INEQUALITIES FOR SYMMETRIC MEANS

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Outline

(1) Symmetric means
(2) Classical inequalities
(3) New inequalities
(4) Work in progress

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Symmetric Means

Theorem: (Maclaurin) For all $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0}$,

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt{\frac{a_1 a_2 + \cdots + a_{n-1} a_n}{\binom{n}{2}}} \geq \cdots \geq \sqrt[n]{a_1 \cdots a_n}.$$

Notation

$$\mathbf{x} = (x_1, \ldots, x_n) = \text{ vector of } n \text{ variables},$$

$$g(\mathbf{x}) = \text{ homogeneous degree } d \text{ symmetric function},$$

$$G(\mathbf{x}) = G(\mathbf{x}) = \frac{g(\mathbf{x})}{g(1, \ldots, 1)} = \text{ term normalization of } g(\mathbf{x}),$$

$$\mathfrak{G}(\mathbf{x}) = \sqrt[d]{G(\mathbf{x})} = \text{ symmetric mean corresponding to } g(\mathbf{x}).$$

Classical Inequalities

Write $\mathfrak{F}(\mathbf{x}) \leq \mathfrak{G}(\mathbf{x})$ if $\mathfrak{F}(\mathbf{a}) \leq \mathfrak{G}(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{R}^n_{\geq 0}$.

Write $\mathfrak{F} \leq \mathfrak{G}$ if $\mathfrak{F}(\mathbf{x}) \leq \mathfrak{G}(\mathbf{x})$ for all $n \geq 1$.

AM-GM inequality: $\mathfrak{E}_n(\mathbf{x}) \leq \mathfrak{E}_1(\mathbf{x})$.

Maclaurin's inequalities: $\mathfrak{E}_1 \geq \mathfrak{E}_2 \geq \mathfrak{E}_3 \geq \cdots$.

Newton's inequalities: For $k \in \mathbb{N}$, $\quad \mathfrak{E}_{k,k} \geq \mathfrak{E}_{k+1,k-1}$.
(Usually written $E_{k,k} \geq E_{k+1,k-1}$).

More Classical Inequalities

Muirhead: For $|\lambda| = |\mu|$, $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$ iff $\lambda \preceq \mu$.

Schlömilch: For $i, j \in \mathbb{R}$, $\mathfrak{P}_i \leq \mathfrak{P}_j$ iff $i \leq j$.

Gantmacher: For $k \in \mathbb{N}$, $\mathfrak{P}_{k,k} \leq \mathfrak{P}_{k+1,k-1}$.

Popoviciu: For $i, j \in \mathbb{N}$, $\mathfrak{H}_i \leq \mathfrak{H}_j$ iff $i \leq j$.

Schur: For $k \in \mathbb{N}$, $\mathfrak{H}_{k,k} \leq \mathfrak{H}_{k+1,k-1}$.

Generalization of Newton, Maclaurin Inequalities

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash r$,

$$\mathfrak{E}_\lambda(\mathbf{x}) = \sqrt[r]{\frac{e_\lambda(\mathbf{x})}{\binom{n}{\lambda_1} \cdots \binom{n}{\lambda_\ell}}}.$$

Theorem: (C-G-S '06) $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$ iff $\frac{\lambda^\top}{|\lambda|} \preceq \frac{\mu^\top}{|\mu|}$.

When $|\lambda| = |\mu|$, this becomes $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$ iff $\lambda \succeq \mu$.
Thus the poset of elementary means is a union of majorization orders of integer partitions.

