RESEARCH PLAN

MARK SKANDERA

In basic terms, my research in *algebraic combinatorics* concerns nonnegative integers which occur in algebraic settings. Scientific considerations of symmetry, linear systems of equations, and simple counting guarantee that nonnegative integers and algebra have applications in many disciplines. More specifically, one can view character evaluations from four points of view. We consider explicit character evaluations in Section 1, symmetric generating functions in Section 2, coordinate ring generating functions in Section 3, and applications to total nonnegativity in Section 4.

1. Representations of the symmetric group algebra and Hecke algebra

The *symmetric group algebra* $\mathbb{Z}[S_n]$ and the *(Iwahori-) Hecke algebra* $H_n(q)$ have similar presentations as algebras over $\mathbb{Z}$ and $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ respectively, with multiplicative identity elements e and T_e, generators $s_1, \ldots, s_{n-1}$ and $T_{s_1}, \ldots, T_{s_{n-1}}$, and relations

$$\begin{array}{lll} s_i^2 = e & T_{s_i}^2 = (q^{\frac{1}{2}} - q^{\frac{-1}{2}})T_{s_i} + T_e & \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i = s_j s_i s_j & T_{s_i} T_{s_j} T_{s_i} = T_{s_j} T_{s_i} T_{s_j} & \text{for } |i-j| = 1, \\ s_i s_j = s_j s_i & T_{s_i} T_{s_j} = T_{s_j} T_{s_i} & \text{for } |i-j| \geq 2. \end{array}$$

Analogous to the natural basis $\{w \,|\, w \in S_n\}$ of $\mathbb{Z}[S_n]$ is the natural basis $\{T_w \,|\, w \in S_n\}$ of $H_n(q)$, where we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$ whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for w in S_n. We call $\ell = \ell(w)$ the *length* of w. It is known that $\ell(w)$ is equal to $\textsc{inv}(w)$, the number of inversions in the one-line notation $w_1 \cdots w_n$ of w. For $v_1 \cdots v_k \in S_k$, we say that w *avoids the pattern* v if no subsequence $w_{i_1} \cdots w_{i_k}$ of $w_1 \cdots w_n$ consists of letters which appear in the same relative order as $v_1 \cdots v_k$. The specialization of $H_n(q)$ at $q^{\frac{1}{2}} = 1$ is isomorphic to $\mathbb{Z}[S_n]$.

Representations of $\mathbb{Z}[S_n]$ and $H_n(q)$ are often studied in terms of *characters*, the traces of matrix representations. Define the *trace spaces* $\mathcal{T}_n$ and $\mathcal{T}_{n,q}$ to be the $\mathbb{Z}$-span of S_n-characters, and $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of $H_n(q)$-characters, respectively. Each module has rank equal to the number of integer partitions of n. We write $\lambda \vdash n$ or $|\lambda| = n$ to denote that λ is a partition of n. Five well-studied bases of $\mathcal{T}_n$ are the irreducible characters $\{\chi^\lambda \,|\, \lambda \vdash n\}$, induced sign characters $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$, induced trivial characters $\{\eta^\lambda \,|\, \lambda \vdash n\}$, power sum traces $\{\psi^\lambda \,|\, \lambda \vdash n\}$, and monomial traces $\{\phi^\lambda \,|\, \lambda \vdash n\}$. Analogous bases of $\mathcal{T}_{n,q}$ are $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$, $\{\psi_q^\lambda \,|\, \lambda \vdash n\}$, $\{\phi_q^\lambda \,|\, \lambda \vdash n\}$, specializing at $q^{\frac{1}{2}} = 1$ to the $\mathcal{T}_n$-bases.

Irreducible S_n-characters are the most important and least understood basis of $\mathcal{T}_n$: given $w \in S_n$ and $\lambda \vdash n$, there are cancellation-free formulas for $\epsilon^\lambda(w)$, $\eta^\lambda(w)$, $\phi^\lambda(w)$, $\psi^\lambda(w)$, but none for $\chi^\lambda(w)$. On the other hand, all five bases of $H_n(q)$-traces are poorly understood, with no known simple formulas for evaluating the traces on the natural basis of $H_n(q)$. Since S_n- and $H_n(q)$-traces are linear, it would be interesting to evaluate these on *any*

Date: November 13, 2022.

