LINEAR EXTENSIONS OF POSETS AND PERMUTATION STATISTICS

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Abstract. Inspired by results and conjectures involving f-vectors, h-vectors, and the sequences counting linear extensions of posets, we prove three new results which relate these sequences to permutation statistics and $(\mathbf{3}+\mathbf{1})$-free posets.

Our first result extends Dumont's permutation statistic to the rearrangement classes of arbitrary words, creating a generalized statistic which is again Eulerian. As a consequence, we show that for each distributive lattice $J(P)$ which is a product of chains, there is a poset Q such that the f-vector of Q is the h-vector of $J(P)$. This strengthens for products of chains a result of Stanley concerning the flag h-vectors of Cohen-Macaulay complexes [28, Cor. 4.5]. We conjecture that the result holds for all finite distributive lattices.

Our second result settles a long standing open problem by providing a natural Eulerian permutation statistic, stc, whose joint distribution with INV on S_n is the same as that of des with MAJ. We conjecture that the pair (des, stc) is symmetrically distributed on S_n. We also define a set-valued function STC which provides an analog for stc of the descent set function D, and conjecture some stronger equidistribution results.

Lastly, we characterize $(\mathbf{3}+\mathbf{1})$-free posets in terms of their antiadjacency matrices. Generalizing a previous result for unit interval orders, this characterization relates $(\mathbf{3}+\mathbf{1})$-free posets to totally positive matrices and leads to a simple proof that the chain polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros.

INTRODUCTION

Many current problems in algebraic combinatorics concern the integer sequences $a = (a_0, \ldots, a_n)$ related to simplicial complexes, partially ordered sets (posets), and permutation statistics. At best, we have precise characterizations of the sequences which arise in various settings. A classic example is the Schutzenberger-Kruskal-Katona theorem, (See [32, p. 54]) which characterizes those sequences which are f-vectors of simplicial complexes. However, few results of this type are known.

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More often, authors have made progress toward characterization by proving weaker statements. In abundant research papers, authors have attempted to characterize or at least to obtain significant information about them. (See [3], [5], [4], [32, Ch. 2,3].)

Typical results relate the integers $a_0, \ldots, a_n$ by linear or quadratic inequalities. For instance, the sequence a is called *unimodal* if we have

$$a_0 \leq \cdots \leq a_j \geq \cdots \geq a_d, \text{ for some index } j,$$

and *log-concave* if we have

$$a_i^2 \geq a_{i-1}a_{i+1}, \text{ for } i = 1, \ldots, d-1.$$

It is easy to see that log-concavity implies unimodality. A stronger property still is that the polynomial

$$a_0 + a_1 x + \cdots + a_d x^d$$

has only real zeros.

Other typical results relate sequences which arise in one setting to those which arise in another. Such a result can be useful if one of the two settings is easier to study than the other.

Sequence characterization has proven to be quite a difficult problem in many cases. Even many weaker properties of sequences may be stated today only as conjectures. In Chapter 1, we define three sequences of interest and state some of their known properties, including a theorem of Stanley which relates the h-vectors of a special class of simplicial complexes to the f-vectors of other complexes. The proof of a special case of this theorem and the famous Neggers-Stanley conjectures motivate three problems which we consider in greater detail in the remaining chapters.

1. Three sequences associated to a poset

1.1. **Definitions of the three sequences.** Three integer sequences often associated to a finite poset are its f-vector, h-vector, and the sequence counting its linear extensions by descent.

Let P be a finite poset on n elements. If x and y are two elements of P, we will write $x <_P y$ to indicate that x is less than y in P. A *labelling* of P is any bijective function $\phi : P \to [n]$. If $\phi(x) = i$ and $\phi(y) = j$ we will often abuse notation a bit and write $i <_P j$. We will call the labelling ϕ *natural* if it satisfies $\phi(x) < \phi(y)$ whenever $x <_P y$.

Our first sequence of interest is the *f-vector* of P, which counts *chains* by cardinality. A *chain* of P is an ordered k-tuple of elements $(x_1, \ldots, x_k)$ which satisfies

$$x_1 <_P \cdots <_P x_k.$$

We define the f-vector of P to be the sequence

$$f_P = (f_{-1}, f_0, \ldots, f_{d-1}),$$

where f_i is the number of $(i{+}1)$-element chains in P, and d is the maximum cardinality of a chain in P. By convention, we define $f_{-1} = 1$. Note that the definition of f_P is independent of a particular labelling of P.

There is considerable interest in characterizing the integer sequences which are f-vectors of posets, and in identifying those posets P for which f_P is unimodal, log-concave, and for which the generating function

$$f_P(x) = f_{-1} + f_0 x + \cdots + f_{d-1} x^d$$

has only real zeros. We will call the polynomial $f_P(x)$ defined above the *chain polynomial* of P.

Our second sequence of interest is the *h-vector* of P,

$$h_P = (h_0, h_1, \ldots, h_d),$$

which we define in terms of the f-vector by the equation

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}. \tag{1.1}$$

From this definition, it is clear that knowing the h-vector of a poset is equivalent to knowing the f-vector. It is easy to verify that

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}, \tag{1.2}$$

for $k = 0, \ldots, d$, and one may use this fact as an alternative definition of the h-vector. Another definition of the h-vector of a poset expresses its generating function in terms of a linear fractional transformation of the variable x in the chain polynomial. Defining the polynomial $h_P(x)$ by

$$h_P(x) = (1-x)^d f_P\left(\frac{x}{1-x}\right), \tag{1.3}$$

one easily verifies that the coefficients of $h_P(x)$ are are the h-vector of P,

$$h_P(x) = h_0 + h_1 x + \cdots + h_d x^d.$$

Since we have defined the h-vector in terms of the f-vector, it too is independent of a particular labelling of P.

Example 1.1. If P is the poset in Figure 1.1, then we have $f_P = (1, 4, 3)$ and $h_P = (1, 2, -1)$.

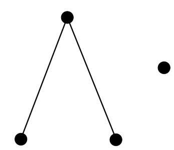


FIGURE 1.1

While the f-vector of a poset is positive by definition, this is not true of the h-vector. There is considerable interest in determining for which posets P the h-vector h_P is nonnegative, unimodal, and log-concave, and for which posets the h-polynomial $h_P(x)$ has only real zeros.

Using (1.3) we observe two facts concerning the f and h-vectors of a given poset.

Observation 1.1. *Let P, P', and P'' be finite posets.*

1. *All zeros of $f_P(x)$ are real if and only if all zeros of $h_P(x)$ are real.*
2. *We have $f_P(x) = f_{P'}(x)f_{P''}(x)$ if and only if we have $h_P(x) = h_{P'}(x)h_{P''}(x)$.*

The polynomial $f_P(x)$ does factor as $f_{P'}(x)f_{P''}(x)$ when P has the form $P = P' \oplus P''$, That is, when P is a disjoint union of the elements of P' and P'', with the relation $<_P$ defined by $x <_P y$ if $x <_{P'} y$ or $x <_{P''} y$ or $x \in P'$, $y \in P''$.

Our third important sequence, which depends upon a particular labelling of P, counts linear extensions by descents. A *linear extension* of a labelled poset is a permutation $\pi = \pi_1 \cdots \pi_n$ in S_n satisfying $i < j$ whenever $\pi_i <_P \pi_j$. The set of linear extensions of a poset P is sometimes called the *Jordan-Hölder set* of P, and is denoted by $\mathcal{L}(P)$. A *descent* of a permutation π is a position i which satisfies $\pi_i > \pi_{i+1}$. We will denote the set of descents of π by $D(\pi)$ and the number of descents by $\mathrm{des}(\pi)$,

$$D(\pi) = \{i \,|\, \pi_i > \pi_{i+1}\},$$
$$\mathrm{des}(\pi) = \#\{i \,|\, \pi_i > \pi_{i+1}\}.$$

The function des allows us partition the Jordan-Hölder set $\mathcal{L}(P)$ into disjoint subsets $\mathcal{L}_0(P), \ldots, \mathcal{L}_{n-1}(P)$, where $\mathcal{L}_k(P)$ is the subset of linear extensions of P having k descents,

$$\mathcal{L}_k(P) = \{\pi \in \mathcal{L}(P) \,|\, \mathrm{des}(\pi) = k\}.$$

Denoting the cardinality of $\mathcal{L}_k(P)$ by g_k, we define the integer sequence

$$g_P = (g_0, \ldots, g_{n-1}),$$

which counts linear extensions of P *by descent.* It would be very interesting to characterize the integer sequences which arise in this manner, or to prove any properties of such sequences. In particular, it appears that for each finite labelled poset P, the sequence g_P is log-concave and unimodal, and the polynomial

$$g_P(x) = g_0 + g_1 x + \cdots + g_{n-1} x^{n-1}$$

has only real roots. These properties have been proven only for special classes of posets. (See Section 1.3. The cases in which P is labelled naturally and arbitrarily correspond to Conjectures 1.4 and 1.5.)

Surprisingly, the sequences which count linear extensions of a poset by descent are closely related to f-vectors and h-vectors. We will describe an explicit relationship in the following section.

1.2. **Distributive lattices and linear extensions of posets.** A very important class of posets is the class of *distributive lattices.* (See [29, Ch 3] for definitions.) Typically defined in terms of lattice join and meet operations, distributive lattices also may be defined in terms of antichains in posets. An *antichain* in a poset P is a subset of elements which are pairwise incomparable in P. For each antichain $X \subset P$ we define the corresponding *order ideal* $\langle X \rangle$ by

$$\langle X \rangle = \{x \in P \,|\, x \leq_P y \text{ for some } y \in X\},$$

and define a new poset $J(P)$ to be the set of order ideals of P, ordered by inclusion.

A distributive lattice is any poset $J(P)$ constructed in this way from an arbitrary poset P. We will denote by $\hat{0}$ and $\hat{1}$ the unique minimal and maximal elements of $J(P)$, which correspond to the order ideals containing no elements of P and all elements of P, respectively.

One important property of a distributive lattice $J(P)$ is that its h-vector is non-negative. Furthermore, the h-vector has a simple combinatorial interpretation which links all three sequences in Section 1.1.

Theorem 1.2. *Let P be a naturally labelled finite poset on n elements, and let $h_{J(P)} = (h_0, \ldots, h_{n+1})$ be the h-vector of the distributive lattice $J(P)$. Then, $h_{J(P)}$ counts linear extensions of P by descents. That is, for $k = 0, \ldots, n+1$ we have*

$$h_k = |\mathcal{L}_k(P)|.$$

Proof. For each subset S of $[n-1]$, define $\alpha(S)$ to be the number of linear extensions of P whose descent set is contained in S, and define $\beta(S)$ to be the number of linear extensions of P whose descent set is equal to S.

$$\alpha(S) = \#\{\pi \in \mathcal{L}(P) | D(\pi) \subset S\},$$
$$\beta(S) = \#\{\pi \in \mathcal{L}(P) | D(\pi) = S\}.$$

The numbers $\alpha(S)$ and $\beta(S)$ are related by

$$\alpha(S) = \sum_{T \subset S} \beta(T),$$

or equivalently, using the principal of inclusion-exclusion, by

$$\beta(S) = \sum_{T \subset S} (-1)^{|S \smallsetminus T|} \alpha(T).$$

Thus we may express the cardinalities of the sets $\mathcal{L}_0(P), \ldots, \mathcal{L}_{n-1}(P)$ in terms of the numbers $\{\alpha(T) \,|\, T \subset [n-1]\}$,

$$\begin{aligned} |\mathcal{L}_k(P)| &= \sum_{S \in \binom{[n-1]}{k}} \beta(S) \\ &= \sum_{S \in \binom{[n-1]}{k}} \sum_{T \subset S} (-1)^{|S \smallsetminus T|} \alpha(T) \\ &= \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i-1}{k-i} \sum_{T \subset \binom{[n-1]}{i}} \alpha(T). \end{aligned}$$

Trivially, $\mathcal{L}_k(P)$ is empty for k greater than $n-1$.

Furthermore, we may express the number $\alpha(T)$ in terms of the f-vector of $J(P) \smallsetminus \{\hat{0}, \hat{1}\}$. Assume that the elements of T are $t_1 < \cdots < t_k$. It is not hard to see that $\alpha(T)$ counts chains in $J(P)$, the ranks of whose elements are are $t_1, \ldots, t_k$,

$$\alpha(T) = \#\{v_1 <_{J(P)} \cdots <_{J(P)} v_k | r(v_i) = t_i, \text{ for } i = 1, \ldots, k\}.$$

Denoting the f-vector of the poset $J(P) \smallsetminus \{\hat{0}, \hat{1}\}$ by $f' = (f'_{-1}, f'_0, \ldots, f'_{n-2})$, we have

$$f'_{i-1} = \sum_{T \subset \binom{[n-1]}{i}} \alpha(T),$$

and therefore

$$|\mathcal{L}_k(P)| = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i-1}{k-i} f'_{i-1} = h'_k,$$

where $h' = (h'_0, \ldots, h'_{n-1})$ is the h-vector of $J(P) \smallsetminus \{\hat{0}, \hat{1}\}$. This is essentially the h-vector of $J(P)$, for by Observation 1.1 (2), we have

$$\begin{aligned} f_{J(P)}(x) &= (1+x)^2 (f'_{-1} + f'_0 x + \cdots f'_{n-2} x^{n-1}), \\ h_{J(P)}(x) &= h'_0 + h'_1 x + \cdots h'_{n-1} x^{n-1} + 0x^n + 0x^{n+1}. \end{aligned}$$

$\square$

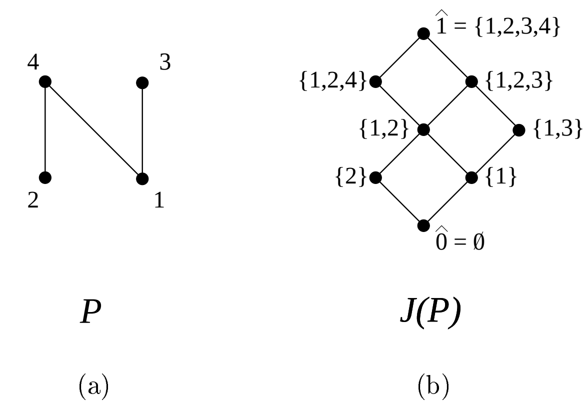


FIGURE 1.2

Example 1.2. Figure 1.2 shows a naturally labelled poset P and the corresponding distributive lattice $J(P)$. The chain polynomial and h-polynomial of $J(P)$ are given by

$$f_{J(P)}(x) = (1+x)^2(1+6x+9x^2+5x^3),$$
$$h_{J(P)}(x) = 1+3x+x^2.$$

The linear extensions of P are $\mathcal{L}(P) = \{1234, 124/3, 13/24, 2/134, 2/14/3\}$. (We have marked descents by slashes.) Therefore, we see that $h_{J(P)}$ counts linear extensions of P by descent.

The astute reader will note from the above example that the polynomial $h_{J(P)}(x)$ is in fact the chain polynomial of another poset Q, whose elements and relations correspond to the "boxes" of the distributive lattice $J(P)$. This raises the question of identifying the distributive lattices whose h-vectors are the f-vectors of other posets. We will return to this question soon.

As we have implied in Section 1.1, the Jordan-Hölder set of a poset P depends upon a particular labelling of P. (Try switching the labels 1 and 2 on the poset P in Figure 1.2, for example.) Theorem 1.2, however, asserts that the cardinalities of the sets $\mathcal{L}_k(P)$ remain constant for all natrual labellings.

1.3. **The Neggers-Stanley conjectures.** One interesting conjecture conscerning f-vectors and h-vectors is due to Neggers [25, p.114].

Conjecture 1.3. *(The Distributive Lattice Conjecture) Let P be any poset on n elements, and let $J(P)$ be the corresponding finite distributive lattice. Then the polynomial*

$$f_{J(P)}(x) = f_{-1} + f_0 x + \cdots + f_n x^{n+1}$$

has only real zeros.

As we have noted in the Introduction, the truth of this conjecture would imply the log-concavity and unimodality of the f-vector $f_{J(P)}$. Various proofs show that the conjecture holds for the special cases in which P is a disjoint sum of chains [27], a Ferrers poset [8], a Gaussian poset [8], and a poset of the form $P' + P''$, where P' and P'' are posets for which the conjecture holds. This last proof implies that the conjecture holds for the class of series-parallel posets. (See [29, Ch 3] for definitions.) In addition, Stembridge has verified the conjecture for all posets P having eight or fewer elements [36]. A more general open problem is to determine whether the conjecture holds for polynomials of the form $f_L(x)$, where L is a *modular* lattice. (See [29, Ch 3].) No counterexamples are known.

By Observation 1.1 (1), Conjecture 1.3 is equivalent to the assertion that the polynomial

$$h_{J(P)}(x) = h_0 + h_1 x + \cdots + h_n x^n$$

has only real zeros. By Theorem 1.2, this polynomial counts linear extensions of P by descent. Therefore, the conjecture may be restated in terms of naturally labelled posets.

Conjecture 1.4. *Let P be any naturally labelled poset on n elements, and define the numbers $g_0, \ldots, g_{n-1}$ by $g_i = |\mathcal{L}_i(P)|$. Then the polynomial*

$$g_P(x) = g_0 + g_1 x + \cdots + g_{n-1} x^n$$

has only real zeros.

Gasharov [**?**] has proven that the sequence g_P is unimodal in the case that P is a naturally labelled poset of rank less than three.

When the poset P in Conjecture 1.4 is *not* naturally labelled, we do not have the equality of sequences $g_P = h_{J(P)}$. Nevertheless, Stanley has conjectured that the same conclusion holds for arbitrary labellings of P. Thus we have the following stronger conjecture. (See [34, Problem 20].)

Conjecture 1.5. *(Poset conjecture) Let P be any labelled poset, and let g_i be the cardinality of $\mathcal{L}_i(P)$ for $i = 0, \ldots, n-1$. Then, the polynomial*

$$g_P(x) = g_0 + g_1 x + \cdots + g_{n-1} x^{n-1}$$

has only real zeros.

Brenti has proven that Conjecture 1.5 holds if P is a Ferrers poset with a column-strict labelling, or if P is a disjoint sum of chains [8]. Stembridge has verified the conjecture for all posets of 7 elements or less [36].

Many equivalent forms of the Neggers-Stanley conjectures may be found in Brenti [8] and Wagner [38]. In these papers and elsewhere in the literature, the polynomials and sequences in question are named differently. The polynomial denoted by $W_P(x)$ corresponds to our polynomial $g_P(x)$. When P is naturally labelled, the polynomial denoted by $E_P(x)$ corresponds to our $f_{J(P)}(x)$, and otherwise, it corresponds to $(1+x)^d g_P(\frac{x}{1+x})$. (In addition, these polynomials may differ from ours by a multiplicative factor of x.)

Recalling the coincidence from Example 1.2, in which the h-vector of one poset was the f-vector of another, we will consider a theorem in Section 1.4 which addresses a similar phenomenon.

1.4. **A theorem for Balanced Cohen-Macauley complexes.**

The conditions for which the h-vector of one poset is the f-vector of another are unknown. It would be very interesting to obtain any information about this phenomenon. On the other hand, there is a result of this nature involving the f-vectors and h-vectors of simplicial complexes.

The f-vector of a simplicial complex generalizes that of a poset. If Σ is a $(d-1)$-dimensional simplicial complex, we define its f-vector to be the sequence

$$f_\Sigma = (f_{-1}, f_0, f_1, \ldots, f_{d-1}),$$

where f_i counts the number of i-dimensional faces of Σ. By convention, $f_{-1} = 1$. Since the chains of a poset P form a simplicial complex $\Delta(P)$, known as the *order complex* of P, this new definition of the f-vector is compatible with the definition we gave in Section 1.1. That is, we have $f_P = f_{\Delta(P)}$. (See [29, p. 120].)

