TOTALLY NONNEGATIVE f-IMMANANTS RELATED TO THE TEMPERLEY-LIEB ALGEBRA

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Outline

(1) Totally nonnegative matrices and polynomials
(2) The Temperley-Lieb algebra
(3) Products of minors
(4) Intersecting path families

Submatrices and minors

Given an $n \times n$ matrix $A = (a_{i,j})$ and two subsets I, I' of $\{1, \ldots, n\}$, define the (I, I') submatrix and (I, I') minor of A by

$$A_{I,I'} = (a_{i,j})_{i \in I, j \in I'},$$
$$\Delta_{I,I'}(A) = \det(A_{I,I'}).$$

For example,

$$A = \begin{bmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \qquad A_{13,23} = \begin{bmatrix} 6 & 3 \\ 4 & 4 \end{bmatrix},$$
$$\Delta_{13,23}(A) = \det(A_{13,23}) = 12.$$

Total nonnegativity

Definition: A matrix is called *totally nonnegative* (TNN) if each of its minors is nonnegative.

Definition: A polynomial $p \in \mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ is called a *totally nonnegative (TNN) polynomial* if

$$p(A) \underset{\text{def}}{=} p(a_{1,1}, \ldots, a_{n,n}) \geq 0$$

for each TNN matrix A of size at least $n \times n$.

Examples:

(1) $\det(x)$, $\Delta_{I,I'}(x)$, (obvious)

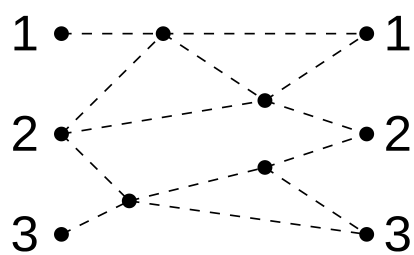
(2) $\mathrm{Imm}_\lambda(X) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$. (JS 91)

Lindström's Lemma

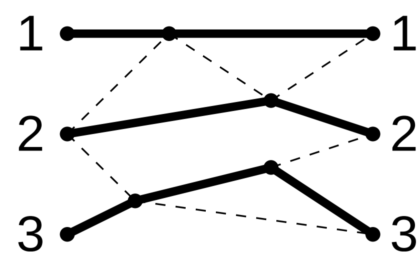
Let G be a planar network with n sources and n sinks. Define $A = [a_{ij}]$ by

$$a_{ij} = \#\text{ paths from source } i \text{ to sink } j.$$

Then A is TNN, every TNN matrix arises this way, and $\det(A) = \#$ nonintersecting path families $\pi = (\pi_1, \ldots, \pi_n)$ in G from all sources to all sinks.



$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$



Question: When is

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \quad \text{TNN?}$$

Answer: (FGJ 01, S 02) + two new answers.

Question: When is $\sum_j c_j \Delta_{I_j,I'_j}(x)\Delta_{\overline{I_j},\overline{I'_j}}(x)$ TNN?

Answer: When it belongs to the cone generated by $\{\mathrm{Imm}_\tau(x) \,|\, \tau \in T_n(2)\}$.

The lattice path criterion

Idea: To the products of minors

$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \qquad \Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x),$$