bases, for instance the *(signless) Kazhdan-Lusztig bases* [9] $\{C'_w(1) \,|\, w \in S_n\}$ of $\mathbb{Z}[S_n]$ and $\{q^{\frac{\ell(w)}{2}} C'_w(q) \,|\, w \in S_n\}$ of $H_n(q)$. These bases expand in the natural bases as

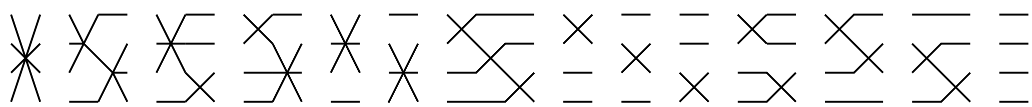
$$q^{\frac{\ell(w)}{2}} C'_w(q) = \sum_{v \in S_n} P_{v,w}(q) T_v \tag{1.1}$$

where coefficients are certain polynomials called *Kazhdan-Lusztig polynomials*.

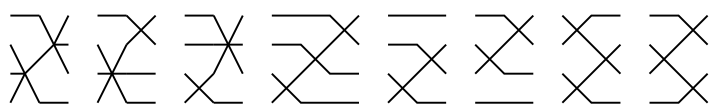
Borrowing a technique called *reversal factorization* from computational biology, the candidate obtained a significant result [15, Thm. 4.3] concerning factorization of Kazhdan-Lusztig basis elements. A *reversal*, denoted $s_{[j,k]}$, is a permutation whose one-line notation contains the letters $k, k-1, \dots, j$ in positions $j, \dots, k$, respectively, and every other letter i in position i. Thus each reversal of the form $s_{[j,j+1]}$ is just the standard generator s_j.

Theorem 1. *If w avoids the patterns 3412 and 4231, then there exists a reversal factorization $w = s_{[i_1,j_1]} \cdots s_{[i_m,j_m]}$ and a rational function f such that the Kazhdan-Lusztig basis element $C'_w(q)$ factors as $C'_w(q) = f(q^{\frac{1}{2}}) C'_{s_{[i_1,j_1]}}(q) \cdots C'_{s_{[i_m,j_m]}}(q)$.*

Since each Kazhdan-Lusztig basis element $C'_{s_{[i,j]}}(q)$ indexed by a reversal has a very simple form, Theorem 1 has led the candidate to advance our ability to combinatorially interpret Kazhdan-Lusztig basis elements. For example, when $n = 4$ the twenty-two Kazhdan-Lusztig basis elements $\{q^{\frac{\ell(w)}{2}} C'_w(q) \,|\, w$ avoids $3412, 4231\}$ can be represented [15] by *zig-zag networks* including the fourteen *descending star networks*

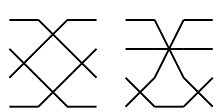
(1.2)  ,

and eight similar networks

(1.3)  .

(In a descending star network, the stars descend from left to right; in a more general zig-zag network, stars can also ascend from left to right, or form a zig-zag pattern as in the last two networks above.) Let $\circ$ denote the concatenation operation on star networks, and let $G_{[a,b]}$ denote the network consisting of a single star formed from wires located in positions $a, \dots, b$ from the bottom of the network. Thus networks 8, 9, 10, 11 in (1.2) are $G_{[3,4]}$, $G_{[2,3]}$, $G_{[1,2]}$, and $G_{[3,4]} \circ G_{[1,2]}$.

Theorem 1 extends to Kazhdan-Lusztig basis elements for some permutations which do not avoid the patterns 3412 and 4231. Specifically, $q^{\frac{\ell(3412)}{2}} C'_{3412}(q)$ and $q^{\frac{\ell(4231)}{2}} C'_{4231}(q)$ can be represented by the star networks

 ,

respectively. Thus the pattern avoidance hypotheses of Theorem 1 appear not to be necessary. On the other hand, some experimentation suggests that it is necessary for w to avoid the pattern 45312. This suggests the following research problem.

Problem 2. *Show that for w avoiding the pattern 45312, the Kazhdan-Lusztig basis element $q^{\frac{\ell(w)}{2}} C'_w(q)$ factors, and state an algorithm to find this factorization.*

The candidate has begun to work on this problem with undergraduate A. Datko, verifying that the desired factorization exists for $w \in S_5 \cup S_6$ [6].

