (5.1) does not belong to $\check{C}_{\mathrm{SNN}}$, because its evaluation on the Jacobi-Trudi matrix H_{2222} expands in the Schur basis as

$$2s_{62} + 2s_{53} + 2s_{521} - s_{44} + 2s_{431} + 2s_{422}.$$

Thus $\check{C}_{\mathbf{B}}$ and $\check{C}_{\mathrm{SNN}}$ are not known to be different.

In order to examine the difference $\check{C}_{\mathrm{TNN}} \smallsetminus \check{C}_{\mathbf{B}}$ more closely, we will use graphs known as planar networks. We define a *planar network of order* n to be an acyclic planar directed multigraph $G = (V, E)$ in which $2n$ boundary vertices are labeled counterclockwise as *sources* $1, \dots, n$ and *sinks* $n, \dots, 1$. In figures, sources will appear on the left and sinks on the right. All edges should be understood to be oriented from left to right. We define a *path family* in a planar network to be an n-tuple $\pi = (\pi_1, \dots, \pi_n)$ of paths from sources $1, \dots, n$, respectively, to sinks $w(1), \dots, w(n)$, respectively, for some permutation $w \in S_n$. We will say that w is the *type* of the path family. We will say also that a path family π *covers* a planar network if each edge in the planar network belongs to at least one of the paths in π.

Planar networks are often used to represent the factorization of permutations into the standard generators $s_1, \dots, s_{n-1}$ of S_n. In this capacity, the planar networks are often refered to as *wiring diagrams*, and vertices are not explicitly drawn. We shall associate planar networks also to reversal factorizations as follows. To represent the reversal $s_{[i,j]}$, we shall draw $i-1$ horizontal lines, above a "star" of $j-i+1$ diagonal lines, above $n-j$ more horizontal lines. To represent a reversal *factorization*, we will concatenate planar networks corresponding to the appropriate reversals. It is easy to see that the connected components of such a planar network correspond to $\oplus$-indecomposable permutations.

The first two planar networks G and G' in Figure 5.1 represent the reversal (zig-zag) factorization

$$s_{[1,5]}s_{[3,5]}s_{[3,6]}s_{[4,6]}s_{[13,15]}s_{[13,13]}s_{[9,13]}s_{[9,10]}s_{[4,10]}s_{[14,15]}s_{[14,18]}s_{[23,25]}s_{[23,24]}s_{[22,24]}s_{[22,22]}s_{[20,22]}$$

of the permutation whose one-line notation is

$$(10, 9, 8, 2, 1, 3, 7, 6, 13, 12, 11, 5, 18, 17, 4, 16, 15, 14, 19, 22, 21, 24, 25, 23, 20).$$

This permutation decomposes as

$$(10, 9, 8, 2, 1, 3, 7, 6, 13, 12, 11, 5, 18, 17, 4, 16, 15, 14) \oplus (1) \oplus (3, 2, 5, 6, 4, 1),$$

or equivalently as $u \oplus e \oplus v$, where

$$\begin{aligned} u &= s_{[1,5]}s_{[3,5]}s_{[3,6]}s_{[4,6]}s_{[13,15]}s_{[13,13]}s_{[9,13]}s_{[9,10]}s_{[4,10]}s_{[14,15]}s_{[14,18]}, \\ e &= s_{[1,1]}, \\ v &= s_{[4,6]}s_{[4,5]}s_{[3,5]}s_{[3,3]}s_{[1,3]}. \end{aligned}$$

Accordingly, both planar networks have three connected components. While G' is smaller than G, it is easy to see that the two planar networks are equivalent in the sense that for any $w \in S_{25}$ there is a bijective correspondence between path families of type w in G and in G'.

For each reversal $s_{[i,j]}$ with $i \neq j$ in a reversal factorization, a "star" is visible in a planar network (e.g. G or G') which represents that factorization. We will call the vertex in the center of that star a *reversal vertex*.

After concatenating planar networks corresponding to reversals, we will find it convenient to delete multiple edges between reversal vertices. Deleting from the planar network G' in Figure 5.1 all but one edge joining each pair of adjacent reversal vertices, we arrive at the planar network H in the same figure. For convenience, we will also delete any vertices having indegree and outdegree equal to one, so that the only vertices remaining in the deleted planar network are sources, sinks, and reversal vertices corresponding to intervals listed in (4.7).

