# RESEARCH PLAN

MARK SKANDERA

In basic terms, the candidate's research in *algebraic combinatorics* concerns nonnegative integers which occur in algebraic settings. Scientific considerations of symmetry, linear systems of equations, and simple counting guarantee that nonnegative integers and algebra have applications in many disciplines. More specifically, the candidate's recent research and future plans concern character evaluations studied from four points of view. After presenting definitions and basic open problems in Section 1, we consider explicit character evaluations in Sections 2 – 3, symmetric generating functions in Section 4, coordinate ring generating functions in Section 5, and applications to total nonnegativity in Section 6.

## 1. Hecke algebras and basic open problems

Given Coxeter group $W$ with generator set $S$, define its *Hecke algebra* $H = H(W)$ to be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of $\{T_w \,|\, w \in W\}$ with multiplicative unit $T_e$ and multiplication defined by

$$T_sT_w = \begin{cases} qT_{sw} + (q-1)T_w & \text{if } sw <_W w, \\ T_{sw} & \text{if } sw >_W w, \end{cases}$$

where $s \in S$, $w \in W$, and $<_W$ is the Bruhat order on $W$. (See [4] for definitions.) This formula guarantees that for $w \in W$ and any reduced expression $s_{i_1} \cdots s_{i_\ell} \in S^\ell$ for $w$, we have $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$. Call $\ell = \ell(w)$ the *length of* $w$ and call $\{T_w \,|\, w \in W\}$ the *natural basis* of $H$. The specialization of $H$ at $q = 1$ is isomorphic to $\mathbb{Z}[W]$. Many results in Coxeter groups concern subsets of $W$. A subgroup of $W$ generated by a subset of the generators $S$ is called *parabolic*. Other subsets are defined in terms of a certain *one-line notation* of elements of $W$ and *pattern avoidance* or *signed pattern avoidance* within the one-line notation. (See [3].)

A second basis [17] of $H$ is the (modified, signless) *Kazhdan–Lusztig basis* $\{\widetilde{C}_w(q) \,|\, w \in W\}$, related to the natural basis by

$$\widetilde{C}_w(q) = \sum_{v \leq_W w} P_{v,w}(q) T_v,$$

where $\{P_{v,w}(q) \,|\, v, w \in W\} \subseteq \mathbb{Z}[q]$ are the *Kazhdan–Lusztig polynomials* whose recursive definition appears in [17]. When $W = \mathfrak{S}_n$ or $\mathfrak{B}_n$ and $w \in W$ avoids the patterns 3412 and 4231, all polynomials $\{P_{v,w}(q) \,|\, v \leq_W w\}$ are identically 1 [17, Thm. A.2], [19].

Representations of $W$ and $H$ are often studied in terms of *characters*, the traces of matrix representations. Define the *trace spaces* $\mathcal{T}(W)$ and $\mathcal{T}_q(W)$ to be the $\mathbb{Z}$-span of $W$-characters and $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of $H$-characters, respectively. When $W = \mathfrak{S}_n$, each module has rank equal to the number of *integer partitions* of $n$, weakly decreasing positive integer sequences summing to $n$. Let $\lambda \vdash n$ or $|\lambda| = n$ denote that $\lambda$ is a partition of $n$. Five well-studied bases of $\mathcal{T}(\mathfrak{S}_n)$ are the irreducible characters $\{\chi^\lambda \,|\, \lambda \vdash n\}$, induced sign characters $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$, induced trivial characters $\{\eta^\lambda \,|\, \lambda \vdash n\}$, power sum traces $\{\psi^\lambda \,|\, \lambda \vdash n\}$, and monomial traces

*Date*: June 19, 2026.

$\{\phi^\lambda \,|\, \lambda \vdash n\}$. Analogous bases of $\mathcal{T}_q(\mathfrak{S}_n)$ are $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$, $\{\psi_q^\lambda \,|\, \lambda \vdash n\}$, $\{\phi_q^\lambda \,|\, \lambda \vdash n\}$, specializing at $q^{\frac{1}{2}} = 1$ to the $\mathcal{T}(\mathfrak{S}_n)$-bases. When $W = \mathfrak{B}_n$, the trace spaces $\mathcal{T}(\mathfrak{B}_n)$, $\mathcal{T}_q(\mathfrak{B}_n)$ have rank equal to the numbers of *integer bipartitions* of $n$, pairs $(\lambda, \mu)$ of partitions with $|\lambda| + |\mu| = n$. Let $(\lambda, \mu) \vdash n$ denote that $(\lambda, \mu)$ is a bipartition of $n$. Commonly studied bases of $\mathcal{T}(\mathfrak{B}_n)$ have the form $\{(\theta\xi)^{\lambda,\mu} \,|\, (\lambda, \mu) \vdash n\}$, where $\{\theta^\lambda \,|\, \lambda \vdash k\}$ and $\{\xi^\mu \,|\, \mu \vdash n-k\}$ are bases of $\mathcal{T}(\mathfrak{S}_k)$ and $\mathcal{T}(\mathfrak{S}_{n-k})$, respectively. Commonly studied bases of $\mathcal{T}_q(\mathfrak{B}_n)$ are $q$-analogs of these.

Many open problems in algebraic combinatorics concern character and trace evaluations.

**Problem 1.** For all $w \in \mathfrak{S}_n$ and partitions $\lambda \vdash n$, state cancellation-free formulas for
(a) $\chi^\lambda(w)$,
(b) $\theta_q^\lambda(T_w)$, where $\{\theta_q^\lambda \,|\, \lambda \vdash n\}$ is any standard basis of $\mathcal{T}_q(\mathfrak{S}_n)$,
(c) $\theta_q^\lambda(\widetilde{C}_w(q))$, where $\{\theta_q^\lambda \,|\, \lambda \vdash n\}$ is any standard basis of $\mathcal{T}_q(\mathfrak{S}_n)$.

**Problem 2.** For all $w \in \mathfrak{B}_n$ and bipartitions $(\lambda, \mu) \vdash n$, state cancellation-free formulas for
(a) $(\chi\chi)^{\lambda,\mu}(w)$,
(b) $(\theta\xi)_q^{\lambda,\mu}(T_w)$, where $\{(\theta\xi)_q^\lambda \,|\, (\lambda, \mu) \vdash n\}$ is any standard basis of $\mathcal{T}_q(\mathfrak{B}_n)$,
(c) $(\theta\xi)_q^{\lambda,\mu}(\widetilde{C}_w(q))$, where $\{(\theta\xi)_q^{\lambda,\mu} \,|\, (\lambda, \mu) \vdash n\}$ is any standard basis of $\mathcal{T}_q(\mathfrak{B}_n)$.