The h-vector of a simplicial complex generalizes that of a poset analogously. We define the h-vector $h_\Sigma = (h_0, h_1, \ldots, h_d)$ in terms of f_Σ, exactly as we did in Section 1.1,

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

For some conditions on a simplicial complex, one can show that its h-vector is the f-vector of another complex. Specifically, we have the following result, due to Stanley [28, Cor. 4.5].

Theorem 1.6. *If Σ is a balanced Cohen-Macaulay complex, then its h-vector is the f-vector of some balanced simplicial complex Γ.*

(The original statement of this theorem asserts the stronger result that the *flag* h-vector of Σ is equal to the *flag* f-vector of Γ. We use the simpler version here to emphasize its similarity to the theorems and conjectures in other sections of this paper.)

We define a simplicial complex to be *Cohen-Macaulay* if it satisfies a certain topological condition ([32, p. 61]), and *balanced* if we can color the vertices with d colors such that no face contains two vertices of the same color ([32, p. 95]). The class of balanced Cohen-Macaulay complexes is quite important because it includes the order complexes of all distributive lattices. We have seen that the distributive lattices, in turn, contain information about all posets.

By placing an additional restriction on the complex Σ, one arrives at a special case of the theorem which has an elegant bijective proof. Let us require that Σ be the order complex of a distributive lattice $J(P)$.

Theorem 1.7. *If $J(P)$ is any finite distributive lattice, then its h-vector is the f-vector of some balanced simplicial complex Γ.*

In this case, $h_\Sigma = h_{J(P)}$ counts the number of linear extensions of P by descents. Therefore, the theorem asserts that there is a bijective correspondence between linear extensions of P with k descents and $(k-1)$-faces of some simplicial complex $\Xi(P)$.

$$\mathcal{L}_k(P) \overset{1-1}{\longleftrightarrow} \{\sigma | \sigma \text{ a } (k-1)\text{-face of } \Xi(P)\}. \tag{1.4}$$

To prove Theorem 1.7, we will construct such a simplicial complex $\Xi(P)$ following [6, Remark 6.6] and [14, Cor. 2.2].

Definition 1.3. Let P be any poset, and define $\Xi(P)$ to be the collection of sets of the form $\{v_1, \ldots, v_k\}$ which satisfy

1. Each element $v_i = v_{i1} \cdots v_{in}$ belongs to $\mathcal{L}_1(P)$.
2. The k descent sets $D(v_i) = \{\ell_i\}$ for $i = 1, \ldots, k$ are distinct.
3. There is some linear extension π in $\mathcal{L}_k(P)$ with descent set $D(\pi) = \{\ell_1, \ldots, \ell_k\}$ such that we have

$$\{\pi_1, \ldots, \pi_{\ell_i}\} = \{v_{i1}, \ldots, v_{i\ell_i}\}, \text{ for } i = 1, \ldots, k. \tag{1.5}$$

To prove that $\Xi(P)$ is a well defined simplicial complex, we observe that for each k-set σ which belongs to the collection, each $(k-1)$-subset of σ does as well.

Proposition 1.8. *For any poset P, the collection $\Xi(P)$ of subsets of $\mathcal{L}_1(P)$ is a simplicial complex.*

Proof. Let P be a poset on n elements. For each pair of indices (i, j) satisfying $1 \leq i < j \leq n$, define the map $\phi_{ij} : S_n \to S_n$ by

$$\pi \mapsto (\pi_1 \cdots \pi_i) \cdot (\overline{\pi_{i+1} \cdots \pi_j}) \cdot (\pi_{j+1} \cdots \pi_n), \tag{1.6}$$

where $\overline{\pi_{i+1} \cdots \pi_j}$ is the strictly increasing rearrangement of the word $\pi_{i+1} \cdots \pi_j$.

It is easy to see that if π is a linear extension of P, then $\phi_{ij}(\pi)$ is as well. Moreover, if π belongs to $\mathcal{L}_k(P)$ and has descent set $D(\pi) = \{\ell_1, \ldots, \ell_k\}$, then $\phi_{\ell_{j-1}\ell_{j+1}}(\pi)$ belongs to $\mathcal{L}_{k-1}(P)$ and has descent set $D(\pi) \setminus \{\ell_j\}$. $\square$

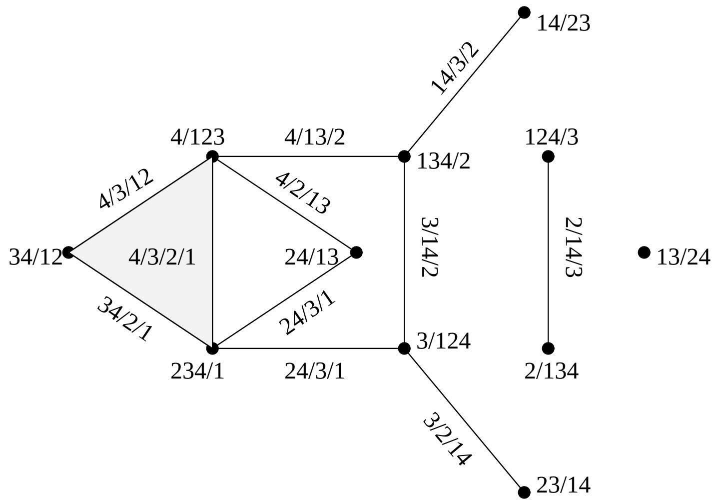


FIGURE 1.3. The complex Ξ_4

It is not difficult to see that the collection $\Xi(P)$ of sets satisfies the conditions (1.4). Definition 1.3 (3) essentially defines a bijection $\Psi : \mathcal{L}(P) \to \Xi(P)$ which identifies a linear extension π with a simplex

$$\Psi(\pi) = \{v_1, \dots, v_k\}$$

in $\Xi(P)$, where $k = \mathrm{des}(\pi)$. Let us call this bijection the *simplex map* and define it explicitly in terms of k *vertex maps*.

Definition 1.4. Let P be an n-element poset and let $\pi = \pi_1 \cdots \pi_n$ be a linear extension of P having descent set $D(\pi) = \{\ell_1, \dots, \ell_k\}$. For each descent ℓ_i in $D(\pi)$, define the *vertex map* $\psi_{\ell_i} : \mathcal{L}(P) \to \mathcal{L}_1(P)$ by

$$\psi_{\ell_i}(\pi) = (\overline{\pi_1 \cdots \pi_{\ell_i}}) \cdot (\overline{\pi_{\ell_i+1} \cdots \pi_n}).$$

Define the *simplex map* $\Psi : \mathcal{L}(P) \to 2^{\mathcal{L}_1(P)}$ by

$$\Psi(\pi) = \{\psi_{\ell_1}(\pi), \dots, \psi_{\ell_k}(\pi)\}.$$

By Definition 1.4, the image of the restriction of Ψ to $\mathcal{L}_k(P)$ lies in the set of $(k-1)$-simplices of $\Xi(P)$, and by Definition 1.3 it is bijective. By Proposition 1.8, we see that the complex $\Xi_n \stackrel{\text{def}}{=} \Xi(n\mathbf{1})$, where $n\mathbf{1}$ is the n-element antichain $\mathbf{1} + \cdots + \mathbf{1}$, has the property that for each n-element poset P, the complex $\Xi(P)$ is a subcomplex of Ξ_n.

Example 1.5. Figure 1.3 shows the complex Ξ_4.

For any poset P, the complex $\Xi(P)$ is balanced. Each vertex may be colored according to its single descent. However, the complex is not in general an *order complex*, so Theorem 1.7 does not imply that for each distributive lattice $J(P)$ there exists a poset $Q = Q(P)$ which satisfies $f_Q = f_{\Xi(P)} = h_{J(P)}$. The fact that the complex Ξ_4 is not an order complex is apparent from Figure 1.3. While Ξ_4 contains the faces $\{2413, 2341\}$, $\{2413, 4123\}$, and $\{2341, 4123\}$, it is *missing* the face $\{2413, 2341, 4123\}$. Clearly no three poset elements can be pairwise comparable without forming a chain.

In order for a simplicial complex to be an order complex it is necessary that each maximal non-face (or *missing face*) have at most two elements. This condition characterizes the class of *flag complexes*, which contains the class of order complexes. (See [32, p. 100].) Other properties which the complexes $\{\Xi_n\}_{n>0}$ lack are connectivity and acyclicity. The problem of calculating the homology groups of these complexes is still open. (See [14, Question 4.15].)

Returning to Theorems 1.6 and 1.7, we note that the hypothesis of Theorem 1.7 is much stronger than that of Theorem 1.6, and yet the conclusions of the two theorems are the same. In principle, a stronger hypothesis should allow us to assert a stronger conclusion. This observation suggests the following questions, which we will discuss further in Section 2.7.

Question 1.6. For which posets P is there a flag complex Γ such that $f_\Gamma = h_{J(P)}$?

Question 1.7. For which posets P is there a poset Q such that $f_Q = h_{J(P)}$?

1.5. **Three Open Problems.** The results and conjectures in Sections 1.1-1.4 point to some intriguing open problems, and to several strategies for solving them.

Theorem 1.6, which refers to the broad class of Balanced Cohen-Macaulay complexes, offers many opportunities for proving stronger results in special cases. In the interest of studying the h-vectors of distributive lattices, we will consider the following problem.

Problem 1.8. Find the most general poset class $\mathcal{P}$ such that for every poset P in $\mathcal{P}$ there exists a poset Q satisfying $f_Q = h_{J(P)}$. More generally, find the most general class $\mathcal{S}$ of simplicial complexes such that for every complex Σ in $\mathcal{S}$, there exists a poset Q satisfying $f_Q = h_\Sigma$.

As a stratgy, the discussion following Theorem 1.7 suggests proving refinements of Theorem 1.6 by using an Eulerian statistic *stat* to associate a simplicial complex to a set T of permutations. It suffices to find a method of diminishing the value of stat on a permutation, as in (1.6), in such a way that the resulting permutation still belongs to T. We will use this strategy in considering Problem 1.8 in Chapter 2.

The proof of Theorem 1.2 suggests that a good permutation statistic to be used in solving Problem 1.8 might be one for which we have an analogy of the descent set. Thus, we could reinterpret the f-vector and h-vector in terms of new interpretations of the numbers $\alpha(S)$ and $\beta(S)$ in the proof of the theorem. A close look at the definition of the Jordan-Hölder set in Section 1.1 suggests that Mahonian statistics and their relations to our chosen Eulerian statistic may also be of some use. We therefore propose a second problem.

Problem 1.9. Find a simple Eulerian permutation statistic *stat* to which we can associate a set-valued function $SETFN : S_n \to 2^{[n-1]}$ satisfying the equation

$$\#\{\pi \in S_n \,|\, SETFN(\pi) = T\} = \#\{\pi \in S_n \,|\, D(\pi) = T\}$$

for all subsets $T \subset [n-1]$. Find equidistribution results for pairs such as $(\text{stat}, \text{MAJ})$ and $(\text{stat}, \text{INV})$, and modify the proof of Theorem 1.2 to use stat and $SETFN$ in place of des and D.

We will consider this problem in Chapter 3, and will state some new results involving joint distributions of permutation statistics.

If permutation statistics were to provide a method for interpreting the h-vector of one poset as the f-vector of second, then one would be tempted to prove the Neggers-Stanley conjectures by demonstrating that the chain polynomial of this second poset has only real zeros. This possibility suggests that it would be worthwhile to obtain as much information as possible about the posets whose chain polynomials have only real zeros.

Problem 1.10. Describe various characterizations for poset classes $\mathcal{P}$ for which $f_P(x)$ has only real zeros whenever the poset P belongs to $\mathcal{P}$.

In Chapter 4 we will see that such posets often have nice characterizations, and that there are many distributive lattices $J(P)$ for which there exists a poset Q satisfying $f_Q = h_{J(P)}$.

All three of the problems above provide many opportunities for new results, and perhaps significant progress on any of them would help solve the Neggers-Stanley conjectures.

2. Dumont's Statistic On Words

2.1. **Eulerian Permutation Statistics.** Let S_n be the symmetric group on n letters. A *permutation statistic* is a function stat : $S_n \to \mathbb{N}$ which maps permutations to nonnegative integers. The *distribution* of a permutation statistic on S_n is simply a count of the number of permutations π in S_n for which $\text{stat}(\pi) = k$, for all possible values of k.

Recall from Section 1.1 that *des* is a statistic counting the number of descents in a permutation. Writing $\pi = \pi_1 \cdots \pi_n$, we call position i a *descent* in π if $\pi_i > \pi_{i+1}$. Therefore,

$$\text{des}(\pi) = \#\{i \,|\, \pi_i > \pi_{i+1}\}.$$

A second statistic *exc* counts the number of excedances in a permutation. We call position i an *excedance* in π if $\pi_i > i$. Therefore,

$$\mathrm{exc}(\pi) = \#\{i \mid \pi_i > i\}.$$

It is well known that the number of permutations in S_n with k descents equals the number of permutations in S_n with k excedances. This number is often denoted $A(n, k+1)$. The numbers $\{A(n, k+1) | n \geq 1; k = 0, \ldots, n-1\}$ are called the *Eulerian numbers* and satisfy the recurrence

$$A(n,k) = (k+1)A(n-1,k) + (n-k-1)A(n-1,k-1), \tag{2.1}$$

subject to the initial conditions

$$A(1,k) = \begin{cases} 1, & \text{for } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.1. The following table shows the Eulerian numbers for $n = 1, \ldots, 5$.

$n \backslash k$	0	1	2	3	4	5
1	1					
2	1	1				
3	1	2	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

The generating function

$$A_n(x) = \sum_{k=0}^{n-1} A(n, k+1)x^{k+1} = \sum_{\pi \in S_n} x^{1+\mathrm{des}(\pi)} = \sum_{\pi \in S_n} x^{1+\mathrm{exc}(\pi)}$$

is called the nth *Eulerian polynomial*, and any permutation statistic *stat* satisfying

$$A_n(x) = \sum_{\pi \in S_n} x^{1+\mathrm{stat}(\pi)}$$

is called *Eulerian*.

A third Eulerian statistic, essentially defined by Dumont [13], counts the number of distinct nonzero letters in the code of a permutation. We define $\mathrm{code}(\pi)$ to be the word $c_1 \cdots c_n$, where

$$c_i = \#\{j > i | \pi_j < \pi_i\}.$$

Denoting Dumont's statistic by *dmc*, we have

$$\mathrm{dmc}(\pi) = \#\{\ell \neq 0 | \ell \text{ appears in } \mathrm{code}(\pi)\}.$$

Example 2.2.

$$\begin{array}{lcccccccccc} \pi & = & 2 & 8 & 4 & 3 & 6 & 7 & 9 & 5 & 1 \\ \mathrm{code}(\pi) & = & 1 & 6 & 2 & 1 & 2 & 2 & 2 & 1 & 0 \end{array}$$

The distinct nonzero letters in $\mathrm{code}(\pi)$ are $\{1, 2, 6\}$. Thus, $\mathrm{dmc}(\pi) = 3$.

Dumont showed bijectively that the statistic dmc is Eulerian, i.e. that

$$\#\{\pi \in S_n | \mathrm{dmc}(\pi) = k\} = \#\{\pi \in S_n | \mathrm{des}(\pi) = k\}.$$

While few researchers have found an application for dmc since [13], Foata [16] proved the following equidistribution result involving the statistics INV (inversions) and MAJ (major index). These two statistics belong to the class of *Mahonian* statistics. (See [16] for further information.)

Theorem 2.1. *The Eulerian-Mahonian statistic pairs* (des, INV) *and* (dmc, MAJ) *are equally distributed on S_n, i.e.*

$$\#\{\pi \in S_n | \mathrm{des}(\pi) = k; \mathrm{INV}(\pi) = p\} = \#\{\pi \in S_n | \mathrm{dmc}(\pi) = k; \mathrm{MAJ}(\pi) = p\}.$$

Note that the statistics des, exc, and dmc are all defined on a permutation π to be the cardinalities of certain sets associated to π. To name these sets, we define the *descent set* and *excedance set* of π to be the sets of descents and excedances in π. We denote these by $D(\pi)$ and $E(\pi)$, respectively. Similarly, we define the *letter set* of a word to be the set of its nonzero letters, and we denote the letter set of $\mathrm{code}(\pi)$ by $LC(\pi)$. Thus,

$$\begin{aligned} \mathrm{des}(\pi) &= |D(\pi)|, \\ \mathrm{exc}(\pi) &= |E(\pi)|, \\ \mathrm{dmc}(\pi) &= |LC(\pi)|. \end{aligned}$$

It is easy to see that for every subset T of $[n-1]$ there are permutations π, σ, and ρ in S_n satisfying

$$T = D(\pi) = E(\sigma) = LC(\rho).$$

In fact, Dumont's original bijection [13] shows that for each such subset T we have

$$\#\{\pi \in S_n | E(\pi) = T\} = \#\{\pi \in S_n | LC(\pi) = T\}.$$

However, the analogous statement involving $D(\pi)$ is not true.

The Eulerian numbers have many interesting combinatorial interpretations in addition to those mentioned here and perhaps can be used to prove variations on Theorems 1.2 and 1.6. In particular, the following method generalizes that of the proof of Theorem 1.7. Let $\mathrm{stat} : S_n \to \{0, 1, \ldots, n-1\}$ be an Eulerian statistic and for each naturally labelled n-element poset P define $\mathcal{A}_k(P)$ to be the set of linear extensions π of P satisfying $\mathrm{stat}(\pi) = k$,

$$\mathcal{A}_k(P) = \#\{\pi \in \mathcal{L}(P) | \mathrm{stat}(\pi) = k\}.$$

Let $\mathcal{P}$ be a class of labelled posets such that for each n-element poset P in $\mathcal{P}$ the cardinalities of the sets $\mathcal{A}_k(P)$ and $\mathcal{L}_k(P)$ are equal for $k = 0, \ldots, n-1$.

To each permutation π in $\mathcal{A}_k(P)$, associate a k-subset $\{\psi_1(\pi), \ldots, \psi_k(\pi)\}$ of $\mathcal{A}_1(P)$ which satisfies the conditions

1. The map $\Psi : \mathcal{L}(P) \to 2^{\mathcal{A}_1(P)}$ defined by $\Psi(\pi) = \{\psi_1(\pi), \ldots, \psi_k(\pi)\}$ is bijective.
2. The collection of subsets $\{\Psi(\pi) \,|\, \pi \in \mathcal{L}(P)\}$ forms a simplicial complex.

Then we have a family of simplicial complexes $\{\Gamma(P) \,|\, P \in \mathcal{P}\}$ such that for each poset $P \in \mathcal{P}$, the h-vector of the distributive lattice $J(P)$ is the f-vector of $\Gamma(P)$. This gives a combinatorial proof of the special case of Theorem 1.7 corresponding to the distributive lattices $\{J(P) \,|\, P \in \mathcal{P}\}$. If the complexes $\{\Gamma(P) \,|\, P \in \mathcal{P}\}$ have any special properties, then this result strengthens the special case of the theorem.

In Section 2.4 we will use Dumont's statistic in just this way. The family of posets $\mathcal{P}$ will be those which are disjoint sums of chains, and the complexes we will construct will be *order* complexes.

2.2. **Word Statistics.** Generalizing permutations on n letters are *words* $w = w_1 \cdots w_m$ on n letters, where $m \geq n$. We will assume that each letter in $[n]$ appears at least once in w. Generalizing the symmetric group S_n, we define the *rearrangement class* of w by

$$R(w) = \{w_{\sigma^{-1}(1)} \cdots w_{\sigma^{-1}(m)} | \sigma \in S_m\}.$$

Each element of $R(w)$ is called a *rearrangement* of w.