The combinatorial interpretations (1.2) – (1.3) of Kazhdan-Lusztig basis elements played a crucial role in the candidate's work with thesis students Shelton, Clearman and postdoctoral assistant Hyatt on $H_n(q)$-trace evaluations [4]. Theorems 3 – 7 depend upon combinatorial objects called *path tableaux*, which are left- and bottom- justified arrays of paths in the networks (1.2). We assume that each path begins at some boundary vertex (source) on the left of the network and terminates at some boundary vertex (sink) on the right side of the network, and that sources and sinks are labeled $1, \dots, n$ from bottom to top. If a family $\pi = (\pi_1, \dots, \pi_n)$ of such paths has the property that path π_i begins at source i and ends at sink w_i for some $w \in S_n$, we say that π has *type* w.

Let $U_1, \dots, U_r$ be the rows of a path tableau U, say that U has *shape* $(\lambda_1, \dots, \lambda_r)$ if $|U_i| = \lambda_i$, and let $U_1 \circ \cdots \circ U_r$ be the one-rowed tableau which is the concatenation of all rows. Let F_w denote the network corresponding to the Kazhdan-Lusztig basis element $q^{\frac{\ell(w)}{2}} C'_w(q)$. Each of the following formulas for a trace evaluation is a sum, over tableaux having certain properties, of powers of q given by functions $\textsc{inv}$, $\textsc{rinv}$ from tableaux to $\mathbb{N}$. For brevity we omit definitions of these properites and functions. (See [4].)

Theorem 3. *Let $w \in S_n$ avoid the patterns 3412 and 4231, and let λ be a partition of n. We have*

$$\eta_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) = \sum_U q^{\textsc{rinv}(U_1 \circ \cdots \circ U_r)}, \tag{1.4}$$

where the sum is over all row-closed, left row-strict F_w-tableaux of shape λ,

$$\epsilon_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) = \sum_U q^{\textsc{inv}(U)}, \tag{1.5}$$

where the sum is over all column-strict F_w-tableaux of type e and shape $\lambda^\top$, where $\lambda^\top$ is the partition obtained by transposing λ ($\lambda^\top_j = \#\{i \,|\, \lambda_i \geq j\}$),

$$\chi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) = \sum_U q^{\textsc{inv}(U)}, \tag{1.6}$$

where the sum is over all standard F_w-tableaux of type e and shape λ.

Haiman [8] conjectured that the monomial traces satisfy $\phi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) \in \mathbb{N}[q]$ for all w. No formula analogous to (1.4) – (1.6) has been conjectured for general $\lambda \vdash n$, even for w avoiding the patterns 3412 and 4231. On the other hand, the candidate has proved a special case [4].

Theorem 4. *Let $w \in S_n$ avoid the patterns 3412 and 4231. For $\lambda \vdash n$ satisfying $\lambda_1 \leq 2$ we have*

$$\phi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) = \sum_U q^{\textsc{inv}(U)}, \tag{1.7}$$

where the sum is over all column-strict F_w-tableaux of type e and shape $\lambda^\top$, assuming no analogous tableaux of shape $\mu \preceq \lambda^\top$ exist.

Posets related to the Kazhdan-Lusztig basis elements for w avoiding the patterns 3412 and 4231 suggest that an alternative formulation of (1.4) may help to find a formula for $\phi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$ for general $\lambda \vdash n$. (See [4].)

Problem 5. *Find a function* STAT *such that for all $w \in S_n$ avoiding the patterns* 3412 *and* 4231 *and all $\lambda \vdash n$, we have*

$$\eta_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) = \sum_U q^{\mathrm{STAT}(U)},$$

where the sum is over all row-semistrict F_w-tableaux of type e and shape λ.

To formulate the desired function, the candidate plans to use the induction ring on $H_n(q)$ characters and a q-extension of results in [16], including the following.

Theorem 6. *For all $w \in S_n$ avoiding the patterns* 3412 *and* 4231 *and all $\lambda \vdash n$ we have*

$$\eta_q^\lambda(C'_w(1)) = \#\text{row-semistrict } F_w\text{-tableaux of type } e \text{ and shape } \lambda.$$

Another strategy for understanding the monomial traces $\{\phi_q^\lambda \mid \lambda \vdash n\}$ is to consider their relationship to the power sum traces $\{\psi_q^\lambda \mid \lambda \vdash n\}$. We can evaluate ψ_q^λ in four ways: two employing properties of F-tableaux proposed by Shareshian and Wachs [14], and two employing properties of F-tableaux proposed by the candidate. Building upon results of Shareshian and Wachs and Athanasiadis [1], the candidate and assistants have proved the following [4]. Let U_i^R be the ith row of tableau U, written in reverse, and let $[m]_q = 1 + q + \cdots + q^{m-1}$. Again for brevity we omit definitions of several tableau properites. (See [4].)