**Problem 3.** For sequences $(w^{(1)}, \dots, w^{(r)})$ of maximal parabolic elements of $W = \mathfrak{S}_n$ or $\mathfrak{B}_n$ and $H = H(W)$, state formulas for
(a) the expansion of each product $\widetilde{C}_{w^{(1)}}(q) \cdots \widetilde{C}_{w^{(r)}}(q)$ in the natural basis of $H(W)$,
(b) the evaluations of standard traces at products $\widetilde{C}_{w^{(1)}}(q) \cdots \widetilde{C}_{w^{(r)}}(q)$.

**Problem 4.** Describe polynomial inequalities of the form $\sum_{w \in \mathfrak{S}_n} \theta(w) a_{1,w_1} \cdots a_{n,w_n} \geq 0$ that are satisfied by the entries of every totally nonnegative matrix $A = (a_{i,j})$.

## 2. Representations of the symmetric group and its Hecke algebra

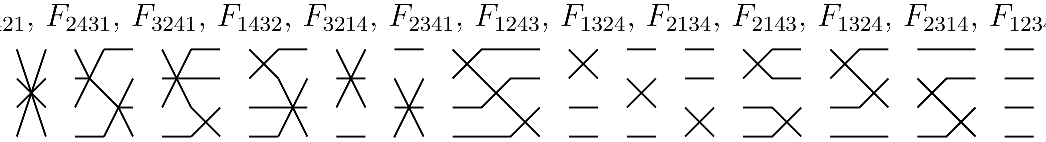
The candidate began to make progress on Problems 1 – 3 with the result [29, Thm. 4.3].

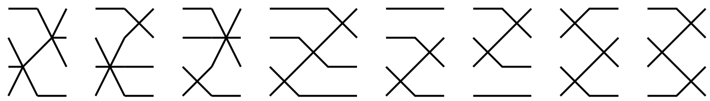
**Theorem 5.** *Each element $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231 factors as a product $w = w^{(1)} \cdots w^{(r)}$ of maximal elements of parabolic subgroups, with $\widetilde{C}_w(q) \in H(\mathfrak{S}_n)$ similarly factoring as $\widetilde{C}_w(q) = \frac{1}{p(q)} \widetilde{C}_{w^{(1)}}(q) \cdots \widetilde{C}_{w^{(r)}}(q)$ for some polynomial $p \in \mathbb{N}[q]$.*

Since each Kazhdan–Lusztig basis element $\widetilde{C}_v(q)$ with $v$ maximal in a parabolic subgroup has a very simple form, Theorem 5 led the candidate to combinatorially interpret Kazhdan–Lusztig basis elements of $H(\mathfrak{S}_n)$ indexed by 3412-avoiding, 4231-avoiding permutations [29].

**Theorem 6.** *For $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, a certain "zig-zag" network $F_w$ combinatorially encodes the natural expansion of $\widetilde{C}_w(q)$ in $H(\mathfrak{S}_n)$. The network $F_w$ is a "descending star network" if $w$ avoids the pattern 312.*

For example, when $n = 4$ the Kazhdan–Lusztig basis elements excluding $\widetilde{C}_{3412}(q)$, $\widetilde{C}_{4231}(q)$ can be encoded by *zig-zag networks of order* 4 including the fourteen *descending star networks* $F_{4321}$, $F_{3421}$, $F_{2431}$, $F_{3241}$, $F_{1432}$, $F_{3214}$, $F_{2341}$, $F_{1243}$, $F_{1324}$, $F_{2134}$, $F_{2143}$, $F_{1324}$, $F_{2314}$, $F_{1234}$,

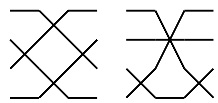
(2.1)  , and eight similar networks $F_{4312}, F_{4213}, F_{4132}, F_{4123}, F_{3124}, F_{1423}, F_{3142}, F_{2413}$,

(2.2)  .

Careful study of zig-zag networks led the candidate to reduce some trace evaluations to the special case of evaluations $\theta_q(\widetilde{C}_v(q))$ for $v$ avoiding the pattern 312 [7, Thm. 3.6, Thm. 4.6].

**Theorem 7.** *For each element $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, there is an element $v \in \mathfrak{S}_n$ avoiding the pattern 312 such that for all traces $\theta_q \in \mathcal{T}_q(\mathfrak{S}_n)$, we have $\theta_q(\widetilde{C}_w(q)) = \theta_q(\widetilde{C}_v(q))$.*

Theorem 5 extends to some Kazhdan–Lusztig basis elements indexed by permutations not avoiding the patterns 3412 and 4231. Specifically, $\widetilde{C}_{3412}(q)$ and $\widetilde{C}_{4231}(q)$ can be encoded by

 ,

respectively. Call arbitrary concatenations of zig-zag networks *star networks*. This led the candidate to pose the problem [29, Question 4.5] of improving the hypotheses of Theorem 5.

**Problem 8.** State conditions on $w \in \mathfrak{S}_n$ which guarantee the Kazhdan–Lusztig basis element $\widetilde{C}_w(q)$ to factor or not to factor as in Theorem 5.

The candidate and a vertically integrated team of assistants made progress on this problem by using [12] to prove the following negative result [27].

**Theorem 9.** *For each element $w \in \mathfrak{S}_n$ avoiding the pattern 4231 and containing the pattern 3412, if all subwords matching the pattern 3412 have first and last letters that differ by at least 2, then $\widetilde{C}_w(q)$ has no reversal factorization.*

Additional work of the candidate and others [1], [9], suggests that avoidance of the pattern 45312 is an important part of the solution to Problem 8.

The combinatorial interpretations $\{F_w \,|\, w \in \mathfrak{S}_n$ avoids the patterns 3412 and 4231$\}$ of Kazhdan–Lusztig basis elements play a crucial role in the evaluation of $H(\mathfrak{S}_n)$ traces. In particular, the candidate and assistants [7] used families of paths in zig-zag networks and objects called *$F_w$-tableaux* to answer a special case of Problem 1(b) as in Theorems 10 – 11.