Example

$\mathfrak{E}_{221}$ is incomparable to $\mathfrak{E}_{3111}$:

$$221^{\top} = 32, \qquad 3111^{\top} = 411.$$

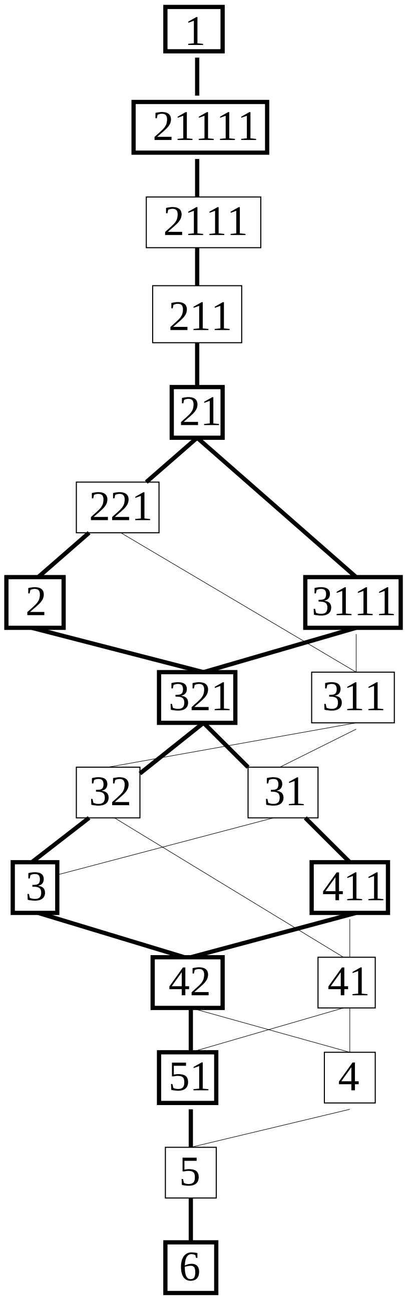
$$\frac{221^{\top}}{5} = \left(\frac{3}{5}, \frac{2}{5}\right), \qquad \frac{3111^{\top}}{6} = \left(\frac{4}{6}, \frac{1}{6}, \frac{1}{6}\right).$$

Partial sums are

$$\frac{3}{5} < \frac{4}{6}$$

$$\frac{3}{5} + \frac{2}{5} = 1 > \frac{4}{6} + \frac{1}{6} = \frac{5}{6}.$$

Poset of Elementary Means



Generalization of Gantmacher, Schlömilch Inequalities

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash r$,

$$\mathfrak{P}_\lambda(\mathbf{x}) = \sqrt[r]{\frac{p_\lambda(\mathbf{x})}{n^\ell}}.$$

Theorem: (C-G-S '06) $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$ iff $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$.

Thus the poset of power sum means is dual to the poset of elementary means,

$$\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu \text{ iff } \mathfrak{E}_\lambda \geq \mathfrak{E}_\mu.$$

Generalization of Muirhead Inequalities

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) = 1^{\alpha_1} 2^{\alpha_2} \cdots r^{\alpha_r} \vdash r$,

$$\mathfrak{M}_\lambda(\mathbf{x}) = \sqrt[r]{\frac{m_\lambda(\mathbf{x})}{\binom{n}{\alpha_1, \ldots, \alpha_r, n-\ell}}}.$$