Many definitions pertaining to S_n generalize immediately to the rearrangement class of any word. In particular, the definitions of descent, descent set, code, letter set of a code, and Dumont's statistic remain the same for words as for permutations. Generalization of excedances requires only a bit of effort.

For any word w, denote by $\bar{w} = \bar{w}_1, \ldots, \bar{w}_m$ the unique *nondecreasing rearrangement* of w. We define position i to be an *excedance* in w if $w_i > \bar{w}_i$. Thus,

$$\mathrm{exc}(w) = \#\{i | w_i > \bar{w}_i\}.$$

If position i is an excedance in word w, we will refer to the letter w_i as the *value* of excedance i. One can see word excedances most easily by associating to the word w the *biword*

$$\begin{pmatrix} \bar{w} \\ w \end{pmatrix} = \begin{pmatrix} \bar{w}_1 \cdots \bar{w}_m \\ w_1 \cdots w_m \end{pmatrix}.$$

Example 2.3. Let $w = 312312311$. Then,

$$\begin{pmatrix} \bar{w} \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 1 \end{pmatrix}$$

Thus, $E(w) = \{1, 3, 4\}$ and $\mathrm{exc}(w) = 3$. The corresponding excedance values are 3, 2, and 3.

We will use biwords not only to expose excedances, but to define and justify maps in Sections 2.4 and 2.5. In particular, if $u = u_1, \ldots, u_m$ and $v = v_1, \ldots, v_m$ are words and y is the biword

$$y = \begin{pmatrix} u \\ v \end{pmatrix},$$

then we will define *biletters* $y_1, \ldots, y_m$ by

$$y_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix},$$

and will define the rearrangement class of y by

$$R(y) = \{y_{\sigma^{-1}(1)} \cdots y_{\sigma^{-1}(m)} | \sigma \in S_m\}.$$

It is well known that just as the permutation word statistics des and exc are equally distributed on S_n, the *word* statistics des and exc are equally distributed on the rearrangement class of any word w. That is,

$$\#\{y \in R(w) | \mathrm{exc}(y) = k\} = \#\{y \in R(w) | \mathrm{des}(y) = k\}.$$

Analogously to the case of permutation statistics, a word statistic *stat* is called *Eulerian* if it satisfies

$$\#\{y \in R(w) | \mathrm{stat}(y) = k\} = \#\{y \in R(w) | \mathrm{des}(y) = k\}$$

for any word w and any nonnegative integer k.

2.3. **Generalizing Dumont's statistic.** As implied in Section 1.1, we define Dumont's statistic on a word w to be the number of distinct nonzero letters in $\mathrm{code}(w)$,

$$\mathrm{dmc}(w) = |LC(w)|.$$

This generalized statistic is Eulerian.

Theorem 2.2. *If $R(w)$ is the rearrangement class of an arbitrary word w and k is any nonnegative integer, then*

$$\#\{v \in R(w) | \mathrm{dmc}(v) = k\} = \#\{v \in R(w) | \mathrm{exc}(v) = k\}.$$

Our bijective proof of the theorem depends upon an encoding of a word which we call the *excedance table.*

Definition 2.4. Let $v = v_1 \cdots v_m$ be an arbitrary word and let let $c = c_1 \cdots c_m$ be its code. Define the excedance table of v to be the unique word $\mathrm{etab}(v) = e_1 \cdots e_m$ satisfying

1. If i is an excedance in v, then $e_i = i$.
2. If $c_i = 0$, then $e_i = 0$.
3. Otherwise, e_i is the c_ith excedance of v having value at least v_i.

Note that $\mathrm{etab}(v)$ is well defined for any word v. In particular, if i is not an excedance in v and if $c_i > 0$, then there are at least c_i excedances in v having value at least v_i. To see this, define

$$k = \#\{j \in [m] | v_j < v_i\}.$$

Since c_i of the letters $\bar{v}_1, \ldots, \bar{v}_k$ appear to the right of position i in v, then at least c_i of the letters $\bar{v}_{k+1}, \ldots, \bar{v}_m$ must appear in the first k positions of v. The positions of these letters are necessarily excedances in v.

An important property of the excedance table is that the letter set of $\mathrm{etab}(v)$ is precisely the excedance set of v.

Example 2.5. Let $v = 514514532$, and define $c = \mathrm{code}(v)$. Calculating $e = \mathrm{etab}(v)$, we have

$$\begin{pmatrix} \bar{v} \\ v \\ c \\ e \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 5 & 5 \\ 5 & 1 & 4 & 5 & 1 & 4 & 5 & 3 & 2 \\ 6 & 0 & 3 & 4 & 0 & 2 & 2 & 1 & 0 \\ 1 & 0 & 3 & 4 & 0 & 3 & 4 & 1 & 0 \end{pmatrix}.$$

Calculation of $e_1, \ldots, e_5$ and e_9 is straightforward since the positions $i = 1, \ldots, 5$ and 9 are excedances in v or satisfy $c_i = 0$. We calculate e_6, e_7, and e_8 as follows. Since $c_6 = 2$, and the second excedance in v with value at least $v_6 = 4$ is 3, we set $e_6 = 3$. Since $c_7 = 2$, and the second excedance in v with value at least $v_7 = 5$ is 4, we set $e_7 = 4$. Since $c_8 = 1$, and the first excedance in v with value at least $v_8 = 3$ is 1, we set $e_8 = 1$.

We prove Theorem 2.2 with a bijection $\theta : R(w) \to R(w)$ which satisfies

$$E(v) = LC(\theta(v)), \tag{2.2}$$

and therefore

$$\mathrm{exc}(v) = \mathrm{dmc}(\theta(v)). \tag{2.3}$$

Definition 2.6. Let $w = w_1 \cdots w_m$ be any word. Define the map $\theta : R(w) \to R(w)$ by applying the following procedure to an arbitrary element v of $R(w)$.

1. Define the biword $z = \begin{pmatrix} v \\ \mathrm{etab}(v) \end{pmatrix}$.

2. Let y be the unique rearrangement of z satisfying $y = \begin{pmatrix} u \\ \mathrm{code}(u) \end{pmatrix}$.
3. Set $\theta(v) = u$.

Construction of y is quite straightforward. Let $e = e_1 \cdots e_m = \mathrm{etab}(v)$, and linearly order the biletters $z_1, \ldots, z_m$ by setting $z_i < z_j$ if

$$\begin{aligned} &v_i < v_j, \text{ or} \\ &v_i = v_j \text{ and } e_i > e_j. \end{aligned}$$

Break ties arbitrarily. Considering the biletters according to this order, insert each biletter z_i into y to the left of e_i previously inserted biletters.

Example 2.7. Let v and e be as in Example 2.5. To compute $\theta(v)$, we define

$$z = \begin{pmatrix} v \\ e \end{pmatrix} = \begin{pmatrix} 5 & 1 & 4 & 5 & 1 & 4 & 5 & 3 & 2 \\ 1 & 0 & 3 & 4 & 0 & 3 & 4 & 1 & 0 \end{pmatrix}.$$

We consider the biletters of z in the order

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix},$$

and insert them individually into y:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 \\ 00 \end{pmatrix}, \begin{pmatrix} 112 \\ 000 \end{pmatrix}, \begin{pmatrix} 1132 \\ 0010 \end{pmatrix}, \begin{pmatrix} 14132 \\ 03010 \end{pmatrix}, \ldots$$

Finally we obtain

$$y = \begin{pmatrix} u \\ e \end{pmatrix} = \begin{pmatrix} 1 & 4 & 5 & 5 & 4 & 1 & 3 & 5 & 2 \\ 0 & 3 & 4 & 4 & 3 & 0 & 1 & 1 & 0 \end{pmatrix},$$

and set $\theta(v) = 145541352$.

It is easy to see that any biword z has at most one rearrangement y satisfying Definition 2.6 (2). Such a rearrangement exists if and only if we have

$$e_i \leq \#\{j \in [m] | v_j < v_i\}, \text{ for } i = 1, \ldots, m, \tag{2.4}$$

or equivalently, if and only if

$$\bar{v}_{e_i} < v_i, \text{ for } i = 1, \ldots, m, \tag{2.5}$$

where we define $\bar{v}_0 = 0$ for convenience.

Observation 2.3. *Let $v = v_1 \cdots v_m$ be any word and let $e = \mathrm{etab}(v)$. Then we have*

$$e_i \leq \#\{j \in [m] | v_j < v_i\}, \text{ for } i = 1, \ldots, m.$$

Proof. If i is an excedance in v, then $e_i = i$ and $\bar{v}_1 \leq \cdots \leq \bar{v}_i < v_i$. If $c_i = 0$, then $e_i = 0$. Otherwise, define

$$k = \#\{j \in [m] | v_j < v_i\}.$$

By the discussion following Definition 2.4, at least c_i of the positions $1, \ldots, k$ are excedances in v with values at least v_i. The letter e_i, being one of these excedances, is therefore at most k. □

Thus the map θ is well defined and satisfies (2.2) and (2.3). We invert θ by applying the procedure in the following proposition.

Proposition 2.4. *Let y be a biword satisfying*

$$y = \binom{u}{c} = \binom{u}{\mathrm{code}(u)}.$$

The following procedure produces a rearrangement z of y satisfying

$$z = \binom{v}{e} = \binom{v}{\mathrm{etab}(v)}.$$

1. *For each letter ℓ in $L(c)$, find the greatest index i satisfying $c_i = \ell$, and define $z_\ell = y_i$. Let S be the set of such greatest indices, and define $T = [m] \smallsetminus S$.*
2. *For each index $i \in T$, define*

$$d_i = \begin{cases} \#\{j \in S | c_j \leq c_i; u_j \geq u_i\}, & \text{if } c_i > 0, \\ 0, & \text{otherwise.} \end{cases}$$

3. *Let $(y_{\sigma^{-1}(i)})_{i \in T}$ be the unique rearrangement of $(y_i)_{i \in T}$ satisfying*

$$(d_{\sigma^{-1}(i)})_{i \in T} = \mathrm{code}((u_{\sigma^{-1}(i)})_{i \in T}).$$

4. *Insert the biletters $(y_{\sigma^{-1}(i)})_{i \in T}$ in order into the remaining positions of z.*

Proof. First we claim that the procedure is well defined. In particular, we may perform step 3 because we have

$$d_i \leq \#\{j \in T | u_j < u_i\}, \text{ for each } i \in T,$$

as required by (2.4). To see that this is the case, let i be an index in T with $c_i > 0$. In step 1 we have placed d_i biletters y_j with $u_j > u_i$ into positions $1, \ldots, c_i$ of z. Thus, at least d_i biletters y_j with $u_j \leq \bar{u}_{c_i}$ have *not* been placed into these positions. The index j of any such biletter belongs to S only if $c_j > c_i$. However, since $\bar{u}_{c_j} < u_j \leq \bar{u}_{c_i} < u_i$, we have $c_j < c_i$. Thus, j belongs to T.

Next we claim that $e = \mathrm{etab}(v)$. The positions $L(c) = \{c_j | j \in S\}$ are excedances in v, because for each index j in S, we have $v_{c_j} = u_j > \bar{u}_{c_j} = \bar{v}_{c_j}$. As required, we also have $e_{c_j} = c_j$.

The positions $L(c)$ are in fact the only excedances in v. For each index j in T, denote by $\phi(j)$ the position of z into which we have placed y_j. Supposing that some indices $\{\phi(j)|j \in T\}$ are excedances in v, choose $i \in T$ so that $\phi(i)$ is the leftmost of these excedances and define

$$k = \#\{j \in [m]|u_j < u_i\}.$$

Then,

$$k > \#\{j \in S|c_j \leq k\} + \#\{j \in T|\sigma(j) < \sigma(i)\}. \tag{2.6}$$

Since $c_i \leq k$ by (2.4), we have

$$\#\{j \in S|c_j \leq k\} = \#\{j \in S|c_j \leq c_i\} + \#\{j \in S|c_i < c_j \leq k\},$$

and by the definition of σ, we have

$$\begin{aligned}\#\{j \in T|\sigma(j) < \sigma(i)\} &= \#\{j \in T|u_j < u_i\} - d_i \\ &= \#\{j \in T|u_j < u_i\} - \#\{j \in S|c_j \leq c_i; u_j \geq u_i\}.\end{aligned}$$

Using these identities to simplify (2.6), we obtain

$$\#\{j \in S|u_j < u_i; c_j > c_i\} > \#\{j \in S|c_i < c_j \leq k\}. \tag{2.7}$$

If j belongs to the set on the left hand side of (2.7) and satisfies $c_j > k$, then we have

$$u_j > \bar{u}_{c_j} \geq \bar{u}_k = u_i - 1,$$

a contradiction. On the other hand, if each index j in this set satisfies $c_j \leq k$, then we have the inclusion

$$\{j \in S|u_j < u_i; c_j > c_i\} \subset \{j \in S|c_i < c_j \leq k\},$$

which contradicts (2.7). We conlcude that the set $\{\phi(j)|j \in T\}$ is precisely the set of non-excedances in v.

To complete the proof, let c' be the code of v. We must show that for each index $i \in T$, we have

$$e_{\phi(i)} = c_i = \begin{cases} \text{the } c'_{\phi(i)}\text{th excedance in } v \text{ having value at least } u_i, & \text{if } c'_{\phi(i)} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

By our definition of the sequence $(d_i)_{i \in T}$, it suffices to show that $c'_{\phi(i)} = d_i$ for each index i. The subword $v_{\phi(i)+1} \cdots v_m$ of v includes d_i letters $v_{\phi(j)}$ with $j \in T$ and $v_{\phi(j)} < v_{\phi(i)}$. On the other hand, any excedance in v to the right of $\phi(i)$ has value *greater* than $v_{\phi(i)}$. We conclude that $c'_{\phi(i)} = d_i$. □

The above procedure inverts θ because the biword z it produces is the *unique* rearrangement of y having the desired properties.

Proposition 2.5. *Let $v = v_1 \cdots v_m$ be an arbitrary word, and define*

$$z = \binom{v}{e} = \binom{v}{\mathrm{etab}(v)}.$$

If there is any rearrangement z' of z satisfying

$$z' = \binom{v'}{e'} = \binom{v'}{\mathrm{etab}(v')},$$

then $z' = z$.

Proof. Let L be the letter set of e. By Definition 2.4, we must have $E(v) = E(v') = L$. Let i be an excedance of v and v'. By condition (1) of Definition 2.4 we must have $e_i = e'_i = i$, and by condition (3) the upper letters v_i and v'_i must be as large as possible. Thus, $(z_i)_{i \in L} = (z'_i)_{i \in L}$.

Let $T = [m] \smallsetminus L$ be the set of non-excedance positions of v and v', and consider the corresponding subsequences of biletters $(z_i)_{i \in T}$ and $(z'_i)_{i \in T}$. By condition (3) of Definition 2.4, the codes of $(v_i)_{i \in T}$ and $(v'_i)_{i \in T}$ are determined by the excedances and excedance values in v and v'. Thus, the two codes must be identical. Applying the argument following Example 2.7, we conclude that $(z_i)_{i \in T} = (z'_i)_{i \in T}$. □

2.4. **An application of Dumont's statistic.** Recall from Theorems 1.6 and 1.7 that the h-vector of any distributive lattice $J(P)$ is also the f-vector of some balanced simplicial complex Γ. It would be interesting to determine when Γ can be chosen to be an *order complex* $\Delta(Q)$ for some poset Q. Then, $h_{J(P)}$ would be the f-vector of Q and one could study the polynomial $h_{J(P)}(x)$ by studying the polynomial $f_Q(x)$. As an application of Dumont's (word) statistic, we will show that this is the case when $J(P)$ is a product of chains (or equivalently, when P is a disjoint sum of chains.)

Theorem 2.6. *Let the distributive lattice $J(P)$ be a product of chains. Then there is a poset Q such that the h-vector of $J(P)$ is the f-vector of Q.*

If $J(P)$ is a product of chains having cardinalities $(p_1+1), \ldots, (p_n+1)$, then P is the disjoint *sum* of chains $(\mathbf{p_1} + \cdots + \mathbf{p_n})$. It is not difficult to see that linear extensions of P are in bijective correspondence with rearrangements of the word $w = 1^{p_1} \cdots n^{p_n}$. Combining this observation with Theorem 2.2, we restate Theorem 2.6 in terms of Dumont's statistic.

Proposition 2.7. *Let w be any word and define the vector $h = (h_0, \ldots, h_d)$ by*

$$h_i = \#\{u \in R(w) | \mathrm{dmc}(u) = i\},$$

where d is the maximum cardinality of $LC(u)$ over all rearrangements u of w. Then, there is a poset Q whose f-vector is h.

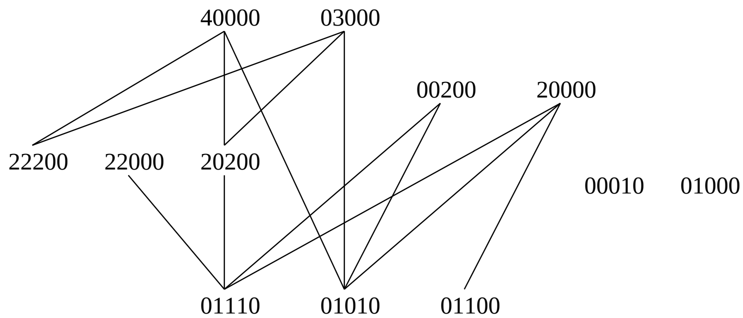


FIGURE 2.1

Let $C(w)$ be the set of codes of all rearrangements of w.

$$C(w) = \{\text{code}(u) | u \in R(w)\}.$$

Proposition 2.7 asserts that for any word w, there is a bijection between k-letter words in $C(w)$ and k-element chains in some poset Q.

$$\{c \in C(w) | c \text{ a } k\text{-letter code}\} \stackrel{1-1}{\longleftrightarrow} \{(v_1 <_Q \cdots <_Q v_k) \in \Delta(Q)\}.$$

To prove the proposition, and therefore Theorem 2.6, we will construct such a poset $Q = Q(w)$ as in Definition 2.8. In Sections 2.5 and 2.6 we will follow the general method outlined in Section 2.1 to give an explicit bijection $\Psi : C(w) \to \Delta(Q)$, taking k-letter codes in $C(w)$ to k-element chains in Q.

Definition 2.8. For any word w, let Q be the subset of one-letter codes in $C(w)$, and let c and c' be codes in Q whose letters are ℓ and ℓ', respectively. Define $c <_Q c'$ if

1. $\ell < \ell'$.
2. The multiplicity of ℓ in c is strictly *greater* than that of ℓ' in c'.
3. For each position i such that $c'_i = \ell'$, we have $c_{i+\ell'-\ell} = \ell$.

Figure 2.1 shows the poset Q corresponding to the word 11223.

2.5. **The chain map Ψ.** Fix a nondecreasing word $w = w_1 \cdots w_m$ on n letters, and define the poset Q as in Definition 2.8. We will define a *chain map* $\Psi : C(w) \to \Delta(Q)$ which will identify a code c with a chain

$$\Psi(c) = v_1 <_Q \cdots <_Q v_k,$$

of elements in Q. If c is a code on the k letters $\ell_1 < \cdots < \ell_k$, then each poset element v_i will be a code whose unique nonzero letter is ℓ_i. Specifically, we will determine v_i by applying a *vertex map* $\psi_{\ell_i} : C(w) \to Q$ to c.

$$v_i = \psi_{\ell_i}(c).$$

After proving that $\psi_{\ell_i}(c) <_Q \psi_{\ell_j}(c)$ whenever $\ell_i < \ell_j$, we will define the chain map to be a product of vertex maps,

$$\Psi(c) = \psi_{\ell_1}(c) <_Q \cdots <_Q \psi_{\ell_k}(c).$$

We begin by observing that several simple operations on codes in $C(w)$ yield other codes in $C(w)$.