**Theorem 10.** *For $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231 and any trace $\theta_q$ in the bases $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$, $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, $\{\psi_q^\lambda \,|\, \lambda \vdash n\}$ of $\mathcal{T}_q(\mathfrak{S}_n)$, there exists a set $\mathcal{S}(\theta_q, w)$ of $F_w$-tableaux and a function $\mathrm{STAT} : \mathcal{S}(\theta_q, w) \rightarrow \mathbb{N}$ such that we have*

$$\theta_q(\widetilde{C}_w(q)) = \sum_{U \in \mathcal{S}(\theta_q, w)} q^{\mathrm{STAT}(U)}. \tag{2.3}$$

Haiman [13] conjectured that the monomial traces satisfy $\phi_q^\lambda(\widetilde{C}_w(q)) \in \mathbb{N}[q]$ for all $w$. This extends the conjecture of Stanley and Stembridge [39], now proved by Hikita [15], that $\phi^\lambda(\widetilde{C}_w(1)) \in \mathbb{N}$ when $w$ avoids the patterns 3412 and 4231. No formula analogous to (2.3) has been conjectured for general $\lambda \vdash n$, even for $q = 1$ and $w$ avoiding the patterns 3412 and 4231. On the other hand, Wolfgang [44, Thm. 2.5.1] proved nonnegativity of $\phi^\lambda(\widetilde{C}_w(1))$ for $\lambda_1 \leq 2$ and $w$ avoiding the patterns 3412 and 4231. The candidate [7, Thm. 10.3] then found a combinatorial interpretation of $\phi_q^\lambda(\widetilde{C}_w(q))$ in this case.

**Theorem 11.** *For $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231 and $\lambda \vdash n$ satisfying $\lambda_1 \leq 2$, there exists a set $\mathcal{S}(\phi_q^\lambda, w)$ of $F_w$-tableaux and a function* $\textsc{stat} : \mathcal{S}(\phi_q^\lambda, w) \to \mathbb{N}$ *such that we have*

$$\phi_q^\lambda(\widetilde{C}_w(q)) = \sum_{U \in \mathcal{S}(\phi_q^\lambda, w)} q^{\textsc{stat}(U)}. \tag{2.4}$$

Solving Problem 3 for $\theta_q = \epsilon_q^\lambda$ (and generalizing Theorems 6 and 10), the candidate and A. Clearwater found formulas for the natural expansions of certain products of Kazhdan–Lusztig basis elements which arise in the study of total nonnegativity [41, §5], and for the evaluations of induced sign characters at these products. Both involve *families* $\pi$ of $n$ left-to-right paths in star networks, and the *type* of each in $\mathfrak{S}_n$, i.e., the permutation of right endpoints of the paths relative to left endpoints [8, Cor. 5.3, Thm. 6.5].

**Theorem 12.** *Let $w^{(1)}, \dots, w^{(r)}$ be a sequence of maximal parabolic elements in $\mathfrak{S}_n$, and let $F = F_{w^{(1)}} \circ \cdots \circ F_{w^{(r)}}$ be the corresponding concatenation of zig-zag networks. Then there are functions* dfct*,* $\textsc{cr}$ *on path families in $F$, and a function* $\textsc{invnc}$ *on $F$-tableaux such that*

$$\widetilde{C}_{w^{(1)}}(q) \cdots \widetilde{C}_{w^{(r)}}(q) = \sum_{v \in \mathfrak{S}_n} \Big( \sum_\pi q^{\mathrm{dfct}(\pi)} \Big) T_v, \tag{2.5}$$

*where the inner sum is over path families of type $v$ in $F$, and*

$$\epsilon_q^\lambda(\widetilde{C}_{w^{(1)}}(q) \cdots \widetilde{C}_{w^{(r)}}(q)) = \sum_\pi q^{\frac{\textsc{cr}(\pi)}{2}} \sum_W q^{\textsc{invnc}(W)}, \tag{2.6}$$

*where the sums are over path families of type $e$ in $F$, and certain $F$-tableaux $W$.*

These results provide progress toward the problem of finding a combinatorial interpretation of the Stembridge–Haiman result [13], [40] that we have

$$\chi_q^\lambda(\widetilde{C}_{w^{(1)}}(q) \cdots \widetilde{C}_{w^{(r)}}(q)) \in \mathbb{N}[q] \tag{2.7}$$

for all sequences $w^{(1)}, \dots, w^{(r)}$ of maximal parabolic elements of $\mathfrak{S}_n$.

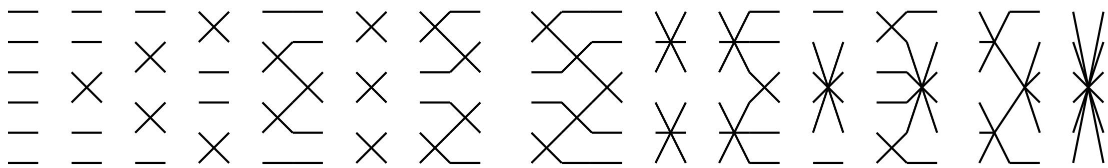
## 3. Representations of the hyperoctahedral group and its Hecke algebra

Motivated by known connections between type-A trace evaluations and type-A Hessenberg varieties, and suspected connections between type-BC trace evaluations and (the distinct) type-B and C Hessenberg varieties, the candidate extended some results from Section 2 to the hyperoctahedral group and its Hecke algebra.

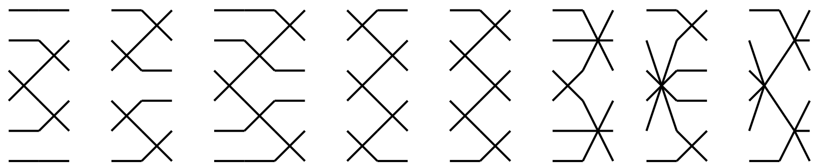
Theorems 5 – 6 extend as follows [33, Thm. 5.18 – Thm. 5.21].