Theorem 7. *Let $w \in S_n$ avoid the patterns* 3412 *and* 4231, *and let λ be a partition of n. We have*

$$\psi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) = \sum_U q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)},$$

where the sum is over all record-free, row-semistrict tableaux of type e and shape λ,

$$\psi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))[\lambda_1]_q \cdots [\lambda_r]_q \sum_U q^{\mathrm{INV}(U_1^R \circ \cdots \circ U_r^R)},$$

where the sum is over all right-anchored, row-semistrict tableaux of type e and shape λ,

$$\psi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) = \sum_U q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)},$$

where the sum is over all cylindrical tableaux of shape λ, and

$$\psi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) = [\lambda_1]_q \cdots [\lambda_r]_q \sum_U q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)},$$

where the sum is over all left-anchored, cylindrical tableaux of shape λ.

As in the motivation for Problem 5, consideration of relevant posets suggests that an alternative formulation of results in Theorem 7 may help to find a formula for $\phi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$ [4].

Problem 8. *Find a function* STAT *for which we have*

$$\psi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) = \sum_U q^{\mathrm{STAT}(U)},$$

where the sum is over all cyclically row-semistrict F_w-tableaux of type e and shape λ.

Again, the candidate plans to use the induction ring on $H_n(q)$ characters and a q-extension of results in [16] to state the desired function.

Generalizing the formula (1.5), the candidate and A. Clearwater [5, Thm. 6.5] have found a formula for evaluating induced sign characters ϵ_q^λ at products of Kazhdan-Lusztig basis elements such as those appearing in Theorem 1, whether or not the products themselves are Kazhdan-Lusztig basis elements. In the following, CR and INVNC are statistics on path families and tableaux, respectively.

Theorem 9. *Fix intervals* $J_1, \dots, J_m \subset [1, n]$ *and let* $G = G_{J_1} \circ \cdots \circ G_{J_m}$. *Then for* $\lambda \vdash n$ *we have*

$$\epsilon_q^\lambda(q^{\frac{|J_1|}{2}} C'_{s_{J_1}}(q) \cdots q^{\frac{|J_m|}{2}} C'_{s_{J_m}}(q)) = \sum_{\pi} q^{\frac{\textsc{cr}(\pi)}{2}} \sum_{W} q^{\textsc{invnc}(W)}, \tag{1.8}$$

where the sums are over path families π *of type e which cover* G, *and column-strict* π*-tableaux* W *of shape* $\lambda^\top$.

This result is a weakening of the problem of finding a combinatorial interpretation of the Stembridge-Haiman result that we have

$$\chi_q^\lambda(q^{\frac{|J_1|}{2}} C'_{s_{J_1}}(q) \cdots q^{\frac{|J_m|}{2}} C'_{s_{J_m}}(q)) \in \mathbb{N}[q] \tag{1.9}$$

for all interval sequences $J_1, \dots, J_m \subset [1, n]$. The next natural step in finding this combinatorial interpretation is the following.

Problem 10. *Fix intervals* $J_1, \dots, J_m \subset [1, n]$, *let* $G = G_{J_1} \circ \cdots \circ G_{J_m}$, *and fix* $\lambda \vdash n$. *Find a formula for*

$$\eta_q^\lambda(q^{\frac{|J_1|}{2}} C'_{s_{J_1}}(q) \cdots q^{\frac{|J_m|}{2}} C'_{s_{J_m}}(q)).$$

When $q = 1$, two nice formulas describe the evaluations in terms of paths families in planar networks and extensions of the permutation statistics exc and des to these. It is likely that for general q, much of the required proof will use path families, bijective methods, and permutations.