Theorem: (C-G-S '07) For $|\lambda| \le |\mu|$,

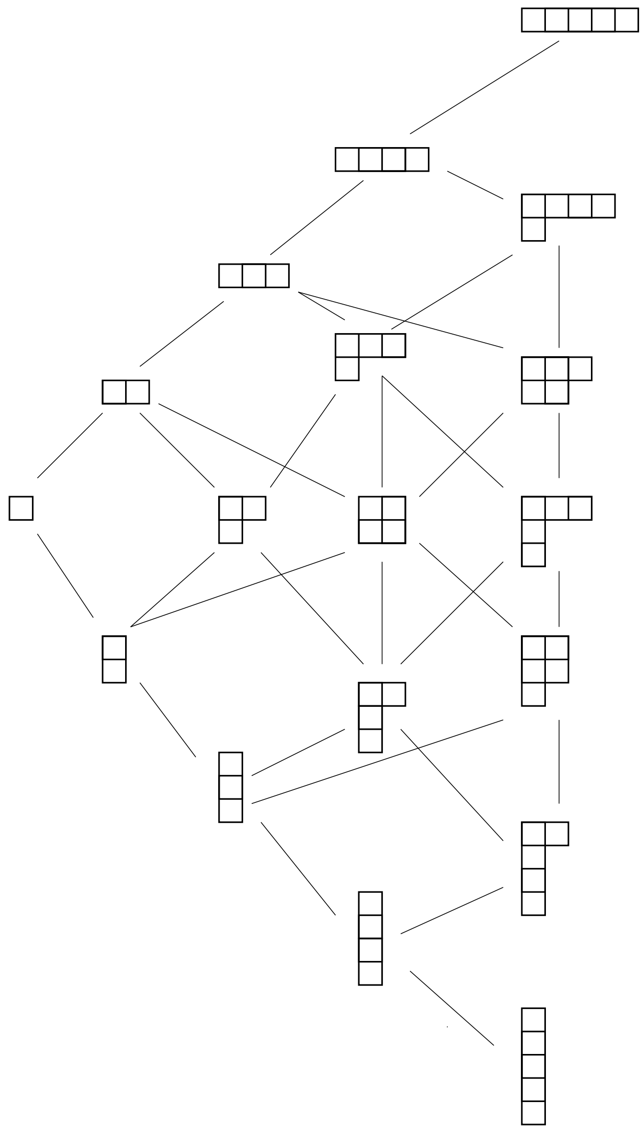
$$\mathfrak{M}_\lambda \le \mathfrak{M}_\mu \iff \frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|} \iff \mathfrak{E}_{\lambda^\top} \ge \mathfrak{E}_{\mu^\top}.$$

Conjecture: For $|\lambda| \ge |\mu|$,

$$\mathfrak{M}_\lambda \le \mathfrak{M}_\mu \iff \frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|} \iff \mathfrak{P}_\lambda \le \mathfrak{P}_\mu.$$

Double Majorization

$$\lambda \trianglelefteq \mu \text{ iff } \frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|} \text{ and } \frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}.$$



Example

31 is incomparable to 32 in double majorization:

$$\frac{31}{4} = \left(\frac{3}{4}, \frac{1}{4}\right) \succeq \frac{32}{5} = \left(\frac{3}{5}, \frac{2}{5}\right), \text{ since}$$
$$\frac{3}{4} > \frac{3}{5}, \quad \text{and} \quad \frac{3}{4} + \frac{1}{4} = 1 = \frac{3}{5} + \frac{2}{5}.$$

$$\frac{31^\top}{4} = \left(\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\right) \not\succeq \frac{32^\top}{5} = \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right), \text{ since}$$
$$\frac{2}{4} > \frac{2}{5}, \quad \text{but} \quad \frac{2}{4} + \frac{1}{4} = \frac{3}{4} < \frac{2}{5} + \frac{2}{5} = \frac{4}{5}.$$

Work in Progress

Theorem: (C-G-S '06) We have

$$\mathfrak{E}_{\lambda^\top} \leq \mathfrak{M}_\lambda \leq \mathfrak{P}_\lambda, \qquad \mathfrak{S}_\lambda \leq \mathfrak{M}_\lambda, \qquad \mathfrak{H}_\lambda \leq \mathfrak{P}_\lambda,$$

$$\mathfrak{M}_\lambda, \mathfrak{H}_\lambda \text{ incomparable}$$

Conjecture: We have $\mathfrak{E}_{\lambda^\top} \leq \mathfrak{S}_\lambda \leq \mathfrak{H}_\lambda$.

Conjecture: For $|\lambda| = |\mu|$, we have

$$\mathfrak{S}_\lambda \leq \mathfrak{S}_\mu \text{ iff } \lambda \leq \mu.$$

Problem: Show that for $|\lambda| = |\mu|$, we have

$$\mathfrak{H}_\lambda \leq \mathfrak{H}_\mu \text{ iff } \lambda \leq \mu.$$