**Theorem 13.** *Each element $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231 factors as a product $w = w^{(1)} \cdots w^{(r)}$ of maximal elements of parabolic subgroups, with $\widetilde{C}_w(q) \in H(\mathfrak{B}_n)$ similarly factoring as $\widetilde{C}_w(q) = \frac{1}{p(q)}\widetilde{C}_{w^{(1)}}(q) \cdots \widetilde{C}_{w^{(r)}}(q)$ for some polynomial $p \in \mathbb{N}[q]$. Furthermore, a certain "zig-zag" network $F_w$ combinatorially encodes the natural expansion of $\widetilde{C}_w(q)$ in $H(\mathfrak{B}_n)$. The network $F_w$ is a "descending star network" if $w$ avoids the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, $312$, $3\overline{1}2$.*

Type-BC zig-zag networks for $\mathfrak{B}_n$ and $H(\mathfrak{B}_n)$ are essentially type-A zig-zag networks of order $2n$ which are symmetric about a horizontal line. For example, when $n = 3$ the twenty-two Kazhdan–Lusztig basis elements $\widetilde{C}_w(q) \in H(\mathfrak{B}_n)$ indexed by $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231 can be encoded by zig-zag networks of order 6, including fourteen type-BC descending star networks (in which stars descend from left to right in the upper half) $F_{123}$, $F_{\overline{1}23}$, $F_{213}$, $F_{132}$, $F_{2\overline{1}3}$, $F_{\overline{1}32}$, $F_{231}$, $F_{23\overline{1}}$, $F_{321}$, $F_{32\overline{1}}$, $F_{\overline{12}3}$, $F_{\overline{1}3\overline{2}}$, $F_{3\overline{12}}$, $F_{\overline{123}}$,



and eight similar networks $F_{\overline{2}13}$, $F_{312}$, $F_{\overline{3}12}$, $F_{\overline{2}31}$, $F_{3\overline{1}2}$, $F_{\overline{3}21}$, $F_{\overline{13}2}$, $F_{\overline{32}1}$,

(3.1)  .

These results allowed the candidate to prove BC-extensions of the $q = 1$ specializations of Theorems 7 and 10. Trace evaluation equalities were established in [33, Thm. 10.5].

**Theorem 14.** *For each element $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231, there is an element $v \in \mathfrak{B}_n$ avoiding the signed patterns $1\overline{2}$, $\overline{2}1$, $\overline{21}$, 312, $3\overline{1}2$ such that for all traces $\theta \in \mathcal{T}(\mathfrak{B}_n)$, we have $\theta(\widetilde{C}_w(1)) = \theta(\widetilde{C}_v(1))$.*

Again, combinatorial interpretations involve paths in networks $F_w$, but now paths are placed into pairs of tableaux which we call *marked $F_w$-bitableax* [33, Thm. 9.6 – Cor. 9.10].

**Theorem 15.** *For $w \in \mathfrak{B}_n$ avoiding the patterns 3412 and 4231 and any trace $\theta$ in the bases $\{(\chi\chi)^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}$, $\{(\eta\eta)^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}$, $\{(\eta\epsilon)^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}$, $\{(\epsilon\eta)^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}$, $\{(\epsilon\epsilon)^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}$, $\{(\psi\psi)^{\lambda,\mu} \,|\, (\lambda,\mu) \vdash n\}$ of $\mathcal{T}(\mathfrak{B}_n)$, there exists a set $\mathcal{S}(\theta, w)$ of marked $F_w$-bitableaux such that we have $\theta(\widetilde{C}_w(1)) = |\mathcal{S}(\theta, w)|$.*

Theorem 13 also has allowed the candidate and a vertically integrated team of assistants to extend Theorem 12 (2.5) to products in $H(\mathfrak{B}_n)$ [16].

**Theorem 16.** *Let $w^{(1)}, \dots, w^{(r)}$ be a sequence of maximal parabolic elements in $\mathfrak{B}_n$, and let $F = F_{w^{(1)}} \circ \cdots \circ F_{w^{(r)}}$ be the corresponding concatenation of zig-zag networks. Then there is a function $\mathrm{dfct}^{\mathsf{BC}}$ on path families in $F$ such that we have*

$$\widetilde{C}_{w^{(1)}}(q) \cdots \widetilde{C}_{w^{(r)}}(q) = \sum_{v \in \mathfrak{B}_n} \Big( \sum_{\pi} q^{\mathrm{dfct}^{\mathsf{BC}}(\pi)} \Big) T_v, \tag{3.2}$$

*where the inner sum is over BC-path families of type $v$ in $F$.*

## 4. Symmetric generating functions for character evaluations

Like traces in $\mathcal{T}(H(\mathfrak{S}_n))$, elements of the module $\Lambda_n$ of homogeneous degree-$n$ symmetric functions in $x = (x_1, x_2, \dots)$ are usually expressed in terms of bases indexed by partitions of $n$. (See, e.g., [38, Ch. 7].) Common bases are the *Schur* basis $\{s_\lambda \,|\, \lambda \vdash n\}$, *monomial* basis $\{m_\lambda \,|\, \lambda \vdash n\}$, *elementary* basis $\{e_\lambda \,|\, \lambda \vdash n\}$, *homogeneous* basis $\{h_\lambda \,|\, \lambda \vdash n\}$, *power sum* basis $\{p_\lambda \,|\, \lambda \vdash n\}$, and *forgotten* basis $\{f_\lambda \,|\, \lambda \vdash n\}$. It is therefore natural to define a symmetric generating function for the character evaluations of each element $D \in H(\mathfrak{S}_n)$,

$$Y_q(D) := \sum_{\lambda \vdash n} \epsilon_q^\lambda(D) m_\lambda = \sum_{\lambda \vdash n} \phi_q^\lambda(D) e_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(D) s_\lambda = \sum_{\lambda \vdash n} b_\lambda \psi_q^\lambda(D) p_\lambda = \sum_{\lambda \vdash n} \eta_q^\lambda(D) f_\lambda,$$

where equalities follow from transition matrices for the bases, and where $\{b_\lambda \,|\, \lambda \vdash n\}$ are certain rational numbers. This function has the following universal property [30, Prop. 2.3].

**Proposition 17.** *Every element of* $\mathbb{Z}[q] \otimes \Lambda_n$ *has the form* $Y_q(D)$ *for some* $D \in \mathbb{Q}(q) \otimes H(\mathfrak{S}_n)$*.*

A well-known problem concerns *chromatic* symmetric functions $X_{P,q} \in \mathbb{Z}[q] \otimes \Lambda_n$ defined in terms of posets $P$ called *natural unit interval orders* in [28]. (See also [37], [39].) The expansion of $X_{P,q}$ as a polynomial in $q$ with symmetric function coefficients describes an action of $\mathfrak{S}_n$ on an algebraic variety called the type-$\mathsf{A}$ *Hessenberg variety* associated to $P$. (See [42]–[43].) These functions are conjectured [28], [37], [39] to be $\mathbb{N}[q]$-linear combinations of elementary symmetric functions. The candidate and graduate student B. Shelton proved that the expansion of $X_{P,q}$ in each standard symmetric function basis yields coefficients which are equal to $H(\mathfrak{S}_n)$-trace evaluations at Kazhdan–Lusztig basis elements [7, Sec. 7].