associate lattice paths

$$P(I,I') \qquad P(J,J'),$$

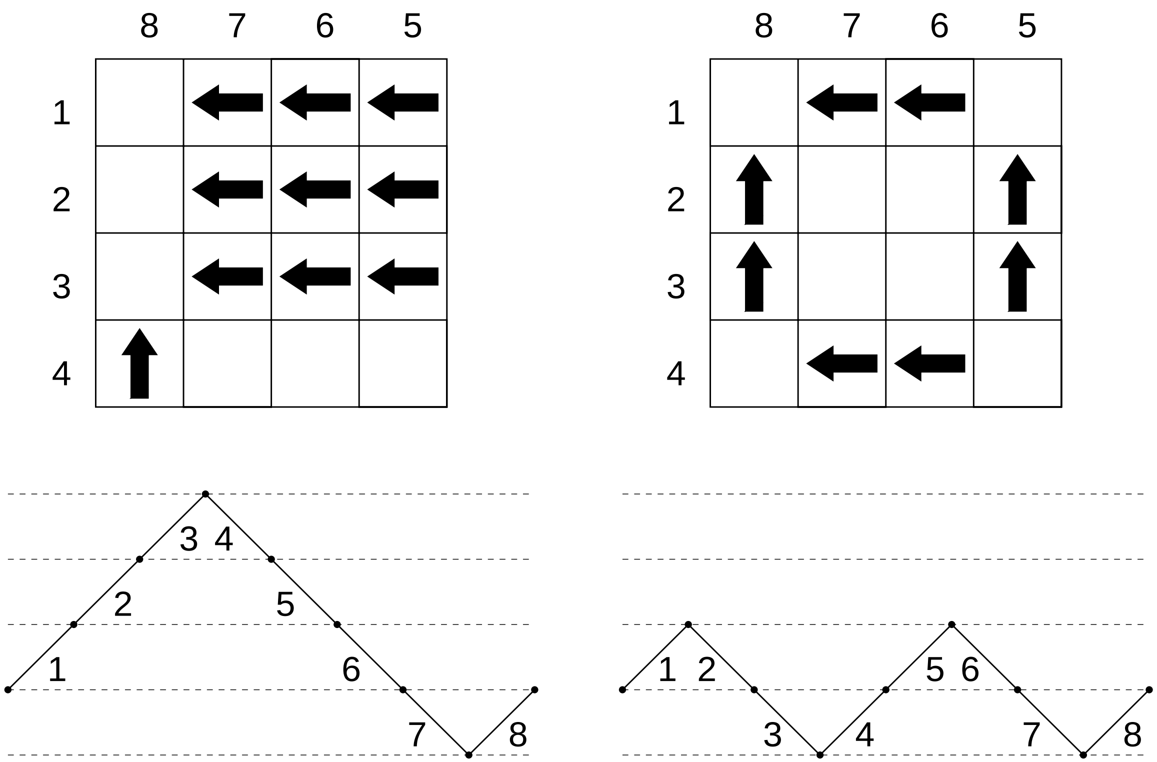
and set partitions

$$\Pi(I,I') \qquad \Pi(J,J').$$

If $\Pi(I,I')$ refines $\Pi(J,J')$, then

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \quad \text{is TNN.}$$

$$\Delta_{123,234}(x)\Delta_{4,1}(x) \qquad \Delta_{14,23}(x)\Delta_{23,14}(x)$$



$$\Pi(123,234) = 16|25|34|78, \qquad \Pi(14,23) = 1256|3478.$$

$$\Delta_{14,23}(x)\Delta_{23,14}(x) - \Delta_{123,234}(x)\Delta_{4,1}(x) \text{ is TNN.}$$

The Temperley-Lieb algebra

Define $T_n(2)$ to be the $\mathbb{C}$-algebra generated by elements $t_1, \ldots, t_{n-1}$ subject to the relations

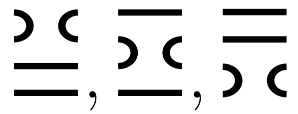
$$\begin{aligned} t_i^2 &= 2t_i, && \text{for } i = 1, \ldots, n-1, \\ t_i t_j t_i &= t_i, && \text{if } |i-j| = 1, \\ t_i t_j &= t_j t_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

$$\mathbb{C}[S_n]/(1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1) \cong T_n(2)$$
$$\theta : s_i \mapsto t_i - 1$$

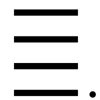
$\dim_{\mathbb{C}} T_n(2) = C_n = \frac{1}{n+1}\binom{2n}{n}$.

Kauffman's version of $T_4(2)$

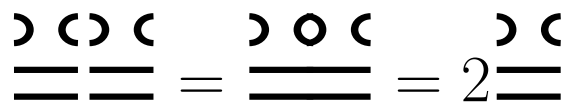
generators t_1, t_2, t_3 :



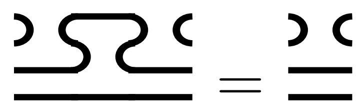
identity 1 :



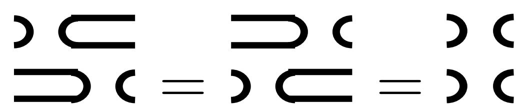
$t_1^2 = 2t_1$:



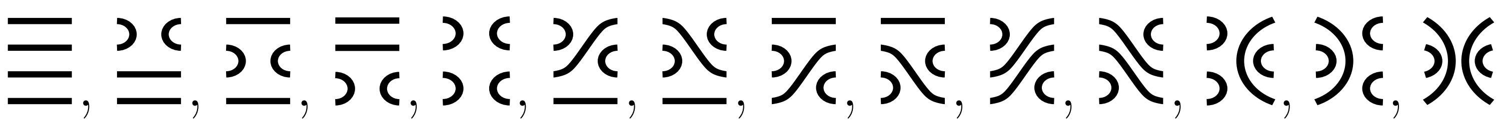
$t_1 t_2 t_1 = t_1$:



$t_1 t_3 = t_3 t_1$:



basis elements:



The Temperley-Lieb criterion

Idea: To the products of minors

$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \qquad \Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x),$$

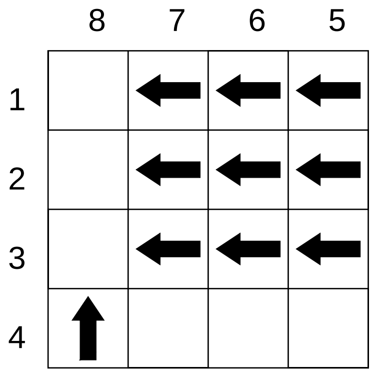
associate sets of Temperley-Lieb diagrams

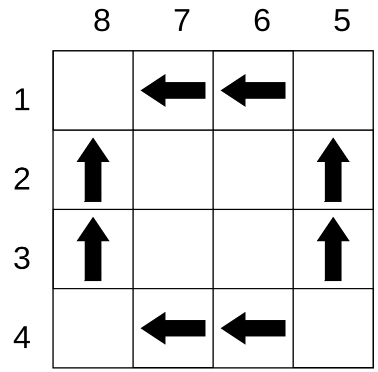
$$\Phi(I,I') \qquad\qquad \Phi(J,J').$$

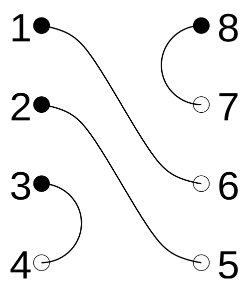
If $\Phi(J,J')$ contains $\Phi(I,I')$, then

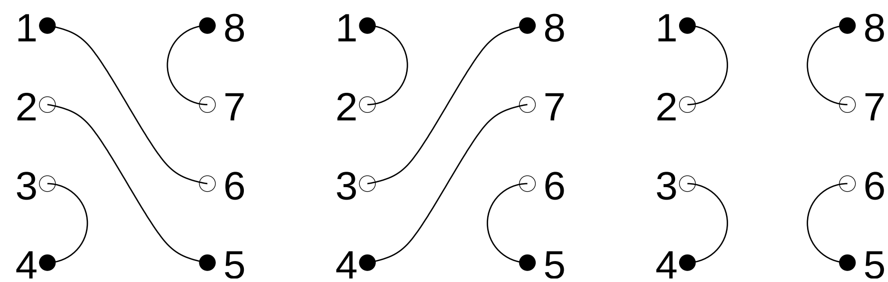
$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \quad \text{is TNN.}$$

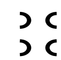
$$\Delta_{123,234}(x)\Delta_{4,1}(x) \qquad\qquad \Delta_{14,23}(x)\Delta_{23,14}(x)$$









$\Phi(123, 234) = \{$  $\}$ $\qquad$ $\Phi(14, 23) = \{$ , ,  $\}$

$$\Delta_{14,23}(x)\Delta_{23,14}(x) - \Delta_{123,234}(x)\Delta_{4,1}(x) \quad \text{is TNN.}$$

The Temperley-Lieb immanants

Using the isomorphism

$$\begin{aligned}\theta : \mathbb{C}[S_n]/I &\longrightarrow T_n(2),\\ s_i &\mapsto t_i - 1,\end{aligned}$$

define the function

$$\begin{aligned}f_\tau : S_n &\longrightarrow \mathbb{Z}\\ \sigma &\mapsto \text{coefficient of } \tau \text{ in } \theta(\sigma).\end{aligned}$$