Observation 2.8. *Let u be a rearrangement of w and let $c = \mathrm{code}(u)$.*

1. *If $c_i > c_{i+1}$, then the word*
$$c_1 \cdots c_{i-1} \cdot c_{i+1} \cdot (c_i - 1) \cdot c_{i+2} \cdots c_m$$
belongs to $C(w)$.
2. *If for some $r > i$, c_i is strictly less than $c_{i+1}, \ldots, c_r$, and $c_i > c_{r+1}$, then the word*
$$c_1 \cdots c_{i-1} \cdot c_{r+1} \cdot c_{i+1} \cdots c_r \cdot (c_i - 1) \cdot c_{r+2} \cdots c_m$$
belongs to $C(w)$.
3. *If $c_i < c_{i+1}$, or if $c_i = c_{i+1}$ and $u_i < u_{i+1}$, then the word*
$$c_i \cdots c_{i-1} \cdot (c_{i+1} + 1) \cdot c_i \cdot c_{i+2} \cdots c_m$$
belongs to $C(w)$.

Proof. (1) This is the code of the word obtained by switching the letters u_i and u_{i+1}, in the case that $u_i > u_{i+1}$.

(2) This is the code of the word obtained by switching the letters u_i and u_{r+1}, in the case that u_i is less than $u_{i+1}, \ldots, u_r$, c_i is less than $c_{i+1}, \ldots, c_r$, and $u_i > u_{r+1}$.

(3) This is the code of the word obtained by switching the letters u_i and u_{i+1}, in the case that $u_i < u_{i+1}$. □

Using this observation we will define two families of maps from $C(w)$ to itself, $\lambda_1, \ldots, \lambda_{m-1}$ and $\mu_1, \ldots, \mu_{m-1}$. Then, composing maps from these two families, we will define the family of vertex maps $\psi_1, \ldots, \psi_{m-1}$.

The map $\lambda_{\ell_i} : C(w) \to C(w)$ removes from a code c all letters ℓ_j which are greater than ℓ_i. It essentially changes each such letter ℓ_j to ℓ_i and moves it $\ell_j - \ell_i$ places to the right in c. If we identify c with the k-element chain $v_1 <_Q \cdots <_Q v_k$, then we will identify $\lambda_{\ell_i}(c)$ with the i-element subchain $v_1 <_Q \cdots <_Q v_i$.

Definition 2.9. Let ℓ be a nonzero letter. Define the map $\lambda_\ell : C(w) \to C(w)$ by performing the following procedure on a code c.

For $i = m, m-1, \ldots, 1$, if $c_i > \ell$, then

1. Set $\delta = c_i - \ell$.
2. Redefine $c = c_1 \cdots c_{i-1} \cdot c_i \cdots c_{i+\delta} \cdot \ell \cdot c_{i+\delta+1} \cdots c_m$.

Analogous to λ_{ℓ_i}, the map $\mu_{\ell_i} : C(w) \to C(w)$ removes all letters which are *less* than ℓ_i. It simply changes each lower letter to 0. Thus, if we identify c with the k-element chain $v_1 <_Q \cdots <_Q v_k$, then we will identify $\mu_{\ell_i}(c)$ with the $(k-i+1)$-element subchain $v_i <_Q \cdots <_Q v_k$.

Definition 2.10. Let ℓ be a nonzero letter. Define the map $\mu_\ell : C(w) \to C(w)$ by $\mu_\ell(c) = a_1 \cdots a_m$, where

$$a_i = \begin{cases} 0, & \text{if } c_i \leq \ell, \\ c_i, & \text{otherwise.} \end{cases}$$

The maps $\lambda_1, \ldots, \lambda_{m-1}$, and $\mu_1, \ldots, \mu_{m-1}$ are well defined, for their definitions are merely repeated applications of Observation 2.8 (1) and (2). Note that the composition $\mu_\ell \lambda_\ell$ produces a code on the single letter ℓ. This code is an element of Q, and a *vertex* of $\Delta(Q)$.

Definition 2.11. Let ℓ be a nonzero letter. Define the *vertex map* $\psi_\ell : C(w) \to Q$ by

$$\psi_\ell = \mu_\ell \lambda_\ell.$$

It is easy to see that $\lambda_\ell^2 = \lambda_\ell$, and therefore that $\psi_\ell \lambda_\ell = \psi_\ell$. These and the following relations will be essential in establishing a bijection between $C(w)$ and $\Delta(Q)$.

Proposition 2.9. *Let ℓ and ℓ' be letters, $1 \leq \ell < \ell' \leq n$. The maps $\lambda_\ell, \lambda_{\ell'}, \psi_\ell$, and $\psi_{\ell'}$ satisfy the relations*

1. $\lambda_{\ell'} \lambda_\ell = \lambda_\ell \lambda_{\ell'} = \lambda_\ell$.
2. $\psi_\ell \lambda_{\ell'} = \psi_\ell$.
3. $\psi_\ell(c) <_Q \psi_{\ell'}(c)$, *if c contains both letters.*

Proof. (1) Let $c = \text{code}(u)$ be an element of $C(w)$. By the comments following Definition 2.10, we may interpret $\lambda_\ell(c)$ as follows. Define $b = b_1 \cdots b_m$ by

$$b_i = \begin{cases} \ell, & \text{if } c_i > \ell, \\ c_i, & \text{otherwise,} \end{cases}$$

and rearrange the biword

$$\begin{pmatrix} u \\ b \end{pmatrix}$$

as

$$\begin{pmatrix} u' \\ b' \end{pmatrix}$$

so that $b' = \text{code}(u')$. Then, $b' = \lambda_\ell(c)$.

It is not hard to see that there is a unique such rearrangement. Using this interpretation, it is easy to see that $\lambda_{\ell'}\lambda_\ell$, $\lambda_\ell\lambda_{\ell'}$, and λ_ℓ describe the same procedure.

(2) Using (1), we have $\psi_\ell\lambda_{\ell'} = \mu_\ell\lambda_\ell\lambda_{\ell'} = \mu_\ell\lambda_\ell = \psi_\ell$.

(3) We may assume that ℓ' is the greatest letter in c. (Otherwise, we define $d = \lambda_{\ell'}(c)$ and note that $\psi_\ell(c) = \psi_\ell(d)$ and $\psi_{\ell'}(c) = \psi_{\ell'}(d)$.) Let $e = \psi_\ell(c)$ and $e' = \psi_{\ell'}(c)$. Clearly, the multiplicity of ℓ in e is strictly greater than that of ℓ' in e', for

$$\#\{i|e_i = \ell\} = \#\{i|c_i \geq \ell\} > \#\{i|c_i \geq \ell'\} = \#\{i|e'_i = \ell'\}.$$

Next, we show that for any position i of e' satisfying $e'_i = \ell$, we must have $e_{i+\ell'-\ell} = \ell$. Since by assumption, ℓ' is the greatest letter in c, we have $e'_i = \ell'$ if and only if $c_i = \ell'$. To find e, we first calculate $\lambda_\ell(c)$ by the procedure of Definition 2.9. At each iteration i such that $c_i = \ell'$, we place the letter ℓ into position $i + \ell' - \ell$ of $\lambda_\ell(c)$. This position will not be altered by iterations $i-1, \ldots, 1$, since all letters of c are no greater than ℓ'. Finally, since $e = \mu_\ell\lambda_\ell(c)$, and μ_ℓ changes only those letters less than ℓ, we see that $e_{i+\ell'-\ell} = \ell$ for every position i such that $e_i = \ell'$. $\square$

Now we may define the map Ψ.

Definition 2.12. Define the *chain* map $\Psi : C(w) \to \Delta(Q)$ by

$$\Psi(c) = \psi_{\ell_1}(c) <_Q \cdots <_Q \psi_{\ell_k}(c),$$

where $\ell_1 < \cdots < \ell_k$ are the distinct nonzero letters in c.

2.6. **Inverting Ψ.** We will define a map $\Phi : \Delta(Q) \to C(w)$ which takes a k-element chain in Q to a k-letter code in $C(w)$. By demonstrating that Φ inverts Ψ, we will complete the proof of Proposition 2.7.

We begin by defining an operation $\vee : C(w) \times Q \to C(w)$ which joins a new letter to a code.

Definition 2.13. Let $d \in Q$ be a code whose unique nonzero letter is ℓ', and let $c \in C(w)$ be a code whose greatest letter is ℓ. Assume that $\psi_\ell(c) <_Q d$. Define the code $e = c \vee d$ by the following procedure.

1. For each i such that $d_i = \ell'$, set $e_i = \ell'$ and cross out the ℓ in position $i + \delta$ of c.
2. Fill the remaining positions of e with the remaining components of c, in order.

Note that $L(e) = L(c) \cup \{\ell\}$. Therefore, we may map a chain of k one-letter codes to a single k-letter code by iterating the join operation.

Definition 2.14. Let $v_1 <_Q \cdots <_Q v_k$ be a chain of one-letter codes on the letters $\ell_1 < \cdots < \ell_k$, respectively. Define the map $\Phi : \Delta(Q) \to C(w)$ by

$$\Phi(v_1 <_Q \cdots <_Q v_k) = (\cdots((v_1 \vee v_2) \vee v_3) \cdots) \vee v_k.$$

The following proposition shows that the join operation is well defined. It follows that Φ is well defined also.

Proposition 2.10. *If c and d are codes in $C(w)$ satisfying the hypotheses of Definition 2.13, then $c \vee d$ also belongs to $C(w)$.*

Proof. Let u and y be words in $R(w)$ whose codes $c = \mathrm{code}(u)$ and $d = \mathrm{code}(y)$ satisfy the conditions of Definition 2.13. Consider the leftmost position i in c such that $c_i = \ell$ and $d_{i-\delta} = \ell'$. By assumption, $c_{i-1} \leq c_i$. If $c_{i-1} < c_i$, or if $c_{i-1} = c_i$ and $u_{i-1} < u_i$, then we may apply Observation 2.8 (3) $\ell' - \ell$ times to obtain the word

$$c_1 \cdots c_{i-\delta-1} \cdot \ell' \cdot c_{i-\delta+1} \cdots \hat{c}_i \cdots c_m,$$

which belongs to $C(w)$. (Here, $\hat{c}_i$ means that the letter c_i is omitted.) Repeating this process for each such position i, we redefine the join operation. Therefore it suffices to show that for every position i satisfying $d_{i-\delta-1} = 0$, $d_{i-\delta} = \ell'$, and $c_{i-1} = c_i = \ell$, we have $u_{i-1} < u_i$.

Let i be such a position and suppose that $u_{i-1} = u_i$. Since $d_{i-\delta} = \ell'$ and $d_{i-\delta-1} = 0$, there are exactly $i - \delta - 1 + \ell' = i + \ell - 1$ letters in y which are strictly less than y_i. In particular, we have

$$w_{i+\ell-1} < w_{i+\ell}. \tag{2.8}$$

Let k be the number of positions preceeding i such that $u_{i-k} = u_{i-k+1} = \cdots = u_i$ and $c_{i-k} = c_{i-k+1} = \cdots = \ell$. Then there are exactly $i - k - 1 + \ell$ letters in u which are strictly less than u_i $(= u_{i-1} = \cdots = u_{i-k})$. In particular, we have

$$w_{i-k-1+\ell} < w_{i-k+\ell} = w_{i-k+1+\ell} = \cdots = w_{i+\ell},$$

which contradicts (2.8). We conclude that $u_{i-1} < u_i$, and therefore that $c \vee d$ belongs to $C(w)$. □

To begin demonstrating that Φ inverts Ψ we note the following relations satisfied by ψ, λ and $\vee$.

Proposition 2.11. *The pair of maps (ψ, λ) inverts the operation $\vee$ in the following sense.*

1. *Let $c \in C(w)$ be a code with greatest letter ℓ, and let $d \in Q$ be a code with letter $\ell' > \ell$ and satisfying $\psi_\ell(c) <_Q d$. Then we have*
$$\psi_{\ell'}(c \vee d) = d,$$
$$\lambda_\ell(c \vee d) = c.$$
2. *Let $c \in C(w)$ be a code whose greatest two letters are $\ell < \ell'$. Then we have*
$$\lambda_\ell(c) \vee \psi_{\ell'}(c) = c.$$

Proof. (1) Let S be the set of positions of d containing the letter ℓ', and let $\delta = \ell' - \ell$.

Define the words $e = c \vee d$, $d' = \psi_{\ell'}(c \vee d)$, and $c' = \lambda_\ell(c \vee d)$. Calculating e, we have

$$(e_i)_{i \in S} = \ell' \cdots \ell',$$
$$(e_i)_{i \notin S} = (c_i)_{i-\delta \notin S}.$$

Since e contains no letters greater than ℓ', we have $d' = \psi_{\ell'}(e) = \mu_{\ell'}(e)$. Thus, $d' = d$:

$$(d'_i)_{i \in S} = \ell' \cdots \ell',$$
$$(d'_i)_{i \notin S} = 0 \cdots 0.$$

Calculating $c' = \lambda_\ell(e)$, we change each occurrence of ℓ' in e to ℓ, and move it δ positions to the right. Since $\psi_\ell(c) <_Q d$, we see that $c' = c$:

$$(c'_i)_{i-\delta \in S} = \ell \cdots \ell = (c_i)_{i-\delta \in S},$$
$$(c'_i)_{i-\delta \notin S} = (c_i)_{i-\delta \notin S}.$$

(2) Similar. □

Completing the proof of Proposition 2.7, the following proposition shows that Ψ is bijective.

Proposition 2.12. *Let $c \in C(w)$ be a code on the letters $\ell_1 < \cdots < \ell_k$, and let $v_1 <_Q \cdots <_Q v_k$ be a k-element chain in Q, where the letter of v_i is ℓ_i for each i. The maps Ψ and Φ satisfy*

1. $\Psi\Phi(v_1 <_Q \cdots <_Q v_k) = v_1 <_Q \cdots <_Q v_k$.
2. $\Phi\Psi(c) = c$.

Proof. (1) By Definition 2.14, we have

$$\Phi(v_1 <_Q \cdots <_Q v_k) = (\cdots((v_1 \vee v_2) \vee v_3) \cdots) \vee v_k.$$

Applying $\Psi = \psi_{\ell_1} \times \cdots \times \psi_{\ell_k}$ to this code, we calculate $\psi_{\ell_i}((\cdots(v_1 \vee v_2) \vee \cdots) \vee v_k)$, for $i = 1, \ldots, k$. By Proposition 2.11 (1), we have

$$\begin{aligned}\psi_{\ell_i}((\cdots(v_1 \vee v_2) \vee \cdots) \vee v_k) &= \psi_{\ell_i}\lambda_{\ell_i}\lambda_{\ell_{i+1}} \cdots \lambda_{\ell_k}((\cdots(v_1 \vee v_2) \vee \cdots) \vee v_k) \\ &= \psi_{\ell_i}((\cdots(v_1 \vee v_2) \vee \cdots) \vee v_i) \\ &= v_i,\end{aligned}$$

as desired.

(2) By Definition 2.12, we have

$$\Psi(c) = \psi_{\ell_1}(c) <_Q \cdots <_Q \psi_{\ell_k}(c).$$

Applying Φ to this chain, we join vertices one at a time. Noting that $\psi_{\ell_1}(c) = \lambda_{\ell_1}(c)$, we use Proposition 2.11 (2) to calculate

$$\begin{aligned} \lambda_{\ell_i}(c) \vee \psi_{\ell_{i+1}}(c) &= \lambda_{\ell_i}\lambda_{\ell_{i+1}}(c) \vee \psi_{\ell_{i+1}}\lambda_{\ell_{i+1}}(c) \\ &= \lambda_{\ell_i}(\lambda_{\ell_{i+1}}(c)) \vee \psi_{\ell_{i+1}}(\lambda_{\ell_{i+1}}(c)) \\ &= \lambda_{\ell_{i+1}}(c). \end{aligned}$$

Thus, after $k-1$ join iterations, we recover c. □

2.7. **Open questions.** Since the class of balanced Cohen-Macaulay complexes contains so many widely studied classes of complexes, there are many possibilities to refine Theorem 1.6. In Theorem 2.6, we have required that Σ be an order complex of the form $\Delta(J(P))$, where P is a disjoint sum of chains. One could also ask if the theorem holds for more general classes of posets. (See [29], [32] for definitions in the questions that follow.) For instance, the following questions are open.

Question 2.15. If P is a forest, then is there another poset Q such the h-vector of $J(P)$ is the f-vector of Q?

Question 2.16. If P is a series-parallel poset, then is there another poset Q such the h-vector of $J(P)$ is the f-vector of Q?

We conjecture that the answers to both questions are affirmative. In fact, we conjecture that the answer remains affirmative for *any* choice of a poset P.

Conjecture 2.13. *Let $J(P)$ be any distributive lattice. Then there is another poset Q such that the h-vector of $J(P)$ is the f-vector of Q.*

This conjecture has been tested by computer for all distributive lattices $J(P)$ arising from posets P having up to seven elements. Other open questions place requirements on Γ instead of on Σ.

Question 2.17. For which balanced Cohen-Macaulay complexes Σ is h_Σ the f-vector of a *graded* poset (or $(\mathbf{3}+\mathbf{1})$-free poset, or flag complex)?

To begin to answer Questions 2.15-2.17, it would be interesting to define posets on $\mathcal{L}_1(P)$ by applying the general method of Section 2.1 with various permutation statistics. One might also consider a variation of this method based upon objects other than permutations, such as Motzkin paths or either of the tree representations in [29, pp. 23-25].

Björner and Wachs [7] have obtained a result similar to Theorem 2.2 (in the sense that word rearrangements correspond to linear extensions of certain posets) which states that the statistics INV and MAJ are equally distributed on the linear extensions of posets which are postorder labelled forests. Perhaps Theorem 2.2 could be extended similarly.

Question 2.18. For what conditions on a poset are the statistics des and *dmc* equidistributed on $\mathcal{L}(P)$?

One might apply another variation of the general method of Section 2.1 by defining a rule which maps each poset P to a subset $\mathcal{K}(P)$ of S_n, which is *not* a set of linear extensions of P, but which has the property that the elements π in $\mathcal{K}(P)$ satisfying $\mathrm{dmc}(\pi) = k$ are in bijective correspondence with $\mathcal{L}_k(P)$. In hopes of solving some of these open questions, we will study another Eulerian statistic in the following section.

3. Joint Distributions of Permutation Statistics on S_n

3.1. **Introduction.** Closely related to the Jordan-Hölder set of a poset is the permutation statistic INV which counts *inversions*. We define a pair (i, j) to be an inversion in a permutation π if $i < j$ and $\pi_i > \pi_j$. Thus,

$$\mathrm{INV}(\pi) = \#\{(i,j) | i < j \text{ and } \pi_i > \pi_j\}.$$

Redefining the Jordan-Hölder set as

$$\mathcal{L}(P) = \{\pi \mid (i,j) \text{ an inversion in } \pi \Rightarrow \pi_i \not>_P \pi_j\},$$

we see that the study of linear extensions of a poset by descent is essentially the study of the *pair* of permutation statistics $(\mathrm{des}, \mathrm{INV})$.