**Theorem 18.** *For each $n$-element unit interval order $P$, there exists a 312-avoiding permutation* $w \in \mathfrak{S}_n$ *such that* $X_{P,q} = Y_q(\widetilde{C}_w(q))$*.*

It is natural to ask how Theorem 18 or its $q = 1$ specialization can be extended to combinatorially describe type-$\mathsf{BC}$ character evaluations. In particular it is possible to define for each element $D \in H(\mathfrak{B}_n)$ a type-$\mathsf{BC}$ symmetric function which is a generating function

$$Y_q^{\mathsf{BC}}(D) = \sum_{(\lambda,\mu)} (\epsilon\epsilon)_q^{\lambda,\mu}(D) m_\lambda(x) m_\mu(y) \tag{4.1}$$

for character evaluations at $D$, where $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are two sets of variables. It is also possible to define type-$\mathsf{BC}$ chromatic symmetric functions $X_P^{\mathsf{BC}}$ for type-$\mathsf{BC}$ analogs of posets. The candidate has extended Proposition 17 and Theorem 18 to the following type-$\mathsf{BC}$ analogs [33, Prop. 11.3, Thm. 11.4].

**Proposition 19.** *Every degree-$n$ homogeneous type-$\mathsf{BC}$ symmetric function with integer coefficients is* $Y_q^{\mathsf{BC}}(D)$ *for some element* $D \in \mathbb{Q}(q) \otimes H(\mathfrak{B}_n)$*.*

**Theorem 20.** *For each type-$\mathsf{BC}$ unit interval order $P$ there exists an element* $w \in \mathfrak{B}_n$ *avoiding the signed patterns* $1\overline{2}$, $\overline{2}1$, $\overline{21}$, $312$, $3\overline{1}2$ *such that we have* $X_P^{\mathsf{BC}} = Y_1^{\mathsf{BC}}(\widetilde{C}_w(1))$*.*

It would be interesting to prove a $q$-analog of this result [33, Prob. 11.5].

**Problem 21.** Define a $q$-extension $X_{P,q}^{\mathsf{BC}}$ of the type-$\mathsf{BC}$ chromatic symmetric function $X_P^{\mathsf{BC}}$ which satisfies $X_{P,q}^{\mathsf{BC}} = Y_q^{\mathsf{BC}}(\widetilde{C}_w(q))$ when $P$ and $w$ correspond as in Theorem 20.

It would be even more interesting if this extension $X_{P,q}^{\mathsf{BC}}$ were related to type-$\mathsf{B}$ or $\mathsf{C}$ Hessenberg varieties as $X_{P,q}$ is related to type-$\mathsf{A}$ Hessenberg varieties [6].

A second well-known problem concerns symmetric functions $\mathrm{LLT}_{P,q} \in \mathbb{Z}[q] \otimes \Lambda_n$. (See [20].) These functions are known to be $\mathbb{N}[q]$-linear combinations of Schur functions, but no combinatorial rule for the resulting coefficients is known. The candidate and coauthors [25] found an algebraic description of these coefficients which may eventually lead to a combinatorial description.

**Theorem 22.** *In all standard expansions of* $\mathrm{LLT}_{P,q}$*, the resulting coefficients are evaluations of certain (nonstandard)* $H(\mathfrak{S}_n)$*-traces at Kazhdan–Lusztig basis elements indexed by 312-avoiding permutations.*

## 5. Generating functions in coordinate rings

Often it is useful to have a generating function which records values $\{\theta(w)\,|\,w \in \mathfrak{S}_n\}$ for some $\mathfrak{S}_n$-character $\theta$, i.e.,

$$\mathrm{Imm}_\theta(t) := \sum_{w \in \mathfrak{S}_n} \theta(w) t_{1,w_1} \cdots t_{n,w_n}, \tag{5.1}$$

where $t = (t_{i,j})$ may be thought of as an $n \times n$ matrix of variables. In particular, a simple expression for such a generating function can serve as a satisfactory formula for $\theta(w)$. The candidate and Konvalinka stated generating functions for the $H(\mathfrak{S}_n)$-characters $\{\epsilon_q^\lambda\,|\,\lambda \vdash n\}$, $\{\eta_q^\lambda\,|\,\lambda \vdash n\}$ [18, Thm. 5.4] in a certain quotient $\mathcal{A}_n(q)$ of the noncommutative ring $\mathbb{Z}\langle t \rangle$. Each generating function is expressed in terms of submatrices of $t$, polynomials called the $q$-determinant and $q$-permanent, and *ordered set partitions of type* $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, sequences $(I_1, \dots, I_r)$ of disjoint subsets of $[1,n] := \{1, \dots, n\}$ satisfying $|I_j| = \lambda_j$. Our formulas are $q$-extensions of the Littlewood–Merris–Watkins generating functions [22], [24] in $\mathbb{Z}[t]$ for $\mathfrak{S}_n$-characters.

**Theorem 23.** *Fix partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$. We have in $\mathcal{A}_n(q)$ that*

$$\mathrm{Imm}_{\epsilon_q^\lambda}(t) = \sum_{(I_1,\dots,I_r)} \mathrm{det}_q(t_{I_1,I_1}) \cdots \mathrm{det}_q(t_{I_r,I_r}), \tag{5.2}$$

$$\mathrm{Imm}_{\eta_q^\lambda}(t) = \sum_{(I_1,\dots,I_r)} \mathrm{per}_q(t_{I_1,I_1}) \cdots \mathrm{per}_q(t_{I_r,I_r}), \tag{5.3}$$

*where $t_{K,K} := (t_{i,j})_{i,j \in K}$, and sums are over ordered set partitions of $[1,n]$ of type $\lambda$.*

The candidate also generalized the identities (5.2) – (5.3) to create generating functions for induced one-dimensional characters of wreath products $\mathcal{G}_{n,d} := \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ [31, Thm. 3.1]. Such characters correspond to $d$-tuples $\boldsymbol{\beta}^{\boldsymbol{\lambda}} = (\beta_0^{\lambda^0}, \dots, \beta_{d-1}^{\lambda^{d-1}})$ of symmetric group characters with $\beta_i \in \{\eta, \epsilon\}$ and $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{d-1})$ a $d$-tuple of partitions with $|\lambda^0| + \cdots + |\lambda^{d-1}| = n$. Elements of $\mathcal{G}_{n,d}$ have the form $g = (g_1, \dots, g_n) = (\zeta^{\gamma_1} w_1, \dots, \zeta^{\gamma_n} w_n)$ for $\zeta$ a $d$th root of unity, $w_1 \cdots w_n \in \mathfrak{S}_n$, and $(\gamma_1, \dots, \gamma_n) \in \mathbb{Z}/d\mathbb{Z}^n$. The new generating functions belong to the ring $\mathbb{C}[t]$ in $dn^2$ indeterminates $t = \{t_{i,\zeta^k p}\,|\,i, p \in [1,n], k \in \mathbb{Z}/d\mathbb{Z}\}$ and are defined by

$$\mathrm{Imm}_{\boldsymbol{\beta}^{\boldsymbol{\lambda}}}^{\mathcal{G}_{n,d}}(t) := \sum_{g \in \mathcal{G}_{n,d}} \boldsymbol{\beta}^{\boldsymbol{\lambda}}(g) t_{1,g_1} \cdots t_{n,g_n}.$$