We will refer to this network H as the *deleted* planar network corresponding to a reversal factorization. In general, for any planar network G representing a reversal factorization and the corresponding deleted planar network H, there exists a constant c such that for any permutation w, the number of path families of type w in G is equal to c times the number of path families of type w in H. In order to state a precise formula for c, we will partially order the reversal vertices in G by defining $v \leq v'$ if there exists a directed path from v to v' in G. (This is precisely the poset defined in the discussion of Figure 3.1.)

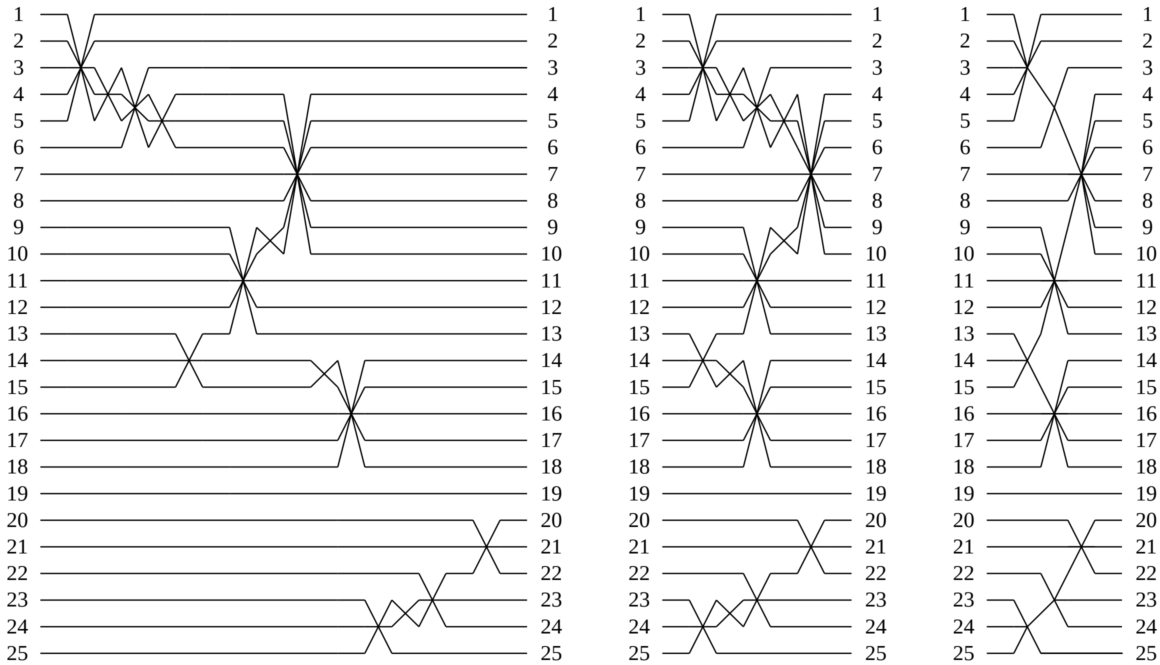

Figure 5.1. Two equivalent planar networks corresponding to a reversal factorization, and a third "deleted" planar network.

Observation 5.2. *Fix a planar network G representating a reversal factorization, let reversal vertices v and v' correspond to reversals $s_{[i,j]}$ and $s_{[i',j']}$, and suppose that v is covered by v'. Define $k = |[i,j] \cap [i',j']|$ and construct a new planar network G'' from G by deleting all but one of the k paths between v and v'. Then for any permutation $w \in S_n$, the number of path families of type w covering G is equal to $k!$ times the number of path families of type w covering G''.*

For any deleted planar network H corresponding to a zig-zag factorization of a 3412-avoiding, 4231-avoiding permutation w, we will show that the number of path families of type v in H can be nonzero only when $v \leq w$.

To each planar network G of order n, we associate a path matrix $A = (a_{i,j})$ by defining $a_{i,j}$ to be the number of paths from source i to sink j. For any permutation $v \in S_n$, the product $a_{1,v_1} \cdots a_{n,v_n}$ is equal to the number of path families of type v in G (not necessarily covering G). A celebrated result in the theory of total nonnegativity [20], [24] asserts the path matrix of a planar network to be TNN. (For more information, see [14] and references there.)