2. Symmetric generating functions for character evaluations

Elements of the module Λ_n of homogeneous degree-n symmetric functions in $y_1, y_2, \dots$ are usually expressed in terms of bases indexed by partitions of all nonnegative integers. (See, e.g., [21, Ch. 7].) Common bases are the *Schur* basis $\{s_\lambda \,|\, \lambda \vdash n\}$, *monomial* basis $\{m_\lambda \,|\, \lambda \vdash n\}$, *elementary* basis $\{e_\lambda \,|\, \lambda \vdash n\}$, *homogeneous* basis $\{h_\lambda \,|\, \lambda \vdash n\}$, *power sum* basis $\{p_\lambda \,|\, \lambda \vdash n\}$, and *forgotten* basis $\{f_\lambda \,|\, \lambda \vdash n\}$. The candidate has defined a symmetric generating function for the character evaluations of each element $g \in H_n(q)$,

$$Y_q(g) := \sum_{\lambda \vdash n} \epsilon_q^\lambda(g) m_\lambda = \sum_{\lambda \vdash n} \phi_q^\lambda(g) e_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(g) s_\lambda = \sum_{\lambda \vdash n} d_\lambda \psi_q^\lambda(g) p_\lambda = \sum_{\lambda \vdash n} \eta_q^\lambda(g) f_\lambda,$$

where equalities follow from transition matrices for the bases, and where $\{d_\lambda \,|\, \lambda \vdash n\}$ are certain rational numbers. This function has the following universal property [16, Prop. 2.3].

Proposition 11. *Every element of* $\mathbb{Z}[q] \otimes \Lambda_n$ *has the form* $Y_q(g)$ *for some* $g \in \mathbb{Q}(q) \otimes H_n(q)$.

A well-known problem concerns symmetric functions $X_{P,q} \in \mathbb{Z}[q] \otimes \Lambda_n$ defined in terms of certain posets P called *natural unit interval orders* in [14]. (See also [20], [22].) The expansion of $X_{P,q}$ as a polynomial in q with symmetric function coefficients describes an action of S_n on an algebraic variety called the (type-A) *Hessenberg variety* associated to P. (See [24]–[23].) These functions are conjectured [14], [20], [22] to be $\mathbb{N}[q]$-linear combinations of elementary symmetric functions. The candidate and graduate student B. Shelton proved that the expansion of $X_{P,q}$ in each standard symmetric function basis yields coefficients which are equal to $H_n(q)$-trace evaluations at Kazhdan-Lusztig basis elements [4, Sec. 7].

Theorem 12. *For each n-element unit interval order P, there exists a 312-avoiding permutation $w \in S_n$ such that $X_{P,q} = Y_q(q^{\frac{\ell(w)}{2}} C'_w(q))$.*

It is natural to ask how Theorem 12 or its $q = 1$ specialization can be extended to combinatorially describe type-B character evaluations. In particular, it is possible to define type-B symmetric functions which are generating functions

$$Y^B_q(g) = \sum_{(\lambda,\mu)} \epsilon^{\lambda,\mu}_q(g) m_\lambda(x) m_\mu(y) \tag{2.1}$$

for characters of the Hecke algebra $H^B_n(q)$ of the hyperoctahedral group S^B_n. We write Y^B for the specialization Y^B_1. It is also possible to define type-B chromatic symmetric functions X^B_P for structures which are a type-B analog of posets. The candidate has extended Proposition 11 and Theorem 12 to the following type-B analogs [18].

Proposition 13. *Every degree-n homogeneous type-B symmetric function with integer coefficients is $Y^B_q(g)$ for some element $g \in \mathbb{Q}(q) \otimes H^B_n(q)$.*

Theorem 14. *For each type-B unit interval order there exists a 3412-avoiding, 4231-avoiding element $w \in S^B_n$ such that we have $X^B_P = Y^B(C'_w(1))$.*

It would be interesting to provide a q-extension of X^B_P which is a generating function for trace evaluations at type-B Kazhdan-Lusztig basis elements. The candidate proposes to do the following.

Problem 15. *Define a q-extension $X^B_{P,q}$ of the type-B chromatic symmetric function X^B_P which combinatorially interprets type-B trace evaluations such as $(\epsilon\epsilon)^{\lambda,\mu}_q(q^{\frac{\ell(w)}{2}} C'_w(q))$.*

It would be even more interesting if this result were related to the type-B Hessenberg variety as $X_{P,q}$ is related to the type-A Hessenberg variety.

3. Generating functions in coordinate rings

Often it is useful to have a generating function which records values $\{\theta(w) \,|\, w \in S_n\}$ for some S_n-character θ, i.e.,

$$\mathrm{Imm}_\theta(x) := \sum_{w \in S_n} \theta(w) x_{1,w_1} \cdots x_{n,w_n}. \tag{3.1}$$

In particular, a simple expression for such a generating function can serve as a satisfactory formula for $\theta(w)$.