Each is a sum of products of $\mathfrak{S}_{|\lambda^0|}, \dots, \mathfrak{S}_{|\lambda^{d-1}|}$ character immanants (specializations of (5.2) – (5.3) at $q=1$) of matrices $Q_0(t), \dots, Q_{d-1}(t)$ defined by $Q_k(t) = (q_{i,j,k}(t))_{i,j \in [1,n]}$, where

$$q_{i,j,k}(t) = t_{i,j} + \zeta^{-k} t_{i,\zeta j} + \zeta^{-2k} t_{i,\zeta^2 j} + \cdots + \zeta^{-(d-1)k} t_{i,\zeta^{(d-1)} j}.$$

**Theorem 24.** *Fix $\mathcal{G}_{n,d}$-character $\boldsymbol{\beta}^{\boldsymbol{\lambda}} = (\beta_0^{\lambda^0}, \dots, \beta_{d-1}^{\lambda^{d-1}})$. We have in $\mathbb{C}[t]$ that*

$$\mathrm{Imm}_{\boldsymbol{\beta}^{\boldsymbol{\lambda}}}^{\mathcal{G}_{n,d}}(t) = \sum_{(I_0,\dots,I_{d-1})} \mathrm{Imm}_{\beta_0^{\lambda^0}}(Q_0(t)_{I_0,I_0}) \cdots \mathrm{Imm}_{\beta_{d-1}^{\lambda^{d-1}}}(Q_{d-1}(t)_{I_{d-1},I_{d-1}}), \tag{5.4}$$

*where the sum is over all ordered set partitions of $[1,n]$ of type $(|\lambda^0|, \dots, |\lambda^{d-1}|)$.*

Theorem 23 is an essential ingredient in the proofs of Theorems 15 and 18. Similarly, since we have $\mathfrak{B}_n \cong \mathcal{G}_{n,2}$, Theorem 24 is essential in the proof of Theorem 20. To solve Problem 21, it would be helpful to state a $q$-extension of the $d = 2$ case of Theorem 24.

**Problem 25.** State formulas in an appropriate ring for $\mathrm{Imm}^{\mathcal{G}_{n,2}}_{(\epsilon\epsilon)_q^{\lambda,\mu}}(t)$, $\mathrm{Imm}^{\mathcal{G}_{n,2}}_{(\epsilon\eta)_q^{\lambda,\mu}}(t)$, $\mathrm{Imm}^{\mathcal{G}_{n,2}}_{(\eta\epsilon)_q^{\lambda,\mu}}(t)$, $\mathrm{Imm}^{\mathcal{G}_{n,2}}_{(\eta\eta)_q^{\lambda,\mu}}(t)$ to provide type-$\mathsf{BC}$ analogs of Theorem 23.

Such a result will also lead to a $\mathsf{BC}$-analog of Theorem 12, combinatorially interpeting evaluations of $(\epsilon\epsilon)_q^{\lambda,\mu}$ at products of Kahzdan–Lusztig basis elements of $H(\mathfrak{B}_n)$.

Yet another generalization [25] of the identities (5.2) – (5.3) provides generating functions in $\mathcal{A}_n(q)$ for evaluations of the nonstandard traces mentioned in Theorem 22. Two such nonstandard trace bases $\{\epsilon^\lambda_{q,\mathrm{LLT}} \,|\, \lambda \vdash n\}$, $\{\eta^\lambda_{q,\mathrm{LLT}} \,|\, \lambda \vdash n\}$ are analogous to induced sign and trivial characters. Given square submatrix $t_{I,J}$ of $t$, define $\mathrm{ldprod}_q(t_{I,J})$ to be the left-to-right product of its diagonal entries, and define $\mathrm{rdprod}_q(t_{I,J})$ to be the right-to-left product of its diagonal entries. (These are not in general equal in $\mathcal{A}_n(q)$.)

**Theorem 26.** *Fix partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$. We have in $\mathcal{A}_n(q)$ that*

$$\mathrm{Imm}_{\epsilon^\lambda_{q,\mathrm{LLT}}}(t) = \sum_{(I_1,\dots,I_r)} \mathrm{ldprod}_q(t_{I_1,I_1}) \cdots \mathrm{ldprod}_q(t_{I_r,I_r}), \tag{5.5}$$

$$\mathrm{Imm}_{\eta^\lambda_{q,\mathrm{LLT}}}(t) = \sum_{(I_1,\dots,I_r)} \mathrm{rdprod}_q(t_{I_1,I_1}) \cdots \mathrm{rdprod}_q(t_{I_r,I_r}), \tag{5.6}$$

*where sums are over ordered set partitions of $[1,n]$ of type $\lambda$.*

## 6. Applications to total nonnegativity

Call a matrix $A = (a_{i,j})$ *totally nonnegative* if each square submatrix $A_{I,J} := (a_{i,j})_{i\in I, j\in J}$ satisfyies $\det(A_{I,J}) \geq 0$. Such matrices appear in many areas of mathematics, and in particular, the polynomial inequalities satisfied by their entries (Problem 4) interest researchers in the area of quantum groups. Results in Sections 2 – 5 have applications to the study of such inequalities. For instance Theorem 11 allowed the candidate and D. Soskin [34] to extend earlier work of Barrett and Johnson [2] on positive semidefinite matrices.