Lemma 5.3. *Let H be the deleted planar network corresponding to the zig-zag factorization of a 3412-avoiding, 4231-avoiding permutation w in S_n and let A be the path matrix of H. Then H and A have the following properties.*

(1) Every path family in H covers H.
(2) Every entry of A is zero or one.
(3) We have

$$a_{1,u_1} \cdots a_{n,u_n} = \begin{cases} 1 & \text{if } u \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that two permutations u and v have reversal factorizations represented by planar networks with path matrices A and B, respectively. It is easy to see that the reversal factorization of $u \oplus v$ formed by concatenating those of u and v is represented by a planar network which is a disjoint union of the two planar networks and whose path matrix has the block diagonal form

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

We may therefore assume that w is $\oplus$-indecomposable.

(1) Suppose that π is a path family in H which does not cover some edge e of H. If e is an edge from a source to a reversal vertex, or from a reversal vertex to a sink, then π is not a path family. Thus e must be an edge from reversal vertex v to an adjacent reversal vertex v'. Let $s_{[i,j]}, s_{[i',j']}$ be the corresponding reversals. If $i < i'$, then paths in π must connect sources $1, \ldots, j$ to sinks $1, \ldots, i'-1$ while $j > i'-1$, contradicting the existence of e. Similarly, if $i > i'$, then paths in π must connect sources $i, \ldots, n$ to sinks $j', \ldots, n$ while $n - i + 1 > n - j'$, again contradicting the existence of e.

(2) Reversal vertices in H correspond to a subsequence $(s_{J_1}, \ldots, s_{J_p})$ of the reversal factorization of w. Let $v(J_1), \ldots, v(J_p)$ be the corresponding reversal vertices in H. Suppose that there are at least two paths in H from source i to sink j. Let k be the smallest index such that i belongs to J_k and let ℓ be the greatest index such that j belongs to J_ℓ. Then there must be at least two paths in H from $v(J_k)$ to $v(J_\ell)$. This is impossible however, since paths in H must follow the partial order (3.5).

(3) Let $I = (I_1, \ldots, I_p)$ be the *full* sequence of intervals in the zig-zag factorization of w. Construct a planar network G to represent this factorization by concatenating planar networks $G(I_1), \ldots, G(I_p)$ corresponding to the reversals.

Suppose that $u \leq w$. By Proposition 3.8, u can be factored as $u = t_1 \cdots t_p$ with $t_j \leq s_{I_j}$ for $j = 1, \ldots, p$. It is therefore clear that there is at least one path family of type u which covers G. By Observation 5.2 there is at least one path family of type u which covers H. By statements (1) and (2) above there is at most one, i.e. $a_{1,u_1} \cdots a_{n,u_n} = 1$. Now suppose that $a_{1,u_1} \cdots a_{n,u_n} = 1$. By Observation 5.2 there is at least one path family π of type u which covers G. Thus π may be viewed as the concatenation of p families $\pi^{(1)}, \ldots, \pi^{(p)}$ of n subpaths, with $\pi^{(j)}$ being the restriction of π to $G(I_j)$. Since $\mathrm{type}(\pi) = \mathrm{type}(\pi^{(1)}) \cdots \mathrm{type}(\pi^{(p)})$, Proposition 3.8 implies that $\mathrm{type}(\pi) \leq w$.

Thus the statements $u \leq w$ and $a_{1,u_1} \cdots a_{n,u_n} = 1$ are equivalent. By statement (2) we must also have the equivalence of $u \not\leq w$ and $a_{1,u_1} \cdots a_{n,u_n} = 0$. □

The path matrices we have constructed serve as "indicator" matrices for Kahzdan-Lusztig immanants in the following sense. (For a different description of the relationship between a 3412-avoiding, 4231-avoiding permutation w and the path matrix of its deleted planar network, see [17] and [31].)

Theorem 5.4. *Let H be the deleted planar network corresponding to a zig-zag factorization of a 3412-avoiding, 4231-avoiding permutation w in S_n, and let A be the path matrix of H. Then we have*

$$\mathrm{Imm}_u(A) = \begin{cases} 1 & \text{if } u = w, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using Lemma 5.3 we have

$$\begin{aligned} \mathrm{Imm}_u(A) &= \sum_{u \leq v \leq w} (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1) a_{1,v_1} \cdots a_{n,v_n} \\ &= \sum_{u \leq v \leq w} (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1). \end{aligned}$$

Since w avoids the patterns 3412 and 4231, we have $P_{v,w}(q) = 1$ for all $v \leq w$, and we may replace each number $P_{w_0 v, w_0 u}(1)$ in the above sum by $P_{w_0 v, w_0 u}(1) P_{v,w}(1)$. Now by Kazhdan and Lusztig's inversion formula [22, Sec. 3]

$$\sum_{u \leq v \leq w} (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1) P_{v,w}(1) = \begin{cases} 1 & \text{if } u = w, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the desired result. □

The existence of these indicator matrices allows us to compare the dual canonical cone with the dual cone of total nonnegativity as follows.