In the past few decades, many result about permutation statistics have involved *joint* distributions on S_n of k-tuples of statistics. In 1954, Carlitz [10] essentially found a generating function for the joint distribution of $(\mathrm{des}, \mathrm{MAJ})$ [11]. This distribution has some particularly interesting symmetries. Foata [16] demonstrated the equidistribution of the pairs $(\mathrm{des}, \mathrm{INV})$ and $(\mathrm{dmc}, \mathrm{MAJ})$, and the symmetry of the distribution of the pair $(\mathrm{INV}, \mathrm{MAJ})$. Extending joint distribution results beyond pairs of statistics, Gessel [23] found a generating function for the joint distribution of $(\mathrm{des}, \mathrm{MAJ}, \mathrm{INV})$, and Garsia and Gessel [20] found a generating function for the joint distribution of $(\mathrm{des}, \mathrm{ides}, \mathrm{MAJ}, \mathrm{IMAJ})$.

The statistic INV arises often in Combinatorics, and an equidistribution result conspicuously absent from those above is one which states that for some Eulerian statistics *stat*, the pairs $(\mathrm{stat}, \mathrm{INV})$ and $(\mathrm{des}, \mathrm{MAJ})$ are equidistributed on S_n. We solve this well known problem in the following sections and state some related conjectures which may be of use in solving Conjecture 1.3.

3.2. **Eulerian-Mahonian statistic pairs.** The generating function for the number of permutations π in S_n with k inversions is the q-analog of n-factorial,

$$\sum_{\pi \in S_n} q^{\mathrm{INV}(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}).$$

The permutation statistics distributed this way on S_n have been named *Mahonian statistics* [16], in honor of MacMahon. Following Clark and Steingrimmson [?], we will write Eulerian statistics with lowercase letters and Mahonian statistics with capitals.

Here are the distributions of INV on S_n for $n = 1, \ldots, 5$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
1	1										
2	1	1									
3	1	2	2	1							
4	1	3	5	6	5	3	1				
5	1	4	9	15	20	22	20	15	9	4	1

Another Mahonian Statistic is MAJ, the sum of the descents of a permutation,

$$\mathrm{MAJ}(\pi) = \sum_{i \in D(\pi)} i.$$

The generating function for the joint distribution of $(\mathrm{des}, \mathrm{MAJ})$ on S_n is given by Carlitz's q-Eulerian polynomial $B_n(t, q)$ [10]. (We follow the notation of Foata and Zeilberger [17])

$$\sum_{\pi \in S_n} t^{\mathrm{des}(\pi)} q^{\mathrm{MAJ}(\pi)} = B_n(t, q).$$

Writing $B_n(t, q)$ as

$$B_n(t) = \sum_{k=0}^{n} B_{n,k}(q) t^k$$

Carlitz [10] showed that the coefficients $B_{n,k}(q)$ satisfy the recurrence relation

$$B_{n,k}(q) = [k+1]_q B_{n-1,k}(q) + q^k B_{n-1,k-1}(q),$$

with initial conditions $B_{0,k}(q) = \delta_{0,k}$.

Example 3.1. Here is the joint distribution on S_5 of (des, MAJ). Note the interesting symmetries.

des\MAJ	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		4	9	9	4						
2				6	16	22	16	6			
3							4	9	9	4	
4											1

A second pair of statistics, (exc, DEN) was conjectured by Denert [12] and shown by Foata and Zeilberger [17] and Han [24] to have the same joint distribution as (des, MAJ).

$$\sum_{\pi \in S_n} t^{\text{exc}(\pi)} q^{\text{DEN}(\pi)} = \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{MAJ}(\pi)} = B_n(t, q).$$

In other words,

$$\#\{\pi \in S_n | \text{exc}(\pi) = k; \text{DEN}(\pi) = p\} = \#\{\pi \in S_n | \text{des}(\pi) = k; \text{MAJ}(\pi) = p\}.$$

Denert's statistic, denoted DEN, is a Mahonian statistic with a slightly peculiar definition. Given a permutation $\pi = \pi_1 \cdots \pi_n$, we define the *Denert table* of π to be the word

$$\text{dentab}(\pi) = d_1 \cdots d_n,$$

where d_i counts the number of indices $j < i$ such that π_j appears in the subword

$$\pi_i \cdots i$$

of the cyclic permutation

$$\pi_i \cdots n \cdot 1 \cdots (\pi_i - 1).$$

(Note that each letter d_i in the Denert table is no greater than $i-1$. We will consider such words in greater detail in the next section.) We define DEN to be the function which maps a permutation π to the sum of the components of its Denert table.

$$\text{DEN}(\pi) = \sum_{i=1}^{n} d_i.$$

Foata and Zeilberger gave an equivalent definition which avoids calculation of the Denert table. Let $E(\pi)$ be the excedance set of π. Then $\text{DEN}(\pi)$ is the sum of the elements of $E(\pi)$ plus the number of inversions (i,j) in π such that both i and j are excedances or both i and j are not excedances.

$$\mathrm{DEN}(\pi) = \sum_{i \in E(\pi)} i + \mathrm{INV}\big((\pi_i)_{i\in E(\pi)}\big) + \mathrm{INV}\big((\pi_i)_{i\notin E(\pi)}\big).$$

Example 3.2. Let $\pi = 72835146$. Computing the Denert table, we have

$$\begin{pmatrix} \pi \\ \mathrm{dentab}(\pi) \end{pmatrix} = \begin{pmatrix} 7 & 2 & 8 & 3 & 5 & 1 & 4 & 6 \\ 0 & 0 & 1 & 0 & 0 & 3 & 2 & 2 \end{pmatrix}.$$

The nonzero components of dentab(π) are

$$\begin{aligned} d_3 &= 1, \text{ because } 2 \text{ occurs in the word } 8123, \\ d_6 &= 3, \text{ because } 2,3,5 \text{ occur in the word } 123456, \\ d_7 &= 2, \text{ because } 5,7 \text{ occur in the word } 4567, \\ d_8 &= 2, \text{ because } 7,8 \text{ occur in the word } 678. \end{aligned}$$

Thus, $\mathrm{DEN}(\pi) = 1+3+2+2 = 8$. Alternatively, we may calculate $\mathrm{DEN}(\pi)$ by adding the excedances of π to the numbers of inversions in the subwords 78 and 235146,

$$\mathrm{DEN}(\pi) = (1+3)+0+4 = 8.$$

Denert conjectured [12] and Foata, Zeilberger, and Han proved [17] [24] the following bivariate distribution result for excedances and Denert's statistic.

Theorem 3.1. *The pairs of permutation statistics* (exc, DEN) *and* (des, MAJ) *are equidistributed on* S_n

Thus the table in Example 3.1 counts permutations in S_5 by excedances and Denert's statistic.

Since the statistic INV arises so often in combinatorics, one might hope for a natural Eulerian statistic stat such that (stat, INV) has this same joint distribution. There is, in fact, a natural Eulerian statistic with this property. We will call it *stc*, and will define it in Section 3.4

3.3. **Codes and major index tables.** Denote by E_n the set of words $w = w_1 \cdots w_n$ on the letters $\{0,\ldots,n-1\}$ in which each letter w_i is at most $n-i$.

$$E_n = \{w = w_1 \cdots w_n | w_i \le n-i; \text{ for } i = 1,\ldots,n.\}$$

We will refer to the strictly decreasing word

$$(n-1)\cdot(n-2)\cdots 0$$

as the *stair word* of length n, and to the elements of E_n as *sub-stair words* of length n. Clearly, the cardinality of the set E_n is $n!$, and bijections between E_n and S_n arise frequently in combinatorics.

One such bijection sends a permutation to its *code.* Recall from Section 2.1 that $\mathrm{code}(\pi)$ is the word $c_1 \cdots c_n$, where

$$c_i = \#\{j \in [n] | j > i; \pi_j < \pi_i\}.$$

Example 3.3.

$$\begin{array}{rcccccccccc} \pi & = & 2 & 8 & 4 & 3 & 6 & 7 & 9 & 5 & 1 \\ \mathrm{code}(\pi) & = & 1 & 6 & 2 & 1 & 2 & 2 & 2 & 1 & 0 \end{array}$$

We will denote the bijection by γ,

$$\begin{aligned} \gamma : S_n &\to E_n \\ \pi &\mapsto \mathrm{code}(\pi). \end{aligned}$$

An important property of the code of a permutation is that the components of $\mathrm{code}(\pi)$ sum to $\mathrm{INV}(\pi)$.

We pause here to comment that the definition of Dumont's statistic in terms of the code (see Section 2.1) is somewhat arbitrary. One would arrive at different Eulerian statistics *dmd, dmi* by counting distinct nonzero letters in the Denert table or inversion table ($= \mathrm{code}(\pi^{-1})$).

Another bijection from S_n to E_n sends a permutation π to its *major index table*, a word whose components sum to $\mathrm{MAJ}(\pi)$. To define the major index table, we will denote by $\pi^{(i)}$ the restriction of π to the letters $i, \ldots, n$. (e.g. if $\pi = 284367951$, then $\pi^{(4)} = 846795$, the restriction of π to the letters $4, \ldots, 9$.)

Definition 3.4. Let π be a permuation in S_n. Define the major index table of π to be the word

$$\mathrm{majtable}(\pi) = m_1 \cdots m_n,$$

where

$$m_i = \begin{cases} 0, & \text{if } i = n, \\ \mathrm{MAJ}(\pi^{(i)}) - \mathrm{MAJ}(\pi^{(i+1)}), & \text{otherwise.} \end{cases}$$

If we imagine building the permutation π by inserting the letters $[n]$ in decreasing order, m_i is the amount by which the major index increases with the insertion of i.

Example 3.5. Let $\pi = 284367951$. To calculate the major index table we build π one letter at a time in the order $9, \ldots, 1$ and note each increase in the major index.

(Slashes below indicate descents.)

i	$\pi^{(i)}$	$\text{MAJ}(\pi^{(i)})$	m_i
9	9	0	0
8	89	0	0
7	8/79	1	1
6	8/679	1	0
5	8/679/5	5	4
4	8/4679/5	6	1
3	8/4/3679/5	9	3
2	28/4/3679/5	12	3
1	28/4/3679/5/1	20	8

Thus we have $\text{majtable}(\pi) = m_1 \cdots m_9 = 833140100$.

It is not hard to see that for each permutation π in S_n, the word $\text{majtable}(\pi)$ belongs to E_n. (See Lemma 3.3.) Furthermore, the map taking a permutation to its major index table is known to be a bijection. (See [11].)

Theorem 3.2. *The map $\mu : S_n \to E_n$ defined by $\mu(\pi) = \text{majtable}(\pi)$ is a bijection.*

Proof. To invert μ we apply the following procedure to a word $m = m_1 \cdots m_n$ in E_n.

1. Define $w^{(n)}$ to be the one-letter word n.
2. For $i = n-1, \ldots, 1$, let $w^{(i)}$ be the unique word obtained by inserting the letter i into the word $w^{(i+1)}$ in such a way that $\text{MAJ}(w^{(i)}) - \text{MAJ}(w^{(i+1)}) = m_i$.
3. Set $\pi = w^{(1)}$.

It is clear that if the word π exists, then it satisfies $\pi^{(i)} = w^{(i)}$ for $i = 1, \ldots, n$, so that $\text{majtable}(\pi) = m$. By the following lemma, π does exist and is unique. In particular, given any permutation w on the letters $\{i+1, \ldots, n\}$ and any integer ℓ in the interval $\{0, \ldots, n-i\}$, then there is a unique permutation w' obtained by inserting the letter i into w in such a way that $\text{MAJ}(w') - \text{MAJ}(w) = \ell$. $\square$

Lemma 3.3. *Let $\pi = \pi_1 \cdots \pi_{n-i}$ be a permutation on the letters $\{i+1, \ldots, n\}$, and suppose that π has k descents. Let $d_{k-1} < \cdots < d_0$ be the positions of these k descents, let $d_k = 0$, and let $a_{k+1} < \cdots < a_{n-i} = n-i$ be the remaining positions of π.*

1. *Let ℓ be an integer satisfying $0 \le \ell \le k$ and define π' to be the permutation obtained by inserting the letter i into position $d_\ell + 1$ of π. Then,*

$$\text{des}(\pi') = \text{des}(\pi),$$
$$\text{MAJ}(\pi') = \text{MAJ}(\pi) + \ell.$$

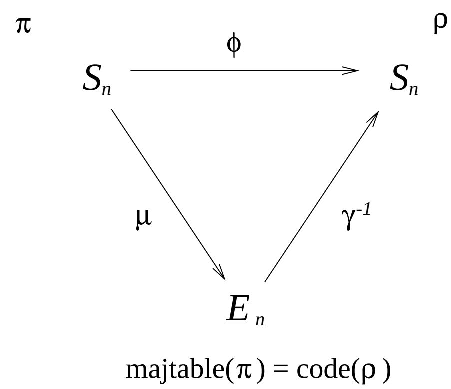


FIGURE 3.1

2. *Let ℓ be an integer satisfying $k < \ell \leq n - i$ and define π' to be the permutation obtained by inserting the letter i into position $a_\ell + 1$ of π. Then,*

$$\mathrm{des}(\pi') = \mathrm{des}(\pi) + 1,$$
$$\mathrm{MAJ}(\pi') = \mathrm{MAJ}(\pi) + \ell.$$

Proof. (1) The descent set of π' is

$$D(\pi') = \{d_{k-1}, \ldots, d_\ell, d_{\ell-1} + 1, \ldots, d_0 + 1\}.$$

(2) Let p be the least number such that $d_p < a_\ell$. Then,

$$D(\pi') = \{d_{k-1}, \ldots, d_p, a_\ell, d_{p-1} + 1, \ldots, d_0 + 1\},$$

and $a_\ell = (\ell - k) + (k - p) = \ell - p$.

□

Combining the bijections γ and μ, we prove the equidistribution on S_n of MAJ and INV as a corollary. (See Figure 3.1.)

Corollary 3.4. *The permutation statistics* INV *and* MAJ *are equally distributed on S_n.*

Proof. Define the bijection $\phi : S_n \to S_n$ by $\phi = \gamma^{-1}\mu$. Since ϕ satisfies

$$\mathrm{majtable}(\pi) = \mathrm{code}(\phi(\pi)),$$

it also satisfies

$$\mathrm{MAJ}(\pi) = \mathrm{INV}(\phi(\pi)).$$

□

Surprisingly, the letter order $n, n-1, \ldots, 1$ preceding Definition 3.4 is not crucial for the construction of the major index table. In fact *any* letter order $\sigma_1, \ldots, \sigma_n$ induces a bijection $S_n \to E_n$ as in Theorem 3.2 [22]. Let us reconsider Theorem 3.2, Lemma 3.3, and Corollary 3.4 in terms of a more general major index table.

Fix a permutation $\sigma = \sigma_1 \cdots \sigma_n$, in S_n and denote by $\pi^{(i)}$ the restriction of π to the letters $\sigma_i, \ldots, \sigma_n$. (e.g. if $\sigma = 852739461$ and $\pi = 284367951$, then $\pi^{(4)} = 436791$, the restriction of π to the letters $\{\sigma_4, \ldots, \sigma_9\} = \{1, 3, 4, 6, 7, 9\}$.) We will call the sequence $\pi^{(1)}, \ldots, \pi^{(n)}$ defined in this way the *sequence of restricted permutations corresponding to σ.*

Definition 3.6. Fix a permutation σ in S_n. For any permutation π in S_n, construct the sequence $\pi^{(1)}, \ldots, \pi^{(n)}$ of restricted permutations corresponding to σ and define the σ-major index table of π to be the word

$$\sigma\text{-majtable}(\pi) = m_1 \cdots m_n,$$

where

$$m_i = \begin{cases} 0, & \text{if } i = n, \\ \textsc{maj}(\pi^{(i)}) - \textsc{maj}(\pi^{(i+1)}), & \text{otherwise.} \end{cases}$$

Note that if σ is the permutation $1 \cdots n$, then the sequence $\pi^{(1)}, \ldots, \pi^{(n)}$ has the same meaning as preceding Definition 3.4, and the σ-major index table is just the major index table.

Theorem 3.5. *Fix a permutation σ in S_n. The map $\mu_\sigma : S_n \to E_n$ defined by $\mu_\sigma(\pi) = \sigma\text{-majtable}(\pi)$ is a bijection.*

Proof. Similar to the proof of Theorem 3.2. Use the following lemma instead of Lemma 3.3. ☐

Lemma 3.6. *Fix a permutation σ in S_n. Let π be a word on the letters $\{\sigma_{i+1}, \ldots, \sigma_n\}$, and suppose that π has k descents. Let $d_{k-1} < \cdots < d_0$ be the positions of these k descents, let $d_k = 0$, and let $d_{-1} = n - i$. Define the positions $d'_k < \cdots < d'_0$ by*

$$d'_\ell = \begin{cases} d_\ell, & \textit{if } \pi_{d_\ell+1} > \sigma_i, \\ \max\{j \in [n-i] | d_\ell < j \le d_{\ell-1}; \pi_j < \sigma_i\}, & \textit{otherwise.} \end{cases}$$

Let $a'_{k+1} < \cdots < a'_{n-i}$ be the positions

$$\{0, 1, \ldots, n-i\} \smallsetminus \{d'_0, \ldots, d'_k\}.$$

1. *Let ℓ be an integer satisfying $0 \le \ell \le k$ and define π' to be the permutation obtained by inserting the letter σ_i into position $d'_\ell + 1$ of π. Then,*

$$\mathrm{des}(\pi') = \mathrm{des}(\pi),$$
$$\textsc{maj}(\pi') = \textsc{maj}(\pi) + \ell.$$

2. *Let ℓ be an integer satisfying $k < \ell \leq n - i$ and define π' to be the permutation obtained by inserting the letter σ_i into position $a_\ell + 1$ of π. Then,*

$$\begin{aligned}\operatorname{des}(\pi') &= \operatorname{des}(\pi) + 1,\\ \textsc{maj}(\pi') &= \textsc{maj}(\pi) + \ell.\end{aligned}$$

Proof. Identical to the proof of Lemma 3.3. □

As a corollary of Theorem 3.5, we have $n!$ bijections of the form

$$\phi_\sigma = \gamma^{-1}\mu_\sigma : S_n \to S_n$$

which satisfy

$$\textsc{maj}(\pi) = \textsc{inv}(\phi_\sigma(\pi))$$

and therefore prove that

$$\#\{\pi \in S_n | \textsc{inv}(\pi) = k\} = \#\{\pi \in S_n | \textsc{maj}(\pi) = k\}.$$

We shall see that the bijections $\{\phi_\sigma | \sigma \in S_n\}$ prove other equidistribution results as well.

It is not difficult to show that if $\sigma = \sigma_1 \cdots \sigma_n$ and $\sigma' = \sigma'_1 \cdots \sigma'_n$ are two different permutations, then the bijections μ_σ and $\mu_{\sigma'}$ are identical if and only if $\sigma_n = \sigma'_{n-1}$ and $\sigma_{n-1} = \sigma'_n$. Further discussion omitted.

3.4. **The statistic stc.** We will define a simple Eulerian statistic stc which satisfies

$$\#\{\pi \in S_n | \operatorname{stc}(\pi) = k; \textsc{inv}(\pi) = p\} = \#\{\pi \in S_n | \operatorname{des}(\pi) = k; \textsc{maj}(\pi) = p\},$$

and will give a bijective proof of this fact in Theorem 3.8. To begin, we introduce a function $st : E_n \to \mathbb{N}$ and relate this function to the Eulerian numbers.

Definition 3.7. Define $st : E_n \to \mathbb{N}$ to be the function which maps a sub-stair word $v = v_1 \cdots v_n$ to the greatest number ℓ such that v contains a subsequence $v_{i_1}, \ldots, v_{i_\ell}$ which is (componentwise) strictly greater than the stair word of length ℓ,

$$v_{i_1} \cdots v_{i_\ell} > (\ell - 1) \cdot (\ell - 2) \cdots 0.$$

While v may contain several such subsequences of maximum length, identifying one and calculating this maximum length is quite easy. Starting from the rightmost position of v and reading left, we circle the first letter which is at least one, the next which is at least two, etc., until we cannot continue. The number of circled positions of v is then $st(v)$.