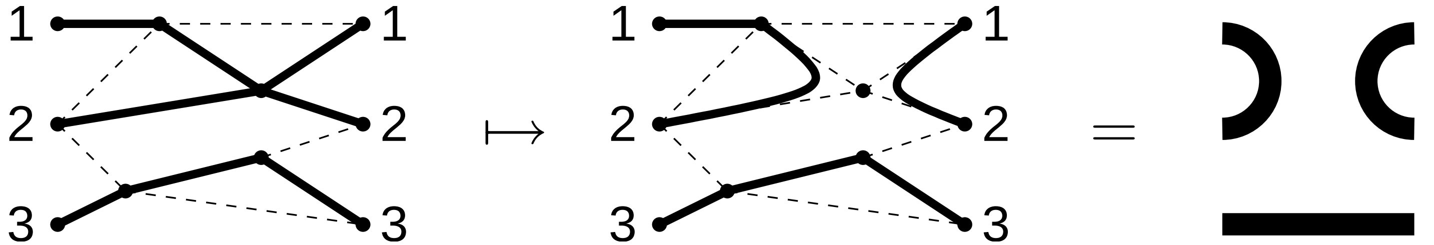
For each basis element τ of $T_n(2)$, define

$$\mathrm{Imm}_\tau(x) = \sum_{\sigma \in S_n} f_\tau(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}.$$

Note: $\mathrm{Imm}_1(x) = \det(x)$.

Combinatorial interpretation of TL immanants

Define the map $\phi : \{(\pi_1, \ldots, \pi_n)\} \to T_n(2)$ by $\times \mapsto$)(,



Theorem: Let A be the path matrix of G. Then $\mathrm{Imm}_\tau(A) = \#$ path families $\pi = (\pi_1, \ldots, \pi_n)$ in G from all sources to all sinks, which satisfy $\phi(\pi) = \tau$.

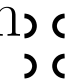
$\mathrm{Imm}_{\underline{)(}}(A)$ counts path families like the one above.

Corollary: For all $\tau \in T_n(2)$, $\mathrm{Imm}_\tau(x)$ is TNN.

Theorem:

$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) = \sum_{\tau\in\Phi(I,I')} \mathrm{Imm}_\tau(x).$$

So,

$\Delta_{14,23}(x)\Delta_{23,14}(x) = \mathrm{Imm}$$(x) + \mathrm{Imm}$$(x) + \mathrm{Imm}$(x),

$\Delta_{123,234}(x)\Delta_{4,1}(x) = \mathrm{Imm}$(x),

and the difference of these two products is

Imm$(x) + \mathrm{Imm}$(x).

Theorem: An immanant of the form

$$\sum_j c_j \Delta_{I_j, I'_j}(x) \Delta_{\overline{I_j}, \overline{I'_j}}(x)$$

is TNN iff it is equal to a nonnegative linear combination

$$\sum_\tau d_\tau \mathrm{Imm}_\tau(x).$$

Proof idea: For each basis element τ of $T_n(2)$, there exists a matrix $A = A(\tau)$ which satisfies

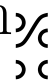
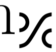
$$\mathrm{Imm}_\xi(A(\tau)) = \begin{cases} 1 & \text{if } \xi = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

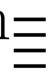
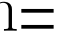
Fun fact:

Define $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$; recall that $\det(A) = \det(B)\det(D)$.

If $\tau = \begin{matrix}\tau_1 \\ \tau_2\end{matrix}$, then we have $\mathrm{Imm}_\tau(A) = \mathrm{Imm}_{\tau_1}(B)\mathrm{Imm}_{\tau_2}(D)$.

Example: If B is 3×3, D is 2×2, then

Imm  (A) = Imm  (B) Imm  (D),

Imm  (A) = Imm  (B) Imm  (D).

Open questions

Definition: Call a polynomial $p \in \mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ MNN (SNN) if for every $n \times n$ Jacobi-Trudi matrix A, the symmetric function $p(A)$ is monomial nonnegative (Schur nonnegative).

Question: Temperley-Lieb immanants are MNN. Are they SNN?

Question: Can we generalize these results to products of k minors?