**Theorem 27.** *Given partitions $\lambda \vdash n$, $\mu \vdash n$, the inequality*

$$\frac{\mathrm{Imm}_{\epsilon^\lambda}(A)}{\epsilon^\lambda(e)} \leq \frac{\mathrm{Imm}_{\epsilon^\mu}(A)}{\epsilon^\mu(e)}$$

*holds for all totally nonnegative matrices $A$ if and only if we have $\mu \preceq \lambda$ in the majorization order, i.e., $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ for all $i$.*

Similarly, Theorem 10 allowed the candidate to state two new combinatorial interpretations of the permanent of a TNN matrix. It is known that a matrix $A = (a_{i,j})$ is TNN if and only if it may be interpreted as the path matrix of a weighted planar network $F$, with $a_{i,j}$ being the sum of weights of all paths in $F$ between vertices called *source* $i$ and *sink* $j$ [5]. The following [30] is a permanental analog of Lindström's Lemma [21].

**Theorem 28.** *Let $A$ be the path matrix of planar network $F$. Then* $\mathrm{per}(A)$ *equals the number, over path families $\pi = (\pi_1, \dots, \pi_n)$ of type $e$ in $F$, of path rearrangements $\pi_{w_1} \cdots \pi_{w_n}$ having the "descent-free" property* [30], *or alternatively, the "excedance-free" property* [30].

Theorem 28 may be useful in proving a permanental analog of Theorem 27.

**Problem 29.** Decide if for partitions $\lambda \vdash n$, $\mu \vdash n$, the inequality

$$\frac{\mathrm{Imm}_{\eta^\lambda}(A)}{\eta^\lambda(e)} \geq \frac{\mathrm{Imm}_{\eta^\mu}(A)}{\eta^\mu(e)}$$

holds for all totally nonnegative matrices $A$ if and only if $\mu \preceq \lambda$ in the majorization order.

A characterization of pairs $(\lambda, \mu)$ of partitions of $n$ which satisfy analogous inequalities for irreducible characters and totally nonnegative matrices,

$$\frac{\mathrm{Imm}_{\chi^\lambda}(A)}{\chi^\lambda(e)} \geq \frac{\mathrm{Imm}_{\chi^\mu}(A)}{\chi^\mu(e)},$$

has been considered by Stembridge [41] and appears to be more difficult. Nevertheless, the candidate has used his previous results in [30] to extend earlier work of Heyfron [14] and Merris [23] on equalities which hold for positive semidefinite matrices [32].

**Theorem 30.** *The Heyfron–Merris inequalities*

$$\mathrm{per}(A) = \frac{\mathrm{Imm}_{\chi^n}(A)}{\chi^n(e)} \geq \frac{\mathrm{Imm}_{\chi^{n-1,1}}(A)}{\chi^{n-1,1}(e)} \geq \frac{\mathrm{Imm}_{\chi^{n-2,1,1}}(A)}{\chi^{n-2,1,1}(e)} \geq \cdots \geq \frac{\mathrm{Imm}_{\chi^{1,\dots,1}}(A)}{\chi^{1,\dots,1}(e)} = \det(A),$$

*hold for all $n \times n$ totally nonnegative matrices $A$.*

The candidate has also used his previous results in [10] to extend work on ratios of minors [11], [36] to characterize ratios of permanents which are bounded on the set of totally positive matrices [35]. Given square submatrix $t_{I,J}$ of $t$, define $(t_{I,J})^{e,e}$ to be the product of its diagonal entries.

**Theorem 31.** *Given the rational function*

$$R(t) = \frac{\mathrm{per}(t_{I_1,I_1'})\,\mathrm{per}(t_{I_2,I_2'})\cdots\mathrm{per}(t_{I_r,I_r'})}{\mathrm{per}(t_{J_1,J_1'})\,\mathrm{per}(t_{J_2,J_2'})\cdots\mathrm{per}(t_{J_q,J_q'})}, \tag{6.1}$$

*define matrices $C = (c_{i,j})$, $D = (d_{i,j})$, $C^* = (c^*_{i,j})$, $D^* = (d^*_{i,j})$ by*

$$(t_{I_1,I_1'})^{e,e}\cdots(t_{I_r,I_r'})^{e,e} = \prod t_{i,j}^{c_{i,j}}, \qquad (t_{J_1,J_1'})^{e,e}\cdots(t_{J_q,J_q'})^{e,e} = \prod t_{i,j}^{d_{i,j}},$$

$$c^*_{i,j} = \sum_{k=1}^{i}\sum_{\ell=1}^{j} c_{k,\ell}, \qquad d^*_{i,j} = \sum_{k=1}^{i}\sum_{\ell=1}^{j} d_{k,\ell}.$$

*Then $R(t)$ is bounded above on the set of totally positive matrices if and only if*

*(1) $C$ and $D$ have identical row sums and identical column sums,*
*(2) $C^* \leq D^*$ in the componentwise order.*

*In this case, $R(t)$ is bounded above by $|I_1|!\cdots|I_r|!$ .*

Theorem 28 may also be useful in proving permanental analogs of the Fallat–Gekhtman–Johnson inequalities [11] for products $\det(A_{I,I})\det(A_{\overline{I},\overline{I}})$, where $\overline{I} := [1,n] \smallsetminus I$.

**Problem 32.** Decide if for $k = 1, \dots, n-1$, the permanental inequalities

$$\mathrm{per}(A_{[1,n]\cap 2\mathbb{Z},[1,n]\cap 2\mathbb{Z}})\,\mathrm{per}(A_{[1,n]\smallsetminus 2\mathbb{Z},[1,n]\smallsetminus 2\mathbb{Z}}) \leq \mathrm{per}(A_{[1,k],[1,k]})\,\mathrm{per}(A_{[k+1,n],[k+1,n]})$$

hold for all TNN matrices $A$.

The candidate and coauthors have verified these inequalities to hold when $A$ is the antiadjacency matrix of a unit interval order [26].

## References


[1] R. Agrawal and V. Sotirov. A generalization of Deodhar's framework for questions in Kazhdan–Lusztig theory (2021). Preprint `math.CO/2107.08522` on ArXiv.

[2] W. Barrett and C. Johnson. Majorization monotonicity of symmetrized Fischer products. *Linear Multilinear Algebra*, **34**, 1 (1993) pp. 67–74.

[3] S. Billey and V. Lakshmibai. *Singular Loci of Schubert Varieties*, vol. 182 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA (2000).

[4] A. Björner and F. Brenti. *Combinatorics of Coxeter Groups*, vol. 231 of *Graduate Texts in Mathematics*. Springer, New York (2005).

[5] F. Brenti. Combinatorics and total positivity. *J. Combin. Theory Ser. A*, **71**, 2 (1995) pp. 175–218.

[6] P. Brosnan and T. Chow. Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. *Adv. Math.*, **329** (2018) pp. 955–1001.