Example 3.8. Let $v = 245223010$. Starting from the right, we circle the 1, 3, 5, and 4. Thus, $st(v) = 4$. Note that $v_2 v_3 v_5 v_8 = 4531$ is strictly greater than 3210, the stair word of length 4.

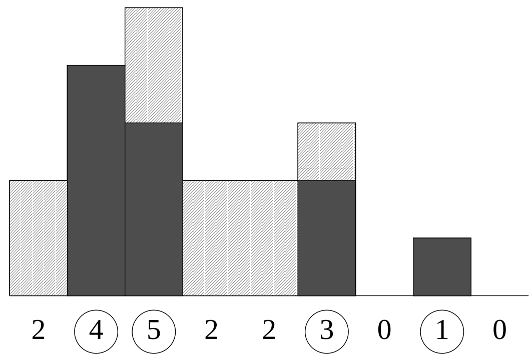


FIGURE 3.2. A histogram interpretation of $st(245223010)$.

Representing words as histograms, we see that $st(v)$ is the number of nonzero stairs which we can place beneath the histogram of v, preserving the order of these stairs. Figure 3.2 shows four nonzero stairs beneath the histogram of the word 245223010.

Let us count the words v in E_n satisfying $st(v) = k$, and denote these numbers by $\alpha(n, k)$. To derive a recursive formula for $\alpha(n, k)$, we note that

$$st(v) = st(v_1 \cdot v_2 \cdots v_n) = \begin{cases} st(v_2 \cdots v_n) + 1, & \text{if } v_1 > st(v_2 \cdots v_n), \\ st(v_2 \cdots v_n), & \text{if } v_1 \leq st(v_2 \cdots v_n). \end{cases} \tag{3.1}$$

Thus we have the recursion

$$\alpha(n, k) = (k+1)\alpha(n-1, k) + (n-k-1)\alpha(n-1, k-1),$$

and initial conditions

$$\alpha(1, k) = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We see that the numbers $\alpha(n, k)$ are in fact the Eulerian numbers.

Observation 3.7. *The number of words v in E_n satisfying $st(v) = k - 1$ is given by the Eulerian number $A(n, k)$.*

By similar reasoning one can show that the statistic dmc is *Eulerian.*

Composing the function st with the bijections γ and μ_σ from Section 3.3, we create a family of permutation statistics.

Definition 3.9. Define the permutation statistics stc and stm_σ by

$$\mathrm{stc}(\pi) = st(\mathrm{code}(\pi)),$$
$$\mathrm{stm}_\sigma(\pi) = st(\sigma\text{-majtable}(\pi)).$$

Observation 3.7 shows that each of these statistics is Eulerian; however, our main theorem reproves this fact bijectively without appealing to the observation. Furthermore, it shows that the statistic stc satisfies

$$\#\{\pi \in S_n | \mathrm{stc}(\pi) = k; \textsc{inv}(\pi) = p\} = \#\{\pi \in S_n | \mathrm{des}(\pi) = k; \textsc{maj}(\pi) = p\}.$$

We prove the main theorem with the bijection $\phi : S_n \to S_n$ from Corollary 3.4, which satisfies

$$\textsc{maj}(\pi) = \textsc{inv}(\phi(\pi)),$$
$$\mathrm{des}(\pi) = \mathrm{stc}(\phi(\pi)).$$

Theorem 3.8. *The pairs of permutation statistics* (des, MAJ) *and* (stc, INV) *are equally distributed on* S_n.

Proof. Fix a permutation σ in S_n, and let $\phi_\sigma : S_n \to S_n$ be the bijection defined in Theorem 3.2. For every permutation π in S_n, the bijection ϕ_σ satisfies

$$\sigma\text{-majtable}(\pi) = \mathrm{code}(\phi_\sigma(\pi)),$$

and therefore also satisfies

$$\textsc{maj}(\pi) = \textsc{inv}(\phi_\sigma(\pi)),$$
$$\mathrm{stm}_\sigma(\pi) = \mathrm{stc}(\phi_\sigma(\pi)).$$

We claim that $\mathrm{stm}_\sigma(\pi) = \mathrm{des}(\pi)$.

Let $m = m_1 \cdots m_n$ be the σ-major index table of π, and let $\pi^{(1)}, \ldots, \pi^{(n)}$ be the sequence of permutations corresponding to σ. Fix $i < n$ and assume that $st(m_{i+1} \cdots m_n) = \mathrm{des}(\pi^{(i+1)})$. By equation (3.1), we have

$$st(m_i \cdot m_{i+1} \cdots m_n) = \begin{cases} st(m_{i+1} \cdots m_n) + 1, & \text{if } m_i > st(m_{i+1} \cdots m_n), \\ st(m_{i+1} \cdots m_n), & \text{otherwise.} \end{cases}$$

By Lemma 3.6, we have

$$\mathrm{des}(\pi^{(i)}) = \begin{cases} \mathrm{des}(\pi^{(i+1)}) + 1, & \text{if } m_i > \mathrm{des}(\pi^{(i+1)}), \\ \mathrm{des}(\pi^{(i+1)}), & \text{otherwise.} \end{cases}$$

Thus, $st(m_i \cdots m_n) = \mathrm{des}(\pi^{(i)})$. Proceding by induction, we obtain $\mathrm{stm}_\sigma(\pi) = \mathrm{des}(\pi)$, as desired. □

3.5. **The stc-set of a permutation.** Theorem 3.8 states an analogy between the statistic pairs $(\mathrm{des}, \textsc{maj})$ and $(\mathrm{stc}, \textsc{inv})$. We will extend this result by associating a set to any permutation in such a way that the relationship of this set to the statistics stc and INV is analogous to that of the descent set to the statistics des and MAJ. Let

$D : S_n \to 2^{[n-1]}$ be the set-valued function which maps a permutation to its descent set. Recall that D satisfies

$$|D(\pi)| = \mathrm{des}(\pi),$$
$$\sum_{i \in D(\pi)} i = \mathrm{MAJ}(\pi).$$

In Definition 3.11 we will define an analogous set-valued function $STC : S_n \to 2^{[n-1]}$ which satisfies

$$|STC(\pi)| = \mathrm{stc}(\pi),$$
$$\sum_{i \in STC(\pi)} i = \mathrm{INV}(\pi).$$

We will call $STC(\pi)$ the *stc-set* of π, and we will show that the functions D and STC are equally distributed on S_n.

Our strategy in defining $STC(\pi)$ for any permutation π is to transform the word $\mathrm{code}(\pi)$, whose components sum to $\mathrm{INV}(\pi)$, into another word in which all the nonzero letters are *distinct* and sum to $\mathrm{INV}(\pi)$. We begin by defining a map $\omega : E_n \to E_n$ which transforms a sub-stair word v into another word whose component sum is the same as that of v, and whose nonzero letters are strictly decreasing. We give an algorithmic description of the map ω.

Definition 3.10. Define the map $\omega : E_n \to E_n$ by performing the following procedure on a sub-stair word $v = v_1 \cdots v_n$.

1. For $i = n-2$ to 1, do
2. For $j = i$ to $n-2$, do
3. If $v_j \leq v_{j+1}$ and these letters are not both zero, then replace them in positions j and $j+1$ by the letters ℓ and ℓ', where

$$(\ell, \ell') = \begin{cases} (v_{j+1}+1, v_j - 1), & \text{if } v_j > 0, \\ (v_{j+1}, v_j), & \text{otherwise.} \end{cases} \tag{3.2}$$

The map ω is essentially a "bubble sort" algorithm, modified so that each time it exchanges two nonzero letters, it increments the letter moving left and decrements the letter moving right. Figure 3.3 shows the computation of ωv for $v = 332110$. Each line in the table represents a single iteration of the algorithm, with positions j and $j+1$ marked below by $\times$ if the corresponding letters are exchanged, and by)(otherwise.

$$\begin{array}{cccccc}
3 & 3 & 2 & 1 & 1 & 0 \\
3 & 3 & 2 & 2 & 0 & 0 \\
3 & 3 & 3 & 1 & 0 & 0 \\
3 & 3 & 3 & 1 & 0 & 0 \\
3 & 4 & 2 & 1 & 0 & 0 \\
3 & 4 & 2 & 1 & 0 & 0 \\
3 & 4 & 2 & 1 & 0 & 0 \\
5 & 2 & 2 & 1 & 0 & 0 \\
5 & 3 & 1 & 1 & 0 & 0 \\
5 & 3 & 2 & 0 & 0 & 0 \\
5 & 3 & 2 & 0 & 0 & 0
\end{array}$$

FIGURE 3.3

It is easy to verify from Definition 3.10 that for any word v in E_n with $\mathrm{stc}(v) = k$, the word $v' = \omega v$ also belongs to E_n and satisfies

$$\begin{aligned} v'_1 > \cdots > v'_k &> 0, \\ v'_{k+1} = \cdots = v'_n &= 0, \\ v'_1 + \cdots + v'_k &= v_1 + \cdots + v_n. \end{aligned}$$

(These properties also follow from Theorem 3.9.) The map ω therefore naturally associates a subset of $[n-1]$ to each sub-stair word v in E_n: the set of nonzero letters of ωv.

Definition 3.11. Define the st-set of a substair word v to be the set

$$ST(v) = \{\ell > 0 | \ell \text{ appears in } \omega v\}.$$

Applying this definition to the code and major index tables of a permutation π, define the stc-set and σ-stm-set of π to be the sets

$$\begin{aligned} STC(\pi) &= ST(\mathrm{code}(\pi)), \\ \sigma\text{-}STM(\pi) &= ST(\sigma\text{-majtable}(\pi)). \end{aligned}$$

We claim that for every subset T of $[n-1]$, the number of permutations in S_n with stc-set T equals the number of permutations in S_n with descent set T. We prove this result with the bijection ϕ from Corollary 3.4 which satisfies

$$D(\pi) = STC(\phi(\pi)).$$

Theorem 3.9. *For every subset T of $[n-1]$, we have*

$$\#\{\pi \in S_n | D(\pi) = T\} = \#\{\pi \in S_n | STC(\pi) = T\}.$$

Proof. Fix a permutation σ in S_n, and let $\phi_\sigma : S_n \to S_n$ be the bijection defined in Theorem 3.2. For every permutation π in S_n, the bijection ϕ_σ satisfies

$$\sigma\text{-majtable}(\pi) = \text{code}(\phi_\sigma(\pi)),$$

and therefore also satisfies

$$\sigma\text{-}STM(\pi) = STC(\phi_\sigma(\pi)).$$

We claim that σ-$STM(\pi) = D(\pi)$.

Let $m = m_1 \cdots m_n$ be the σ-major index table of π, and let $\pi^{(1)}, \ldots, \pi^{(n)}$ be the sequence of permutations corresponding to σ. Fix $i < n$ and let $d_0 > \cdots > d_{k-1}$ be the descents of $\pi^{(i+1)}$. Assume that $\omega(m_{i+1} \cdots m_n) = d_0 \cdots d_{k-1} \cdot 0 \cdots 0$ so that

$$ST(m_{i+1} \cdots m_n) = D(\pi^{(i+1)}).$$

It is easy to see from Definition 3.10 that the map ω satisfies

$$\omega(m_i \cdots m_n) = \omega(m_i \cdot \omega(m_{i+1} \cdots m_n)).$$

Thus, we have

$$\begin{aligned} \omega(m_i \cdots m_n) &= \omega(m_i \cdot d_0 \cdots d_{k-1} \cdot 0 \cdots 0) \\ &= \begin{cases} (d_0+1) \cdots (d_{m_i-1}+1) \cdot d_{m_i} \cdots d_{k-1} \cdot 0 \cdots 0, & \text{if } m_i \le k, \\ (d_0+1) \cdots (d_{j-1}+1) \cdot (m_i - j) \cdot d_j \cdots d_{k-1} \cdot 0 \cdots 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where j is the least integer satisfying $m_i - j > d_j$.

By Lemma 3.3, the nonzero letters in $\omega(m_i \cdots m_n)$ are precisely the descent set of $\pi^{(i)}$. Thus, we have $ST(m_i \cdots m_n) = \text{des}(\pi^{(i)})$. Proceding by induction, we obtain σ-$STM(\pi) = D(\pi)$, as desired. □

Corollary 3.10. *Let π be a permutation in S_n with $\text{stc}(\pi) = k$ and $\text{INV}(\pi) = p$. Then the stc-set of π is a k-subset of $[n-1]$ whose elements sum to p.*

We formulate a second, algebraic, definition of the map ω as follows. Let G be a set of $n-2$ operators $\eta_1, \ldots, \eta_{n-2}$ on E_n, and define the action of η_i on a word $v = v_1 \cdots v_n$ by

$$\eta_i(v) = v_1 \cdots v_{i-1} \cdot \ell \cdot \ell' \cdot v_{i+2} \cdots v_n,$$

where

$$(\ell, \ell') = \begin{cases} (v_i, v_{i+1}) & \text{if } v_i > v_{i+1} \text{ or } v_i = v_{i+1} = 0, \\ (v_{i+1}+1, v_i - 1) & \text{if } 0 < v_i \leq v_{i+1}, \\ (v_{i+1}, v_i) & \text{if } 0 = v_i < v_{i+1}. \end{cases}$$

It is not difficult to see that the operators in the set G satisfy the relations of H_{n-2}, the 0-*Hecke semigroup* on $n-2$ generators:

$$\begin{aligned} \eta_i^2 = \eta_i, &\qquad \text{for } i = 1, \ldots, n, \\ \eta_i \eta_{i+1} \eta_i = \eta_{i+1} \eta_i \eta_{i+1}, &\qquad \text{for } i = 1, \ldots, n-1, \\ \eta_i \eta_j = \eta_j \eta_i, &\qquad \text{for } |i-j| \geq 2. \end{aligned}$$

Thus, an alternative definition of ω is

$$\omega = (\eta_{n-2}\eta_{n-3} \cdots \eta_1) \cdot (\eta_{n-2}\eta_{n-3} \cdots \eta_2) \cdots (\eta_{n-2}\eta_{n-3}) \cdot (\eta_{n-2}).$$

3.6. **More Equidistribution Results.** Since Dumont's statistic has not enjoyed many applications since its definition, the problem of finding a Mahonian partner X for dmc has not received much attention. We will define such a statistic X and two additional statistics Y and Z in order to prove several joint distribution results.

Let $c = c_1 \cdots c_n$ be the code of a permutation π, and for each letter ℓ in $L(c)$, define the number $\xi(\ell)$ by

$$\xi(\ell) = \ell + \#\{i | c_i \geq \ell; \ell \text{ appears in } c_1 \cdots c_{i-1}\}.$$

Define $\mathrm{X}(\pi)$ to be the sum

$$\mathrm{X}(\pi) = \sum_{\ell \in L(c)} \xi(\ell).$$

Example 3.12. Let $\pi = 427196385$. Then, $\mathrm{code}(\pi) = 314042010$. We calculate $\xi(\ell)$ for each letter ℓ in $LC(\pi) = \{1, 2, 3, 4\}$, as the sum of all numbers to its right in the table below.

$\mathrm{code}(\pi)$	=	3	1	4	0	4	2	0	1	0
$\xi(1)$	=		1	1		1	1		1	
$\xi(2)$	=						2			
$\xi(3)$	=	3		1		1				
$\xi(4)$	=			4		1				

Thus, $\mathrm{X}(\pi) = 5 + 2 + 5 + 5 = 17$.

The following definitions of the statistics $\textsc{y}$ and $\textsc{z}$ are perhaps a bit artificial. While one could define them as

$$\textsc{y}(\pi) = \textsc{maj}(\theta'(\pi)),$$
$$\textsc{z}(\pi) = \mathrm{des}(\theta'(\pi)),$$

where θ' is a bijection which will defined shortly, we define the two statistics without referring to θ', in case they might aid in the bijective proof of another equidistribution result.

Let $e = e_1 \cdots e_n$ be the excedance table of a permutation π, and for each letter i, let $\epsilon(i)$ be $e_{\pi^{-1}(i)}$, the letter in e whose position is that of i in π. Define $\textsc{y}(\pi)$ to be the sum

$$\textsc{y}(\pi) = \sum_{i : \epsilon(i+1) > \epsilon(i)} (n - i + 1),$$

and define $\textsc{z}(\pi)$ to be the sum

$$\textsc{z}(\pi) = \#\{i \,|\, \epsilon(i+1) > \epsilon(i)\}.$$

To show that $\textsc{x}$ is a Mahonian partner for Dumont's statistic, we introduce a bijection $\nu : S_n \to E_n$. Interpreting $\nu(\pi)$ as the code of another permutation, we will then create a bijection

$$\theta' = \gamma^{-1}\nu : S_n \to S_n,$$

which satisfies

$$E(\pi) = LC(\theta'(\pi)), \tag{3.3}$$

$$\mathrm{exc}(\pi) = \mathrm{dmc}(\theta'(\pi)), \tag{3.4}$$

$$\textsc{den}(\pi) = \textsc{maj}(\theta'(\pi)). \tag{3.5}$$

Definition 3.13. Define the map $\nu : S_n \to E_n$ by sending a permutation π with excedance table $e = e_1 \cdots e_n$ to the word $a = a_1 \cdots a_n$, where

$$a_{n-\pi_i+1} = e_i.$$

Note that this corresponds to rearranging the biword

$$\begin{pmatrix} \pi \\ \mathrm{etab}(\pi) \end{pmatrix}$$

so that the top letters are in reverse order, and simply reading the resulting bottom word. By equation (2.4) the word $\nu(\pi)$ belongs to E_n.

Example 3.14. To apply ν to the permutation $\pi = 72835146$, we calculate

$$\begin{pmatrix} \pi \\ \mathrm{etab}(\pi) \end{pmatrix} = \begin{pmatrix} 7 & 2 & 8 & 3 & 5 & 1 & 4 & 6 \\ 1 & 1 & 3 & 1 & 3 & 0 & 0 & 0 \end{pmatrix},$$

and rearrange this biword as

$$\begin{pmatrix} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 3 & 1 & 0 & 3 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Thus, we have $\nu(\pi) = 31030110$.

We invert θ' using the procedure in the following proposition.

Proposition 3.11. *Let u be the word $n \cdot (n-1) \cdots 1$, let c be an element of E_n, and define the biword y by*

$$y = \begin{pmatrix} u \\ c \end{pmatrix}.$$

The following procedure produces the unique rearrangement z of y satisfying

$$z = \begin{pmatrix} \rho \\ e \end{pmatrix} = \begin{pmatrix} \rho \\ \mathrm{etab}(\rho) \end{pmatrix}.$$

1. *For each letter ℓ in $L(c)$, find the least index i satisfying $c_i = \ell$, and define $z_\ell = y_i$. Let S be the set of such least indices, and define $T = [n] \smallsetminus S$.*
2. *For each index $i \in T$, define*

$$d_i = \begin{cases} \#\{j \in S | c_j \leq c_i; u_j \geq u_i\}, & \text{if } c_i > 0, \\ 0, & \text{otherwise.} \end{cases}$$

3. *Let $(y_{\sigma^{-1}(i)})_{i \in T}$ be the unique rearrangement of $(y_i)_{i \in T}$ satisfying*

$$(d_{\sigma^{-1}(i)})_{i \in T} = \mathrm{code}((u_{\sigma^{-1}(i)})_{i \in T}).$$

4. *Insert the biletters $(y_{\sigma^{-1}(i)})_{i \in T}$ in order into the remaining positions of z.*

Proof. Similar to the proof of Proposition 2.4. Uniqueness of the rearrangement z follows from Proposition 2.5. □

Proposition 3.12. *The pairs of statistics* (dmc, X) *and* (des, MAJ) *are equidistributed on S_n.*

Proof. By Theorem 3.1, it suffices to show the equidistribution of $(\mathrm{dmc}, \mathrm{X})$ with $(\mathrm{exc}, \mathrm{DEN})$. Let $\pi = \pi_1 \cdots \pi_n$ be a permutation in S_n. The map θ' satisfies $\mathrm{exc}(\pi) = \mathrm{dmc}(\theta''(\pi))$. We claim that it also satisfies $\mathrm{DEN}(\pi) = \mathrm{X}(\theta'(\pi))$.