The candidate and Konvalinka stated generating functions for the $H_n(q)$-characters ϵ^λ_q, η^λ_q [10, Thm. 5.4] in a quotient of the noncommutative ring $\mathbb{Z}\langle x \rangle$, where $x = (x_{i,j})_{i,j \in [1,n]}$ may be thought as a matrix of variables. Each generating function is expressed in terms of submatrices of x, polynomials called the q-determinant and q-permanent, and *ordered set partitions of type* $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, sequences $(I_1, \ldots, I_r)$ of disjoint subsets of $[1, n]$ satisfying $|I_j| = \lambda_j$. The specializations of these formulas at $q = 1$ belong to $\mathbb{Z}[x]$ and are precisely the Littlewood–Merris–Watkins generating functions for S_n-characters [12], [13].

Theorem 16. *For* $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ *we have*

$$\mathrm{Imm}_{\epsilon_q^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}), \tag{3.2}$$

$$\mathrm{Imm}_{\eta_q^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} \mathrm{per}_q(x_{I_1,I_1}) \cdots \mathrm{per}_q(x_{I_r,I_r}), \tag{3.3}$$

where $x_{K,K} = (x_{i,j})_{i,j \in K}$*, and sums are over ordered set partitions of type* λ.

The candidate also generalized the Littlewood–Merris–Watkins identities for induced one-dimensional characters of wreath products $\mathcal{G}_{n,d} := \mathbb{Z}/d\mathbb{Z} \wr S_n$ [17, Thm. 3.1]. Such characters correspond to d-tuples $\boldsymbol{\beta}^{\boldsymbol{\lambda}} = (\beta_0^{\lambda^0}, \ldots, \beta_{d-1}^{\lambda^{d-1}})$ of symmetric group characters with $\beta_i \in \{\eta, \epsilon\}$ and $\boldsymbol{\lambda} = (\lambda^0, \ldots, \lambda^{d-1})$ a d-tuple of partitions with $|\lambda^0| + \cdots + |\lambda^{d-1}| = n$. Elements of $\mathcal{G}_{n,d}$ have the form $g = (g_1, \ldots, g_n) = (\zeta^{\gamma_1} w_1, \ldots, \zeta^{\gamma_n} w_n)$ for ζ a dth root of unity, $w_1 \cdots w_n \in S_n$, and $(\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}/d\mathbb{Z}^n$. The new generating functions belong to the ring $\mathbb{C}[x]$ in the dn^2 indeterminates $x = \{x_{i,\zeta^k p} \,|\, i, p \in [1, n], k \in \mathbb{Z}/d\mathbb{Z}\}$ and are defined by

$$\mathrm{Imm}_{\boldsymbol{\beta}^{\boldsymbol{\lambda}}}^{\mathcal{G}_{n,d}}(x) := \sum_{g \in \mathcal{G}_{n,d}} \boldsymbol{\beta}^{\boldsymbol{\lambda}}(g) x_{1,g_1} \cdots x_{n,g_n}.$$

Each is a sum of products of $S_{|\lambda^0|}, \ldots, S_{|\lambda^{d-1}|}$ character immanants (specializations of (3.2) – (3.3) at $q = 1$) of matrices $Q_0(x), \ldots, Q_{d-1}(x)$ defined by $Q_k(x) = (q_{i,j,k}(x))_{i,j \in [1,n]}$, where

$$q_{i,j,k}(x) = x_{i,j} + \zeta^{-k} x_{i,\zeta j} + \zeta^{-2k} x_{i,\zeta^2 j} + \cdots + \zeta^{-(d-1)k} x_{i,\zeta^{(d-1)} j}.$$

Theorem 17. *Fix* $\mathcal{G}_{n,d}$*-character* $\boldsymbol{\beta}^{\boldsymbol{\lambda}} = (\beta_0^{\lambda^0}, \ldots, \beta_{d-1}^{\lambda^{d-1}})$ *as above. Then we have*

$$\mathrm{Imm}_{\boldsymbol{\beta}^{\boldsymbol{\lambda}}}^{\mathcal{G}_{n,d}}(x) = \sum_{(I_0,\ldots,I_{d-1})} \mathrm{Imm}_{\beta_0^{\lambda^0}}(Q_0(x)_{I_0,I_0}) \cdots \mathrm{Imm}_{\beta_{d-1}^{\lambda^{d-1}}}(Q_{d-1}(x)_{I_{d-1},I_{d-1}}), \tag{3.4}$$

where the sum is over all ordered set partitions of type $(|\lambda^0|, \ldots, |\lambda^{d-1}|)$.