Theorem 5.5. *Let $\mathrm{Imm}_f(x)$ be totally nonnegative and let its expansion in terms of Kazhdan-Lusztig immanants be given by*

$$\mathrm{Imm}_f(x) = \sum_{w \in S_n} d_w \mathrm{Imm}_w(x).$$

Then d_u is nonnegative for each 3412-avoiding, 4231-avoiding permutation u.

Proof. Let u be a 3412-avoiding, 4231-avoiding permutation in S_n, and suppose that d_u is negative. Let H be the deleted planar network corresponding to the zig-zag factorization of u, and let A be the path matrix of H. Then A is TNN and by Theorem 5.4 we have

$$\mathrm{Imm}_f(A) = d_u \mathrm{Imm}_u(A) = d_u < 0,$$

contradicting the total nonnegativity of $\mathrm{Imm}_f(x)$. □

More generally, we have the following.

Theorem 5.6. *Let $W = S_r$, let M, M' be two r-element multisets of $[n]$ and let $p(x)$ be a totally nonnegative element of $\mathcal{A}_r(M, M')$. If the expansion of $p(x)$ in terms of the dual canonical basis is*

$$p(x) = \sum_{w \in S_r} d_w \mathrm{Imm}_w(x_{M,M'}),$$

then d_u is nonnegative for each 3412-avoiding, 4231-avoiding permutation u in S_r.

Proof. By Theorem 2.1, we may assume each coefficient d_w to be zero unless w is a maximal representative of a coset in $W_{\iota(M)} \backslash W / W_{\iota(M')}$. Let u be such a representative which avoids the patterns 3412 and 4231 and suppose that d_u is negative. Let H be the deleted planar network corresponding to the zig-zag factorization $(s_{I_1}, \ldots, s_{I_p})$ of u and let $A = (a_{i,j})$ be the path matrix of H. Then A is TNN.

Furthermore, we claim that

$$a_{i,j} = \begin{cases} a_{i+1,j} & \text{if } m_i = m_{i+1}, \\ a_{i,j+1} & \text{if } m'_j = m'_{j+1}. \end{cases}$$

Fix i and suppose that $m_i = m_{i+1}$. By the maximality of u, we have that $s_i u < u$, or equivalently $u_i > u_{i+1}$ in one-line notation. By Observation 3.2, the first interval I_k in the zig-zag factorization to contain i is also the first interval to contain $i+1$. Thus paths in H from source i to a fixed sink j are in bijection with paths from source $i+1$ to sink j, by swapping the subpaths preceeding the reversal vertex corresponding to s_{I_k}. This implies that we have $a_{i,j} = a_{i+1,j}$. Similarly, the equality $m'_j = m'_{j+1}$ implies that $a_{i,j} = a_{i,j+1}$.

Now let B be the submatrix of A obtained by deleting row i for each index i satisfying $m_i = m_{i+1}$ and by deleting column j for each index j satisfying $m'_j = m'_{j+1}$. It follows that $B_{M,M'} = A$ and since B is a submatrix of A, it too is TNN. By Theorem 5.4 we therefore have

$$p(B) = d_u \mathrm{Imm}_u(A) = d_u < 0,$$

contradicting the total nonnegativity of $p(x)$. □

Theorem 5.5 suggests several problems. Recalling Lakshmibai and Sandhya's result [23] that a permutation w's avoidance of the patterns 3412 and 4231 is equivalent to smoothness of the Schubert variety Γ_w, we have the following.

Problem 5.7. *Find an intuitive reason for the connection between total nonnegativity, the dual canonical basis, and smoothness of Schubert varieties.*

It would also be interesting to understand precisely how the cones mentioned earlier are related.

Problem 5.8. *Find the extremal rays in $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ of the cones $\check{C}_{\mathrm{TNN}}$, $\check{C}_{\mathrm{SNN}}$, and $\check{C}_{\mathrm{SRF}}$, or describe the precise containments satisfied by these cones and $\check{C}_{\mathbf{B}}$.*

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