Theorem:

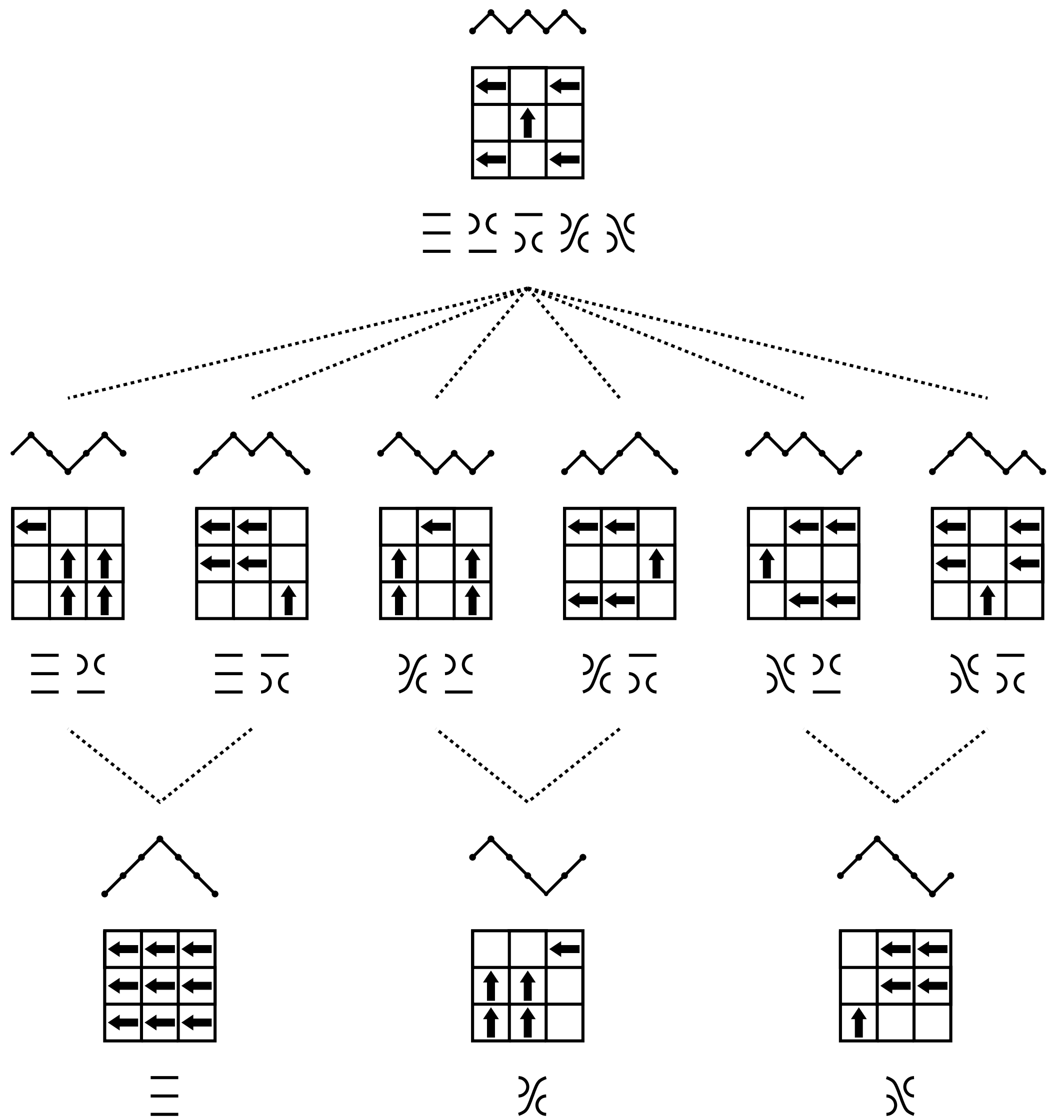
$\dim_{\mathbb{C}} \operatorname{span}\{\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \mid I, I' \subset [n]\} = C_n = \frac{}{n}$

Proof idea:

$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) = \sum_{\tau \in \Phi(I,I')} \operatorname{Imm}_\tau(x). \qquad (\Rightarrow \dim \leq$$

The set $\{\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \mid P(I, I')$ is a Dyck path $\}$

is linearly independent. $(\Rightarrow \dim \geq$



The poset of products of complementary minors of 3×3 matrices.