Using (3.2) and the definition of excedance table, we may express $\text{DEN}(\pi)$ as

$$\begin{aligned}\text{DEN}(\pi) &= \sum_{i\in E(\pi)} i + \text{INV}\big((\pi_i)_{i\in E(\pi)}\big) + \text{INV}\big((\pi_i)_{i\notin E(\pi)}\big)\\ &= \sum_{i\in E(\pi)} i + \sum_{i\in E(\pi)} \#\{j\in E(\pi)|\pi_i>\pi_j; i\le j\} + \sum_{i\in E(\pi)} \#\{j\notin E(\pi)|\pi_i>\pi_j; i\le e_j\}\\ &= \sum_{i\in E(\pi)} i + \sum_{i\in E(\pi)} \#\{j|\pi_i>\pi_j; i\le e_j\}.\end{aligned}$$

Letting $a = a_1\cdots a_n = \nu(\pi)$, we may express $\text{X}(\theta'(\pi))$ as

$$\text{X}(\theta'(\pi)) = \sum_{i\in L(a)} i + \sum_{i\in L(a)} \#\{j|i\le a_j; i \text{ appears in } a_1\cdots a_{j-1}\}.$$

Since $E(\pi) = L(a)$, it suffices to show that for each letter i in $E(\pi)$, we have

$$\#\{j|\pi_i>\pi_j; i\le e_j\} = \#\{j|i\le a_j; i \text{ appears in } a_1\cdots a_{j-1}\}.$$

This is easily demonstrated by the bijection $j\mapsto n-\pi_j+1$. Since $e_j = a_{n-\pi_j+1}$, we have $i\le e_j$ if and only if $i\le a_{n-\pi_i+1}$. Further, the letter i first appears in the word a in position $n-\pi_i+1$, which precedes position $n-\pi_j+1$ if and only if $\pi_i>\pi_j$. □

Proposition 3.13. *The pairs of statistics* (exc, Y) *and* (des, INV) *are equidistributed on* S_n.

Proof. By Theorem 2.2 and Proposition 3.12, it suffices to show the equidistribution of (exc, Y) with (dmc, MAJ). Let $\pi = \pi_1\cdots\pi_n$ be a permutation and let $a = a_1\cdots a_n$ be $\nu(\pi)$ so that $a = \text{code}(\theta'(\pi))$. We simply note that $\epsilon(i+1)$ is greater than $\epsilon(i)$ if and only if $n-i+1$ is a descent in a. This is equivalent to $n-i+1$ being a descent in $\theta'(\pi))$. □

As a corollary, we have an equidistribution result for two quadruples of statistics.

Corollary 3.14. *The two quadruples of permutation statistics* (exc, DEN, Y, Z) *and* (dmc, X, MAJ, des) *are equidistributed on* S_n.

3.7. **Open Equidistribution Questions.** Experience with the statistic stc suggests that a result much stronger than that of Theorem 3.8 is true.

Conjecture 3.15. *The following quadruples of permutation statistics are equidistributed on* S_n.

1. (des, MAJ, INV, stc)
2. (stc, INV, MAJ, des)
3. (dmc, X, MAJ, des)
4. (des, MAJ, X, dmc)

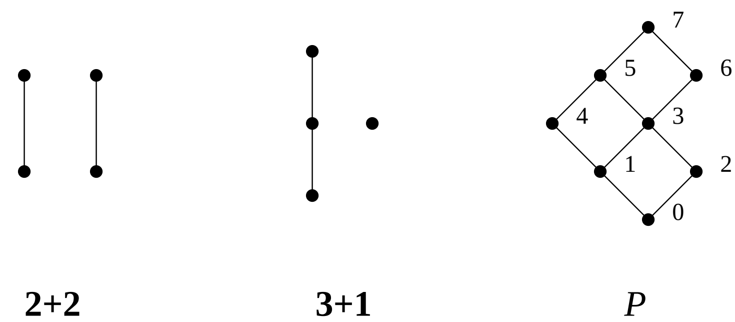


FIGURE 4.1

This conjecture has been tested on the symmetric groups up to S_{10}.

Since the stc-set and the descent set are equidistributed on S_n, perhaps it would be possible to use the statistic stc to modify the proof of Theorem 1.2 or to construct a poset analogous to that in the proof of Proposition 2.7 for a larger class of posets.

Question 3.15. Can we create a poset using major index tables, in which each k-element chain corresponds to a major index table m with $st(m) = k$?

Another strategy for constructing such a complex is to look for new interpretations of Mahonian statistics as we have for Eulerian statistics. The relationship of linear extensions to INV depends on the fact that an inversion of a permutation is a *pair* of numbers. The *inversion set* of a permutation, a subset of $\binom{[n]}{2}$, has no obvious analog for the other Mahonian statistics we have discussed.

Question 3.16. Is there a way to associate a MAJ-set (or X-set or DEN-set) to a permutation which is analogous to the inversion set?

If these sets could be defined easily, they might help associate a set $\mathcal{K}(P)$ of permutations to a poset P in the manner we have described at the end of Section 2.7.

4. Posets Whose chain polynomials have only real zeros

4.1. **Interval characterizations.** For nonnegative integers a and b, we denote by $\mathbf{a}+\mathbf{b}$ the poset which is the disjoint sum of an a-element chain and a b-element chain. A poset is called $(\mathbf{a}+\mathbf{b})$-*free* if it contains no induced subposet isomorphic to $\mathbf{a}+\mathbf{b}$. (See [29, ch. 3] for basic definitions.) For example, the first two posets in Figure 4.1 are $\mathbf{2}+\mathbf{2}$ and $\mathbf{3}+\mathbf{1}$. The third poset P is $(\mathbf{2}+\mathbf{2})$-free but not $(\mathbf{3}+\mathbf{1})$-free, because the subposet induced by the elements $\{2,3,4,6\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$.

Fishburn [15] characterized $(\mathbf{2}+\mathbf{2})$-free posets by showing that any such poset P may be represented as a set of closed intervals of real numbers $[c_i, d_i]$, ordered by

$$[c_i, d_i] <_P [c_j, d_j], \text{ if } d_i < c_j.$$

He also showed that for posets free of *both* $\mathbf{3}+\mathbf{1}$ and $\mathbf{2}+\mathbf{2}$, the intervals may be chosen to have unit length. In honor of these results, posets in the above classes are often called *interval orders* and *unit interval orders*.

Unfortunately, little is known about posets free *only* of $\mathbf{3}+\mathbf{1}$. Characterization of $(\mathbf{3}+\mathbf{1})$-free posets is desirable, because several results and conjectures about posets require avoidance only of $\mathbf{3}+\mathbf{1}$. For instance, Stanley's generalization of the chromatic polynomial [30] is known to be s-positive for the incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets [21], and is conjectured to be e-positive for these graphs as well [30, 33, 35]. Further, the chain polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros. (See [33] and Corollary 4.15).

4.2. **Antiadjacency matrices of unit interval orders.** A well known alternative characterization of unit interval orders involves natural labellings and antiadjacency matrices. (See [**?**].) Let P be a labelled poset on n elements. Recall that any bijective function $\phi : P \rightarrow [n]$ is called a *labelling* of P, and is called *natural* if it satisfies $\phi(x) < \phi(y)$ whenever $x <_P y$. If the elements x and y are labelled as i and j, we will often write $i <_P j$ to mean $x <_P y$. For any labelled poset P, we define the *antiadjacency matrix* $A = [a_{ij}]$ of P by

$$a_{ij} = \begin{cases} 0, & \text{if } i <_P j, \\ 1, & \text{otherwise.} \end{cases}$$

Note that the entries on the diagonal of A are always ones, and the entries below the diagonal are all ones if and only if P is naturally labelled.

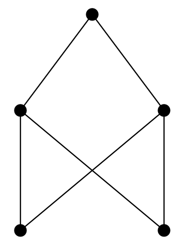
Theorem 4.1. *Let P be a poset on n elements. P is a unit interval order if and only if it may be naturally labelled so that the corresponding antiadjacency matrix A satisfies*

$$a_{jk} \geq a_{i\ell}, \ \textit{for } 1 \leq i \leq j \leq n \textit{ and } 1 \leq k \leq \ell \leq n. \tag{4.1}$$

Since A is a 0-1 matrix, (4.1) implies that the entries of A which are zero form a Ferrers shape in the upper right corner of A. From this fact, it is easy to see that unit interval orders are counted by Catalan numbers.

Example 4.1. Corresponding to any natural labelling of the poset P in Figure 4.2 is the antiadjacency matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$



P

FIGURE 4.2

To prove that a unit interval order may be labelled as in the theorem, we will begin with an arbitrary labelling of P and will *relabel* P to satisfy the theorem. This is equivalent to performing simultaneous row and column permutation on the antiadjacency matrix A which corresponds to the arbitrary labelling. To verify that a poset labelling satisfies the theorem, we will find it convenient to compare matrix columns and rows with vector inequalities. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be any two vectors in $(R)^n$. We define the inequalities $a \leq b$ and $a \lneqq b$ by

$$a \leq b \text{ if } a_i \leq b_i \text{ for } i = 1, \dots, n.$$
$$a \lneqq b \text{ if } a \leq b \text{ and } a_i < b_i \text{ for some } i.$$

Let us denote the ith row and ith column of any matrix M by M_{i*} and M_{*i}.

Observation 4.2. *The following two conditions on any real matrix M are equivalent.*

1. *It is possible to simlutaneously permute the columns and rows of M to obtain a matrix $M' = [m'_{ij}]$ such that the entries of M' weakly increase to the left in rows and downward in columns, i.e.*

$$m'_{jk} \geq m'_{i\ell}, \text{ for } 1 \leq i \leq j \leq n \text{ and } 1 \leq k \leq \ell \leq n.$$

2. *The rows and columns of M corresponding to any pair of indices i and j satisfy one of the following pairs of vector inequalities.*
   (a) $M_{i*} \geq M_{j*}$ *and* $M_{*i} \leq M_{*j}$.
   (b) $M_{i*} \leq M_{j*}$ *and* $M_{*i} \geq M_{*j}$.

The first statement simply says that we may sort the columns of M in weakly decreasing order while simultaneously sorting the rows in weakly increasing order. With a moment's thought, we see that this is possible if and only if the conditions in the second statement are true.

We prove Theorem 4.1 with two propositions. Since the condition (4.1) fails to hold for nonnatural poset labellings, we will consider arbitrary labellings.

Proposition 4.3. *Let P be a labelled poset with antiadjacency matrix A. P is $(\mathbf{2}+\mathbf{2})$-free if and only if the columns of A are pairwise comparable as vectors.*

Proof. Suppose that two columns of A are incomparable as vectors. That is,

$$A_{*i} \not\geq A_{*j} \text{ and } A_{*i} \not\leq A_{*j},$$

for some indices $i \neq j$. Then for some indices $k \neq \ell$ we have

$$1 = a_{ik} > a_{i\ell} = 0 \text{ and } 0 = a_{jk} < a_{j\ell} = 1. \tag{4.2}$$

This implies that $i <_P k$, $j <_P \ell$, $i \not<_P \ell$ and $j \not<_P k$. The four elements must be distinct, and they induce a subposet of P which is isomorphic to $\mathbf{2}+\mathbf{2}$.

Conversely, if four elements i, j, k, and ℓ of P are related as above, then any labelling of P induces an antiadjacency matrix whose entries satisfy (4.2). The columns of this matrix cannot be pairwise comparable. □

Proposition 4.4. *Let P be a labelled $(\mathbf{2}+\mathbf{2})$-free poset with antiadjacency matrix A. P is $(\mathbf{3}+\mathbf{1})$-free if and only if the columns and rows of A satisfy one of the following pairs of vector inequalities.*

1. $A_{i*} \geq A_{j*}$ *and* $A_{*i} \leq A_{*j}$.
2. $A_{i*} \leq A_{j*}$ *and* $A_{*i} \geq A_{*j}$.

Proof. Suppose that have an incorrect pair of comparisons of the form

$$A_{i*} \gneqq A_{j*} \text{ and } A_{*i} \gneqq A_{*j}.$$

That is, P contains elements k and ℓ satisfying

$$1 = a_{i\ell} > a_{j\ell} = 0, \tag{4.3}$$

$$1 = a_{ki} > a_{kj} = 0. \tag{4.4}$$

Thus, we have $k <_P j <_P \ell$, and i must be incomparable to all three of these elements. The subposet induced by these elements is isomorphic to $\mathbf{3}+\mathbf{1}$.

Conversely, if three elements i, j, k, and ℓ are related in P as immediately above, then then any labelling of P induces an antiadjacency matrix whose entries satisfy (4.3) and (4.4). The columns of this matrix do not satisfy our hypotheses. □

An alternative statement of the condition (4.1) relates unit interval orders to totally positive matrices. A real matrix, finite or infinite, is called *totally positive* (or sometimes *totally nonnegative*) if each $k \times k$ minor is nonnegative. Totally positive matrices have many interesting properties and arise frequently in combinatorics. (See [1], [2], [9], [18], [19], [34], [35], [37].) One important property of a finite square totally positive matrix is that all of its eigenvalues are nonnegative and real. (See [2, Thm 1.1]).

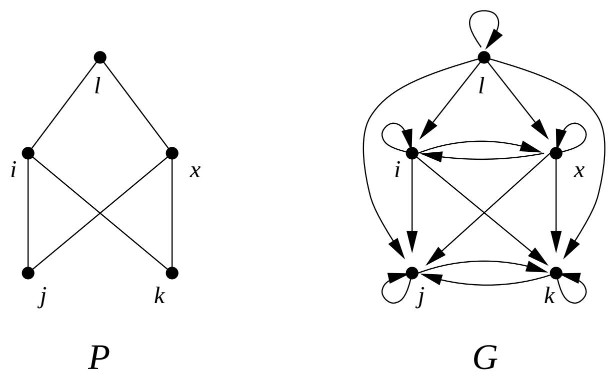


FIGURE 4.3

An infinite matrix which is well known to be totally positive is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 0 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \end{bmatrix}. \tag{4.5}$$

(See [1].) It is easy to see that the matrices satisfying the conditions of Theorem **??** are finite generalized submatrices of C. That is, each such matrix is determined by a finite multiset of columns and a finite multiset of rows of C. The matrix A in Example **??** is the generalized submatrix of C corresponding to rows $(2, 2, 4, 4, 5)$ and columns $(1, 1, 3, 3, 5)$. Since any generalized submatrix of a totally positive matrix is itself totally positive, we see that any unit interval order has a labelling for which the corresponding antiadjacency matrix A is totally positive. It follows that the antiadjacency matrix corresponding to *any* labelling of a unit interval order has only positive real eigenvalues.

4.3. **The squared antiadjacency matrix.** If A is the antiadjacency matrix of a finite labelled poset P on n elements, then the *squared antiadjacency matrix* $B = A^2$ has a simple combinatorial interpretation. Let $G = (P, E)$ be the graph whose *adjacency* matrix is A. The vertex set of G consists of the elements $\{1, \ldots, n\}$ of P, and the edge set consists of the ordered pairs $(i, j) \in P \times P$ such that $i \not<_P j$. (See Figure 4.3.) Clearly, $B = [b_{ij}]$ counts paths of length two in G. That is, b_{ij} is the number of ordered triples (i, x, j), where (i, x) and (x, j) belong to E.

Observation 4.5. *Assume that P is $(\mathbf{3}+\mathbf{1})$-free, and let i, j, k, and ℓ be elements of P.*

1. *If $i <_P j <_P k$, then $b_{ik} = 0$.*

2. *If* $b_{ik} > b_{i\ell}$, *then there is an element* $x <_P \ell$, *such that* $i \not<_P x \not<_P k$
3. *If* $b_{ik} > b_{jk}$, *then there is an element* $y >_P j$, *such that* $k \not>_P y \not>_P i$.

Proof. (1) Assume $b_{ik} > 0$. Then, for some element x of P, (i,x) and (x,k) belong to E, implying that $i \not<_P x \not<_P k$. In fact, x must be incomparable to i and k, for if $x <_P i$, then $x <_P i <_P j <_P k$, and if $k <_P x$, then $i <_P j <_P k <_P x$, both impossible. Similarly, x cannot be comparable to j. Thus, the subposet of P induced by $\{i,j,k,x\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$, contradicting our assumption that P is $(\mathbf{3}+\mathbf{1})$-free.

(2) If $b_{ik} > b_{i\ell}$, then there are more paths of length two in G from i to k than from i to ℓ. It follows that for some element x of P, the pairs (i,x) and (x,k) belong to E, and the pair (x,ℓ) does not.

(3) Apply the argument of (2) to the dual poset of P. □

Elements such as x in Observation 4.5 (2) are central to the proof of Lemma 4.6. To simplify notation, we introduce the following definition.

Definition 4.2. Let i, k, ℓ, and x be elements of P. Call x a (k,ℓ)-*advantage for* i if (i,x) and (x,k) are edges in the graph G and (x,ℓ) is not.

We use the word *advantage*, imagining that x helps us to travel from i to k, but not from i to ℓ. In the language of partially ordered sets, x is a (k,ℓ)-advantage for i if $x <_P \ell$, and $i \not<_P x \not<_P k$. Note that in Figure 4.3, the vertex x is a (k,ℓ)-advantage for i, although it is not a (k,ℓ)-advantage for j.

Lemma 4.6. *Assume that* P *is* $(\mathbf{3}+\mathbf{1})$*-free, and let* i, j, k, *and* ℓ *be elements of* P. *If* $b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell}$, *then one of the following is true:*

1. P *contains an element* x *such that* $j <_P x <_P \ell$ *and* $b_{j\ell} = 0$.
2. P *contains an element* y *such that* $i <_P y <_P k$ *and* $b_{ik} = 0$.

Proof. Let us denote by $\alpha(k,\ell,i)$ the number of elements of P which are (k,ℓ)-advantages for i. Then,

$$b_{ik} - b_{i\ell} = \alpha(k,\ell,i) - \alpha(\ell,k,i).$$

Assuming that $b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell}$, we have

$$\alpha(k,\ell,i) + \alpha(\ell,k,j) > \alpha(k,\ell,j) + \alpha(\ell,k,i),$$

and at least one of the following two inequalities must be true.

$$\alpha(k,\ell,i) > \alpha(k,\ell,j),$$
$$\alpha(\ell,k,j) > \alpha(\ell,k,i).$$

Suppose that $\alpha(k,\ell,i) > \alpha(k,\ell,j)$. Then P contains an element x which is a (k,ℓ)-advantage for i and not a (k,ℓ)-advantage for j. By Definition 4.2, the pairs (i,x)

and (x, k) belong to E and the pairs (x, ℓ) and (j, x) do not. Thus, $j <_P x <_P \ell$ and by Observation 4.5 (1), $b_{j\ell} = 0$. Similarly, if $\alpha(\ell, k, j) > \alpha(\ell, k, i)$, then P contains an element y such that the pairs (y, k) and (i, y) do not belong to E. Thus, $i <_P y <_P k$ and $b_{ik} = 0$.