Theorem 16 is an essential ingredient in the proofs of Theorems 3 – 12. Similarly, since the hyperoctahedral group S_n^B is isomorphic to the wreath product $\mathcal{G}_{n,2}$, Theorem 17 is essential in the proof of Theorem 14.

To help solve Problem 15, it would be helpful to state a q-extension of Theorem 17, at least in the case that $d = 2$.

Problem 18. *State formulas in an appropriate ring for* $\mathrm{Imm}_{(\epsilon\epsilon)_q^{\lambda,\mu}}^{\mathcal{G}_{n,2}}(x)$, $\mathrm{Imm}_{(\epsilon\eta)_q^{\lambda,\mu}}^{\mathcal{G}_{n,2}}(x)$, $\mathrm{Imm}_{(\eta\epsilon)_q^{\lambda,\mu}}^{\mathcal{G}_{n,2}}(x)$, $\mathrm{Imm}_{(\eta\eta)_q^{\lambda,\mu}}^{\mathcal{G}_{n,2}}(x)$ *to provide type-B analogs of Theorem 16.*

It is likely that such a result will also lead to a B-analog of Theorem 9, stating a combinatorial interpretation for evaluations of $(\epsilon\epsilon)_q^{\lambda,\mu}$ at products of type-B Kahzdan-Lusztig basis elements.

4. Applications to total nonnegativity

Results in Sections 1 – 3 have applications to the class of *totally nonnegative* (TNN) matrices, those matrices $A = (a_{i,j})$ satisfying $\det(A_{I,J}) \geq 0$ for all square submatrices $A_{I,J} := (a_{i,j})_{i \in I, j \in J}$. For instance Theorem 4 allowed the candidate to prove inequalities for the average values of products $\det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r})$, taken over all ordered set partitions $(I_1, \ldots, I_r)$ of a fixed type, when A is TNN. This result [19] extends earlier work of Barrett and Johnson [2] for positive semidefinite matrices.

Theorem 19. *Fix $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, $\mu = (\mu_1, \ldots, \mu_s) \vdash n$. The inequality*

$$\sum_{\substack{(I_1,\ldots,I_r) \\ \text{of type } \lambda}} \frac{\det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r})}{\binom{n}{\lambda_1,\ldots,\lambda_r}} \leq \sum_{\substack{(J_1,\ldots,J_s) \\ \text{of type } \mu}} \frac{\det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s})}{\binom{n}{\mu_1,\ldots,\mu_r}}$$

holds for all totally nonnegative matrices A if and only if $\mu \preceq \lambda$ in the majorization order, i.e., $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ for all i.

Theorem 6 allowed the candidate to state two new combinatorial interpretations of the permanent of a TNN matrix. It is known that a matrix $A = (a_{i,j})$ is TNN if and only if it may be interpreted as the path matrix of a weighted planar network G, with $a_{i,j}$ being the sum of weights of all paths in G between vertices called *source* i and *sink* j [3]. The following [16] is a permanental analog of Lindström's Lemma [11]. (See [16] for definitions.)

Theorem 20. *Let A be the path matrix of planar network G. Then* $\mathrm{per}(A)$ *equals the number of path tableaux U of shape n containing a path family π in G of type e, and having one of the properties*

(1) U is descent-free,
(2) U is excedance-free.

Theorem 20 may be useful in proving a permanental analog of Theorem 19.

Problem 21. *Decide if for fixed $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, $\mu = (\mu_1, \ldots, \mu_s) \vdash n$, the inequality*

$$\sum_{\substack{(I_1,\ldots,I_r) \\ \text{of type } \lambda}} \frac{\mathrm{per}(A_{I_1,I_1}) \cdots \mathrm{per}(A_{I_r,I_r})}{\binom{n}{\lambda_1,\ldots,\lambda_r}} \geq \sum_{\substack{(J_1,\ldots,J_s) \\ \text{of type } \mu}} \frac{\mathrm{per}(A_{J_1,J_1}) \cdots \mathrm{per}(A_{J_s,J_s})}{\binom{n}{\mu_1,\ldots,\mu_r}}$$

holds for all totally nonnegative matrices A if and only if $\mu \preceq \lambda$ in the majorization order.

Theorem 20 may also be useful in proving permanental analogs of the Fallat–Gekhtman–Johnson inequalities [7] for products $\det(A_{I,I}) \det(A_{\overline{I},\overline{I}})$, where $\overline{I} := [1, n] \smallsetminus I$.