[7] S. Clearman, M. Hyatt, B. Shelton, and M. Skandera. Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements. *Electron. J. Combin.*, **23**, 2 (2016). Paper 2.7, 56 pages.

[8] A. Clearwater and M. Skandera. Total nonnegativity and Hecke algebra trace evaluations. *Ann. Combin.*, **25** (2021) pp. 757–787.

[9] A. Datko and M. Skandera. Combinatorial interpretation of Kazhdan–Lusztig basis elements indexed by 45312-avoiding permutations in $S_6$. *Pure Math. Appl.*, **30** (2022) pp. 68–74.

[10] B. Drake, S. Gerrish, and M. Skandera. Two new criteria for comparison in the Bruhat order. *Electron. J. Combin.*, **11**, 1 (2004). Note 6, 4 pages (electronic).

[11] S. Fallat, M. Gekhtman, and C. Johnson. Multiplicative principal-minor inequalities for totally nonnegative matrices. *Adv. Appl. Math.*, **30**, 3 (2003) pp. 442–470.

[12] C. Gaetz and Y. Gao. On the minimal power of $q$ in a Kazhdan–Lusztig polynomial. *Adv. Math.*, **457** (2024). Paper No. 109941, 25 pages.

[13] M. Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, **6**, 3 (1993) pp. 569–595.

[14] P. Heyfron. Immanant dominance orderings for hook partitions. *Linear Multilinear Algebra*, **24**, 1 (1988) pp. 65–78.

[15] T. Hikita. On the Stanley-Stembridge conjecture. *Sém. Lothar. Combin.*, **93B** (2025) pp. Art. 31, 12.

[16] G. Hobbs, T. Parisi, M. Skandera, and J. Wang. A defect formula for multiplication of Kazhdan–Lusztig basis elements of the type-BC Hecke algebra (2024). Preprint `math.CO/2506.20099` on ArXiv.

[17] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, **53** (1979) pp. 165–184.

[18] M. Konvalinka and M. Skandera. Generating functions for Hecke algebra characters. *Canad. J. Math.*, **63**, 2 (2011) pp. 413–435.

[19] V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in $SL(n)/B$. *Proc. Indian Acad. Sci. (Math Sci.)*, **100**, 1 (1990) pp. 45–52.

[20] A. Lascoux, B. Leclerc, and J. Y. Thibon. Ribbon tableaux, Hall–Littlewood symmetric functions, quantum affine algebras, and unipotent varieties. *J. Math. Phys.*, **38**, 2 (1997) pp. 1041–1068.

[21] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, **5** (1973) pp. 85–90.

[22] D. E. Littlewood. *The Theory of Group Characters and Matrix Representations of Groups*. Oxford University Press, New York (1940).

[23] R. Merris. Single-hook characters and Hamiltonian circuits. *Linear Multilinear Algebra*, **14** (1983) pp. 21–35.

[24] R. Merris and W. Watkins. Inequalities and identities for generalized matrix functions. *Linear Algebra Appl.*, **64** (1985) pp. 223–242.

[25] A. Morales, M. Skandera, and J. Wang. LLT polynomials and Hecke algebra traces. In *Proceedings of the 13th edition of the conference on Random Generation of Combinatorial Structures. Polyominoes and Tilings*, vol. 403 of *Electron. Proc. Theor. Comput. Sci. (EPTCS)*. EPTCS (2024), pp. 150–155.

[26] S. Pan, M. Skandera, and J. Wang. Permanental inequalities and unit interval orders (2026). To appear in proceedings of GASCom 2026 conference.

[27] T. Parisi, M. Skandera, B. Spahiu, and J. Wang. On Kazhdan–Lusztig basis elements having no reversal factorization (2026). Preprint `math.CO/2605.21733` on ArXiv.

[28] J. Shareshian and M. Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In *Configuration Spaces* (A. Bjorner, F. Cohen, C. De Concini, C. Procesi, and M. Salvetti, eds.). Edizione Della Normale, Pisa (2012), pp. 433–460.

[29] M. Skandera. On the dual canonical and Kazhdan–Lusztig bases and 3412, 4231-avoiding permutations. *J. Pure Appl. Algebra*, **212** (2008).

[30] M. Skandera. Characters and chromatic symmetric functions. *Electron. J. Combin.*, **28**, 2 (2021). Research Paper P2.19, 39 pages.

[31] M. Skandera. Generating functions for monomial characters of wreath products $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$. *Enum. Combin. Appl.*, **1**, 2 (2021). Research Paper S2R10, 10 pages.

[32] M. Skandera. Hook immanantal inequalities for totally nonnegative matrices (2025). Preprint `math.CO/2510.00327` on ArXiv.

[33] M. Skandera. Hyperoctahedral group characters and a type-BC analog of graph coloring. *Algebr. Comb.*, **8**, 6 (2025) pp. 1651–1711.

[34] M. Skandera and D. Soskin. Barrett–Johnson inequalities for totally nonnegative matrices. *Linear Multilinear Algebra*, **73**, 10 (2025) pp. 2320–2343.

[35] M. Skandera and D. Soskin. Permanental inequalities for totally positive matrices. *Electron. J. Combin.*, **32**, 1 (2025). Paper No. 1.14, 16.

[36] D. Soskin and M. Gekhtman. On bounded ratios of minors of totally positive matrices. *Linear Algebra Appl.*, **715** (2025) pp. 46–67.

[37] R. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, **111** (1995) pp. 166–194.

[38] R. Stanley. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge (1999).

[39] R. Stanley and J. Stembridge. On immanants of Jacobi–Trudi matrices and permutations with restricted positions. *J. Combin. Theory Ser. A*, **62** (1993) pp. 261–279.

[40] J. Stembridge. Immanants of totally positive matrices are nonnegative. *Bull. London Math. Soc.*, **23** (1991) pp. 422–428.

[41] J. Stembridge. Some conjectures for immanants. *Canad. J. Math.*, **44**, 5 (1992) pp. 1079–1099.

[42] J. Tymoczko. Permutation actions on equivariant cohomology of flag varieties. In *Toric topology*, vol. 460 of *Contemp. Math.* Amer. Math. Soc., Providence, RI (2008), pp. 365–384.

[43] J. Tymoczko. Permutation representations on Schubert varieties. *Amer. J. Math.*, **130**, 5 (2008) pp. 1171–1194.

[44] H. L. Wolfgang, III. *Two interactions between combinatorics and representation theory: Monomial immanants and Hochschild cohomology*. ProQuest LLC, Ann Arbor, MI (1997). Thesis (Ph.D.)–Massachusetts Institute of Technology.