□

4.4. **A Characterization of $(\mathbf{3}+\mathbf{1})$-free Posets.** To characterize $(\mathbf{3}+\mathbf{1})$-free posets in a manner analogous to Theorem 4.1, we will use squared antiadjacency matrices.

Theorem 4.7. *Let P be a poset on n elements. P is $(\mathbf{3}+\mathbf{1})$-free if and only if it may be naturally labelled so that the squared antiadjacency matrix B satisfies the following two conditions for all integers $1 \leq i \leq j \leq n$ and $1 \leq k \leq \ell \leq n$.*

1. $b_{jk} \geq b_{i\ell}$
2. *If $b_{ik} - b_{i\ell} \neq b_{jk} - b_{j\ell}$, then $b_{i\ell} = 0$ and $b_{ik} < b_{jk} - b_{j\ell}$.*

Example 4.3. Corresponding to any natural labelling of the poset P in Figure 4.3 is the squared antiadjacency matrix

$$B = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 4 & 4 & 2 & 2 & 0 \\ 4 & 4 & 2 & 2 & 0 \\ 5 & 5 & 3 & 3 & 1 \end{bmatrix}.$$

Condition (1) of the theorem says that entries of the squared antiadjacency matrix increase to the left in rows and downward in columns. Following the proof of Theorem 4.1, we will prove that a $(\mathbf{3}+\mathbf{1})$-free poset P may be labelled to satisfy this condition by choosing an arbitrary labelling of P an by performing a simultaneous row and column permutation of the corresponding squared antiadjacency matrix B.

Proposition 4.8. *Any $(\mathbf{3}+\mathbf{1})$-free poset may be naturally labelled so that the entries of its squared antiadjacency matrix weakly increase to the left in rows and downward in columns.*

Proof. Let P be a $(\mathbf{3}+\mathbf{1})$-free poset and assume that for each labelling of P, the corresponding antiadjacency matrix fails to satisfy condition (2) of Observation 4.2. (Trivially, condition (2) fails to hold for each non-natural labelling of P.) We consider two cases for a fixed labelling of P and the corresponding squared antiadjacency matrix B.

Case 1: Two columns of B are incomparable as vectors. That is,

$$B_{*i} \not\geq B_{*j} \text{ and } B_{*i} \not\leq B_{*j},$$

for some indices $i \neq j$. Then for some indices $k \neq \ell$ we have $b_{ik} > b_{i\ell}$ and $b_{jk} < b_{j\ell}$, implying that

$$b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell}.$$

Applying Lemma 4.6 to this inequality, we have $b_{j\ell} = 0$ or $b_{ik} = 0$, both contradictions. The argument for incomparable rows is identical.

Case 2: All columns of B are pairwise comparable as vectors, as are all rows, but for some indices $i \neq j$, we have an incorrect pair of comparisons of the form

$$B_{i*} \gneqq B_{j*} \text{ and } B_{*i} \gneqq B_{*j}.$$

That is, there are elements k and ℓ in P, not necessarily distinct, satisfying $b_{i\ell} > b_{j\ell}$ and $b_{ki} > b_{kj}$.

By Observation 4.5 (3), P contains an element $x <_P j$, such that $k \not<_P x \not<_P i$. By Observation 4.5 (2), P contains an element $y >_P j$, such that $i \not<_P y \not<_P \ell$. Thus, $x <_P j <_P y$ is a chain, and each of these three elements is incomparable to i. This contradicts our assumption that P is $(\mathbf{3}+\mathbf{1})$-free. $\square$

It happens that any labelling of a $(\mathbf{3}+\mathbf{1})$-free poset which satisfies the first condition of Theorem 4.7 also satisfies the second condition.

Proposition 4.9. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset, naturally labelled so that its squared antiadjacency matrix B weakly increases to the left in rows and downward in columns. Let $i, j, k,$ and ℓ be numbers satisfying $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. Then the 2×2 submatrix*

$$\begin{bmatrix} b_{ik} & b_{i\ell} \\ b_{jk} & b_{j\ell} \end{bmatrix}$$

satisfies one of the following two conditions:

1. $b_{ik} - b_{i\ell} = b_{jk} - b_{j\ell}$.
2. $b_{i\ell} = 0$ *and* $b_{ik} < b_{jk} - b_{j\ell}$.

Proof. Suppose that condition (1) is not satisfied.

Case 1: $(b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell})$. We apply Lemma 4.6 to this inequality. If $b_{j\ell} = 0$, then $b_{i\ell} = 0$ and $b_{ik} > b_{jk}$, contraditing our assumptions about weakly increasing entries of B. If instead $b_{ik} = 0$, then $b_{i\ell} = 0$ and $b_{j\ell} > b_{jk}$, another contradiction.

Case 2: $(b_{ik} - b_{i\ell} < b_{jk} - b_{j\ell})$. Again we apply Lemma 4.6. If $b_{jk} = 0$, then all four numbers are zero, a contradiction. We conclude that $b_{i\ell} = 0$ and that condition (2) is satisfied. $\square$

Finally, we show that the only posets satisfying the conditions of Theorem 4.7 are those which are $(\mathbf{3}+\mathbf{1})$-free.

Proposition 4.10. *Let P be a labelled poset containing $\mathbf{3}+\mathbf{1}$ as an induced subposet, and let B be its squared antiadjacency matrix. Then there are two distinct elements i and k such that*

$$b_{ik} \neq 0 \text{ and } b_{ii} - b_{ik} \neq b_{ki} - b_{kk}.$$

Proof. Let $1, 2, 3$, and 4 be four elements of P such that $1 <_P 2 <_P 3$ is a chain, and 4 is incomparable to $1, 2$, and 3. Let $G = (P, E)$ be the graph defined in Section 4.3.

Clearly, $b_{13} \neq 0$, since $(1,4)$ and $(4,3)$ are edges in G. We claim that

$$b_{11} - b_{13} \neq b_{31} - b_{33}.$$

Define the sets

$$\begin{aligned} X &= \{x \in P | (1,x) \in E, (x,1) \in E, (x,3) \notin E\}, \\ Y &= \{x \in P | (x,1) \in E, (x,3) \notin E\}, \end{aligned}$$

and note that

$$\begin{aligned} |X| &= b_{11} - b_{13}, \\ |Y| &= b_{31} - b_{33}. \end{aligned}$$

Certainly X is a subset of Y. Moreover, it is a proper subset, since the element 2 belongs to Y and not to X. Thus, $b_{11} - b_{13} < b_{31} - b_{33}$.

□

As we have considered Theorem 4.1 in terms of totally positive matrices, we now consider Theorem 4.7 in terms of totally positive matrices. The infinite matrix D,

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 2 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 3 & 2 & 1 & 0 & \cdot & \cdot & \cdot \\ 4 & 3 & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \end{bmatrix},$$

is in some sense the square of the infinite matrix C from (4.5). It too is totally positive. It is easy to see that the matrices satisfying the conditions of Theorem 4.7 are generalized submatrices of D: each is determined by a finite multiset of columns and a finite multiset of rows. The matrix B in Example 4.3 is the generalized submatrix of D corresponding to rows $(2, 2, 4, 4, 5)$ and columns $(1, 1, 3, 3, 5)$. Since any generalized submatrix of a totally positive matrix is itself totally positive, we see that any $(\mathbf{3}+\mathbf{1})$-free poset has a labelling for which the corresponding squared antiadjacency matrix B is totally positive. It follows that the squared antiadjacency matrix corresponding to *any* labelling of a $(\mathbf{3}+\mathbf{1})$-free poset has only positive real eigenvalues, and that the antiadjacency matrix has only real eigenvalues.

Just as Theorem 4.1 implies that the Catalan numbers count unit interval orders, it is possible that close inspection of the observation above will yield an analogous result for $(\mathbf{3}+\mathbf{1})$-free posets. No such formula is currently known.

Question 4.4. Does Theorem 4.7 imply a simple formula for the number of $(\mathbf{3}+\mathbf{1})$-free posets on n elements?

4.5. **The altitude sequence.** For each $(\mathbf{3}+\mathbf{1})$-free poset P, it is simple to construct a labelling ϕ which satisfies the conditions of Theorem 4.7, or of Theorem 4.1 if P is a unit interval order. We do so using the principal order ideals and dual principal order ideals of the elements of P.

For any element x in a poset P, we will denote the corresponding principal order ideal and principal dual order ideal by Λ_x and V_x, respectively.

$$\Lambda_x = \{y \in P | y <_P x\}, \tag{4.6}$$

$$V_x = \{y \in P | y >_P x\}. \tag{4.7}$$

It is easy to verify that if P is $(\mathbf{3}+\mathbf{1})$-free, then its order ideals satisfy the following two properties.

1. If $|\Lambda_x| > |\Lambda_y|$, then $|V_x| \leq |V_y|$.
2. If $|V_x| > |V_y|$, then $|\Lambda_x| \leq |\Lambda_y|$.

For each element x of P, we will define its *altitude* to be the difference in cardinality between its principal order ideal and principal dual order ideal.

$$\alpha(x) = |\Lambda_x| - |V_x|.$$

If the element x is labelled as $\phi(x) = i$, we will write Λ_i, V_i, and $\alpha(i)$ in place of Λ_x, V_x, and $\alpha(x)$.

Example 4.5. Let P be the poset in Figure 4.3. Then we have

$$\begin{aligned} \alpha(j) = \alpha(k) &= -3, \\ \alpha(i) = \alpha(x) &= 1, \\ \alpha(\ell) &= 4. \end{aligned}$$

We will say that a labelling ϕ of P *respects altitude* if it satisfies $\phi(x) < \phi(y)$ whenever $\alpha(x) <_P \alpha(y)$. A labelling which respects altitude is necessarily natural.

By the following proposition, the labellings of unit interval orders which satisfy Theorem 4.1 are precisely those which respect altitude.

Proposition 4.11. *Let P be a unit interval order labelled by ϕ and let A be the corresponding antiadjacency matrix. The matrix A satisfies the conditions of Theorem 4.1 if and only if the labelling ϕ respects altitude.*

Proof. Suppose that P has been labelled as in Theorem 4.1, and let x and y be two elements satisfying $\phi(x) < \phi(y)$. Then we have

$$A_{x*} \leq A_{y*} \text{ and } A_{*x} \geq A_{*y}. \tag{4.8}$$

Equivalently, we have

$$|V_x| \geq |V_y| \text{ and } |\Lambda_x| \leq |\Lambda_y|. \tag{4.9}$$

Combining these inequalities, we obtain $\alpha(x) \leq \alpha(y)$.

Now suppose that ϕ is a natural labelling of P such that the corresponding antiadjacency matrix A does not satisfy (4.1). Then for some labels $i < j$ we have

$$A_{i*} \gneqq A_{j*} \text{ or } A_{*i} \lneqq A_{*j}.$$

Equivalently, we have

$$|V_i| < |V_j| \text{ or } |\Lambda_i| > |\Lambda_j|.$$

If $|V_i| < |V_j|$ then we must have $|\Lambda_i| \geq |\Lambda_j|$. Therefore,

$$\alpha(i) - \alpha(j) = |\Lambda_i| - |V_i| - |\Lambda_j| + |V_j| > 0,$$

and ϕ does not respect altitude. Similarly, if $|\Lambda_i| > |\Lambda_j|$ then we must have $|V_i| \leq |V_j|$. Again, ϕ does not respect altitude. $\square$

Similarly, the labellings of $(\mathbf{3}+\mathbf{1})$-free posets which satisfy Theorem 4.1 are precisely those which respect altitude.

Proposition 4.12. *Let P be $(\mathbf{3}+\mathbf{1})$-free poset, labelled by ϕ and let B be the corresponding squared antiadjacency matrix. The matrix B satisfies the conditions of Theorem 4.7 if and only if the labelling ϕ respects altitude.*

Proof. Suppose that P has been labelled as in Theorem 4.7, and let x and y be two elements satisfying $\phi(x) < \phi(y)$. Then we have

$$B_{x*} \leq B_{y*} \text{ and } B_{*x} \geq B_{*y}. \tag{4.10}$$

In particular, if w is any minimal element of P and z is a maximal element, then we have

$$b_{xw} \leq b_{yw} \text{ and } b_{zx} \geq b_{zy}.$$

This implies that

$$|V_x| \geq |V_y| \text{ and } |\Lambda_x| \leq |\Lambda_y|. \tag{4.11}$$

Combining these inequalities, we obtain $\alpha(x) \leq \alpha(y)$.

Now suppose that ϕ is a natural labelling of P such that the corresponding squared antiadjacency matrix B does not satisfy the conditions of Theorem 4.7. Then for some labels $i < j$ we have

$$B_{i*} \gneqq B_{j*} \text{ or } B_{*i} \lneqq B_{*j}.$$

If $B_{*i} \lneqq B_{*j}$, then for some element z of P, we have $b_{zi} < b_{zj}$. By Theorem 4.7 (2), we may assume that z is maximal in P. Thus, we have $|\Lambda_i| > |\Lambda_j|$. It follows that $|V_i| \leq |V_j|$. Therefore,

$$\alpha(i) - \alpha(j) = |\Lambda_i| - |V_i| - |\Lambda_j| + |V_j| > 0,$$

and ϕ does not respect altitude. Similarly, if $B_{i*} \gneqq B_{j*}$, then $|V_i| < |V_j|$ and $|\Lambda_i| \geq |\Lambda_j|$. Again, ϕ does not respect altitude. □

Thus, to demonstrate that a poset is a unit interval order (is $(\mathbf{3}+\mathbf{1})$-free), it suffices to examine the (squared) antiadjacency matrix corresponding to any altitude respecting labelling of P.

Let us define the *altitude sequence* of any poset P to be the sequence of n altitudes, written in weakly increasing order. Is not difficult to show that a unit interval order is uniquely identified by its altitude sequence, although this fact does not hold for posets in general. Furthermore, we conjecture the analogous stronger result for $(\mathbf{3}+\mathbf{1})$-free posets.

Conjecture 4.13. *Two $(\mathbf{3}+\mathbf{1})$-free posets P and Q have the same altitude sequence if and only if they are isomorphic.*

If this conjecture is true, one might use it to count $(\mathbf{3}+\mathbf{1})$-free posets.

Question 4.6. Is there a simple characterization of the altitude sequences arising from $(\mathbf{3}+\mathbf{1})$-free posets on n elements, or a simple formula counting such sequences?

Another conjecture relates the altitude sequences of unit interval orders to pairs of permutations. The following statement of the conjecture, due to Postnikov [26], differs somewhat from its original form due to Kostant [**?**].

Conjecture 4.14. *Let P be a unit interval order on n elements. Then there are two permutations $\pi = \pi_1 \cdots \pi_n$ and $\sigma = \sigma_1 \cdots \sigma_n$ in S_n such that the vector $(\pi_1 - \sigma_1, \ldots, \pi_n - \sigma_n)$ is precisely the altitude sequence of P.*

This conjecture is trivially true for the class of posets of dimension 2 (see [**?**]), and thus holds for all series-parallel posets and many unit interval orders. The analogous statement involving $(\mathbf{3}+\mathbf{1})$-free posets is not true.

4.6. **$(\mathbf{3}+\mathbf{1})$-free posets and distributive lattices.** While $(\mathbf{3}+\mathbf{1})$-free posets and distributive lattices have little in common by definition, the following result suggests a connection between the two classes. By a result of Stanley, we may use the antiadjacency matrix of a poset to determine whether or not the chain polynomial has only real zeros [31]. Letting P be an arbitrarily labelled poset with antiadjacency matrix A, we have

$$f_P(x) = \det(I + xA). \tag{4.12}$$

From this formula we see that $f_P(x)$ has only real zeros if and only if A has only real eigenvalues. Applying the discussion following Proposition 4.10 to Theorem 4.7, we obtain the following corollary. (This result was originally proved in [33, Cor. 2.9], using facts about symmetric functions [21] [33, Thm. 2.8].)

Corollary 4.15. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset. Then the chain polynomial $f_P(x)$ has only real zeros.*

Recall that Stanley and Neggers have conjectured (Conjecture 1.3) the similar statement that the chain polynomial of any distrubutive lattice has only real zeros. Unfortunately, it is not in general possible to label a distributive lattice so that some power of its antiadjacency matrix is totally positive. However, if this were the case for some subclass of distributive lattices, we would have a combinatorial proof that the zeros of the corresponding chain polynomials were roots of unity.

Question 4.7. Let A be the antiadjacency matrix of a labelled distributive lattice $J(P)$. Under what conditions is there a power $k > 0$ for which A^k is totally positive?

Noting that two posets which are not isomorphic can have the same f-vector, we arrive at another possiblility of applying Corollary 4.15 to Conjecture 1.3.

Question 4.8. For which distributive lattices $J(P)$ is there a $(\mathbf{3}+\mathbf{1})$-free poset Q such that $f_{J(P)}(x) = f_Q(x)$ or $h_{J(P)}(x) = f_Q(x)$?

There is some evidence in favor of an affirmative answer to the second equality in Question 4.8, and therefore in favor of Conjecture 2.13. Namely, for all posets P having seven or fewer elements, there is at least one poset Q satisfying

$$f_Q = h_{J(P)}. \tag{4.13}$$

This remains true if we require Q to be a $(\mathbf{3}+\mathbf{1})$-poset, or even a unit interval order. In fact, for a given distributive lattice $J(P)$, the number of unit interval orders Q satisfying (4.13) is often quite large. For each h-vector $h = (h_0, h_1, \ldots, h_n)$, a poset Q which satisfies $f_Q = h$ must have h_1 elements. The following table shows the number of such unit interval orders which satisfy (4.13), averaged over all distinct h-vectors $h_{J(P)}$ for $|P| \leq 7$ with a given value of h_1.

i	average number of posets $Q \in \mathcal{Q}$ satisfying $f_Q = h_{J(P)}$ when $h_1 = i$
1	1
2	1
3	1.33
4	2
5	2.9
6	3.53
7	6
8	8
9	13.43
10	18.51
11	25.97
12	40.36
13	58.53
14	98.53
15	111.97
16	237.72
17	263.30
18	507.75
19	493.10
20	1088.03

Note that the average number of matches is a lower bound for the average number of matches with $(\mathbf{3}+\mathbf{1})$-posets, and a very weak lower bound for the average number of matches with abritrary posets Q. Interestingly, there is no equally strong evidence supporting the first equality in Question 4.8. Many distributive lattices $J(P)$ exist for which $f_{J(P)}$ is not the f-vector of a unit interval order, although $f_{J(P)}(x)$ does have only real roots.

This raises some questions about polynomials with real roots in general. Let $\mathcal{F}$ be the class of polynomials $f(x)$ in $\mathbb{P}[x]$ which have only real zeros and satisfy $f(0) = 1$.

$$\mathcal{F} = \{f(x) = 1 + a_0 x + \cdots a^{n-1} x^n \in \mathbb{P}[x] \,|\, \text{ all zeros of } f(x) \text{ are real}\}.$$

Let $\mathcal{F}_1$ be the subset of $\mathcal{F}$ for which $(1, a_0, \ldots, a_{n-1})$ is the f-vector of some poset P. Let $\mathcal{F}_2$ be the subset of $\mathcal{F}_1$ for which $(1, a_0, \ldots, a_{n-1})$ is the f-vector of some $(\mathbf{3}+\mathbf{1})$-poset P.

Question 4.9. Which, if any, of the inclusions $\mathcal{F}_2 \subset \mathcal{F}_1 \subset \mathcal{F}$ are strict?

Note that if we had the equality $\mathcal{F}_2 = \mathcal{F}_1$, then this would imply a partial converse to Corollary 4.15.

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