Problem 22. *Decide if sets $I, J \subseteq [1, n]$ satisfy*

$$\mathrm{per}(A_{I,I})\mathrm{per}(A_{\overline{I},\overline{I}}) \leq \mathrm{per}(A_{J,J})\mathrm{per}(A_{\overline{J},\overline{J}})$$

for all TNN matrices if and only if they satisfy

$$\det(A_{I,I}) \det(A_{\overline{I},\overline{I}}) \geq \det(A_{J,J}) \det(A_{\overline{J},\overline{J}})$$

for all TNN matrices.

References

[1] C. A. Athanasiadis. Power sum expansion of chromatic quasisymmetric functions. *Electron. J. Combin.*, **22**, 2 (2015). Paper 2.7, 9 pages.

[2] W. W. Barrett and C. R. Johnson. Majorization monotonicity of symmetrized Fischer products. *Linear and Multilinear Algebra*, **34**, 1 (1993) pp. 67–74.

[3] F. Brenti. Combinatorics and total positivity. *J. Combin. Theory Ser. A*, **71**, 2 (1995) pp. 175–218.

[4] S. Clearman, M. Hyatt, B. Shelton, and M. Skandera. Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements. *Electron. J. Combin.*, **23**, 2 (2016). Paper 2.7, 56 pages.

[5] A. Clearwater and M. Skandera. Total nonnegativity and Hecke algebra trace evaluations. *Ann. Combin.*, **25** (2021) pp. 757–787.

[6] A. Datko and M. Skadera. Combinatorial interpretation of Kazhdan-Lusztig basis elements indexed by 45312-avoiding permutations in s_6. *Pure Math. Appl.*, **30** (2022) pp. 68–74.

[7] S. M. Fallat, M. I. Gekhtman, and C. R. Johnson. Multiplicative principal-minor inequalities for totally nonnegative matrices. *Adv. Appl. Math.*, **30**, 3 (2003) pp. 442–470.

[8] M. Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, **6**, 3 (1993) pp. 569–595.

[9] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, **53** (1979) pp. 165–184.

[10] M. Konvalinka and M. Skandera. Generating functions for Hecke algebra characters. *Canad. J. Math.*, **63**, 2 (2011) pp. 413–435.

[11] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, **5** (1973) pp. 85–90.

[12] D. E. Littlewood. *The Theory of Group Characters and Matrix Representations of Groups*. Oxford University Press, New York (1940).

[13] R. Merris and W. Watkins. Inequalities and identities for generalized matrix functions. *Linear Algebra Appl.*, **64** (1985) pp. 223–242.

[14] J. Shareshian and M. Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In *Configuration Spaces* (A. Bjorner, F. Cohen, C. D. Concini, C. Procesi, and M. Salvetti, eds.). Edizione Della Normale, Pisa (2012), pp. 433–460.

[15] M. Skandera. On the dual canonical and Kazhdan-Lusztig bases and 3412, 4231-avoiding permutations. *J. Pure Appl. Algebra*, **212** (2008).

[16] M. Skandera. Characters and chromatic symmetric functions. *Electron. J. Combin.*, **28**, 2 (2021) pp. Research Paper P2.19, approx. 39 pp. (electronic).

[17] M. Skandera. Generating functions for monomial characters of wreath products $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$. *Enum. Combin. Appl.*, **1**, 2 (2021) pp. Research Paper S2R10, approx. 10 pp. (electronic).

[18] M. Skandera. Symmetric generating functions for hyperoctahedral group characters (2021). In progress.

[19] M. Skandera and D. Soskin. Barrett–Johnson inequalities for totally nonnegative matrices (2022). Preprint math.CO/2209.06466 on ArXiv.

[20] R. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, **111** (1995) pp. 166–194.

[21] R. Stanley. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge (1999).

[22] R. Stanley and J. R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted positions. *J. Combin. Theory Ser. A*, **62** (1993) pp. 261–279.

[23] J. S. Tymoczko. Permutation actions on equivariant cohomology of flag varieties. In *Toric topology*, vol. 460 of *Contemp. Math.*. Amer. Math. Soc., Providence, RI (2008), pp. 365–384.

[24] J. S. Tymoczko. Permutation representations on Schubert varieties. *Amer. J. Math.*, **130**, 5 (2008) pp. 1171–1194.