RESEARCH PLAN

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Many current problems in algebra and combinatorics concern the Hecke algebras of Coxeter groups. My work for the past ten years has been to provide formulas for the evaluations of Hecke algebra characters and to explain the connections between these, symmetric functions, and the theory of total nonnegativity. To continue this work I hope to collaborate with faculty in algebra and discrete mathematics and to supervise student research.

1. Generating functions for characters

Let $\{s_1, \ldots, s_{n-1}\}$ and $\{T_{s_1}, \ldots, T_{s_{n-1}}\}$ be the standard generators for the symmetric group algebra $\mathbb{Z}[S_n]$ and the related Hecke algebra $H_n(q)$. As a $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module, $H_n(q)$ is spanned by elements of the form $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$, where $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for $w \in S_n$, and $\ell = \ell(w)$ is called the *length* of w. Characters of $H_n(q)$ form a $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module of rank equal to the number of integer partitions of n and specialize at $q = 1$ to $\mathbb{Z}[S_n]$-characters, since $H_n(1) \cong \mathbb{Z}[S_n]$. Let $\lambda \vdash n$ denote that λ is a partition of n. Three important bases of this module are the irreducible characters $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, induced sign characters $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, and induced trivial characters $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$. (See, e.g., [3].)

The simplicity of known evaluation formulas varies for the bases above and their $q = 1$ specializations. We have cancellation-free formulas for $\epsilon^\lambda(w)$, $\eta^\lambda(w)$, but no such known formulas for $\epsilon_q^\lambda(T_w)$, $\eta_q^\lambda(T_w)$. Algorithms for $\chi^\lambda(w)$, $\chi_q^\lambda(T_w)$ involve cancellation. Alternatively, generating functions give evaluations for $\chi^\lambda(w)$ [7] and $\epsilon^\lambda(w)$, $\eta^\lambda(w)$ [13], [15]. In joint work with Konvalinka, I provided q-analogs of these generating functions [12]. Let $\mathcal{A}_{[n],[n]}$ be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module of degree-$n$ monomials $\{x_{v_1,w_1} \cdots x_{v_n,w_n} \,|\, v, w \in S_n\}$ in noncommuting variables, modulo the subspace generated by the relations $x_{i,\ell}x_{j,k} = x_{j,k}x_{i,\ell}$ and $x_{j,k}x_{i,k} = x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}$ when $i < j$ and $k < \ell$. Define the q-determinant of the $n \times n$ matrix $x = (x_{i,j})$ by

$$\det_q(x) = \sum_{w \in S_n} q^{\frac{-\ell(w)}{2}} x_{1,w_1} \cdots x_{n,w_n},$$

and for subsets A, B of $[n] = \{1, \ldots, n\}$, define the submatrix $x_{A,B} = (x_{i,j})_{i \in A, j \in B}$ of x.

Theorem 1. *For $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ we have*

$$\sum_{w \in S_n} \epsilon_q^\lambda(T_w) q^{\frac{-\ell(w)}{2}} x_{1,w_1} \cdots x_{n,w_n} = \sum_{(I_1, \ldots, I_r)} \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}),$$

where the last sum is over all ordered set partitions $(I_1, \ldots, I_r)$ of $[n]$ satisfying $|I_j| = \lambda_j$.

The generating function for $\eta_q^\lambda(T_w)$ is similar, while that for $\chi_q^\lambda(T_w)$ is more intricate.

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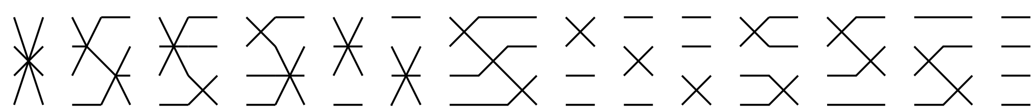
2. Evaluations of characters at Kazhdan-Lusztig basis elements

Considerations in quantum groups led Kazhdan and Lusztig [11] to define another basis $\{\widetilde{C}_w \,|\, w \in S_n\}$ of $H_n(q)$ by

$$\widetilde{C}_w = \sum_{v \leq w} P_{v,w}(q) T_v,$$

where $\leq$ denotes the Bruhat order and $\{P_{v,w}(q) \,|\, v, w \in S_n\}$ are certain recursively defined polynomials in $\mathbb{N}[q]$. Haiman [9] proved that $\chi_q^\lambda(\widetilde{C}_w)$ (and thus $\epsilon_q^\lambda(\widetilde{C}_w)$, $\eta_q^\lambda(\widetilde{C}_w)$) belong to $\mathbb{N}[q]$, but provided no formula. Indeed, since our understanding of irreducible characters and Kazhdan-Lusztig basis elements is algorithmic, one would not expect to have a nice formula for $\chi_q^\lambda(\widetilde{C}_w)$.

Nevertheless, my factorization result [17, Thm 4.3] for $\widetilde{C}_w$ when the one-line notation of w avoids the patterns 3412 and 4231 leads to a combinatorial formula in the case of this pattern avoidance. (See, e.g., [2].) For $w \in S_n$ avoiding the patterns 3412 and 4231, we can represent $\widetilde{C}_w$ by a graph I call a *descending star network*. For example, when $n = 4$, these are the fourteen graphs

(2.1)  ,

One can uniquely cover such a graph by a family $\pi = (\pi_1, \ldots, \pi_n)$ of n noncrossing paths, and can place these paths into a Young tableau which satisfies one or both of the conditions

$$\begin{array}{|c|}\hline \pi_j \\ \hline \pi_i \\ \hline \end{array} \Rightarrow \begin{array}{c}\pi_i \text{ lies entirely} \\ \text{below } \pi_j,\end{array} \qquad \begin{array}{|c|c|}\hline \pi_i & \pi_j \\ \hline \end{array} \Rightarrow \begin{array}{c}\pi_i \text{ intersects or} \\ \text{lies entirely below } \pi_j.\end{array}$$

Call the resulting structure a π-*tableau* which is *column-strict* if it satisfies the first condition, *row-semistrict* if it satisfies the second, and *standard* if it satisfies both. Certain statistics INV, INV$'$ on these π-tableaux then allow for the following formulas, which I proved with two graduate students [5].

Theorem 2. *Let $w \in S_n$ avoid the patterns 3412 and 4231, and let π be the unique non-crossing path family covering the descending star network corresponding to w. For $\lambda \vdash n$ and its transpose partition $\lambda^\top$ we have*

$$\epsilon_q^\lambda(\widetilde{C}_w) = \sum_U q^{\mathrm{INV}(U)}, \qquad \eta_q^\lambda(\widetilde{C}_w) = \sum_U q^{\mathrm{INV}'(U)}, \qquad \chi_q^\lambda(\widetilde{C}_w) = \sum_U q^{\mathrm{INV}(U)},$$

where the sums are over column-strict π-tableaux of shape $\lambda^\top$, row-semistrict π-tableaux of shape λ, and standard π-tableaux of shape λ, respectively.

Haiman [9] has conjectured that for *each* $w \in S_n$ there exists a set $\{v^{(1)}, \ldots, v^{(k)}\}$ of 3412-avoiding, 4231-avoiding permutations which satisfy $\theta_q(\widetilde{C}_w) = \theta_q(\widetilde{C}_{v^{(1)}}) + \cdots + \theta_q(\widetilde{C}_{v^{(k)}})$ for each $H_n(q)$-character θ_q. Thus it is conceivable that Theorem 2 could be extended to the entire Kazhdan-Lusztig basis, although it is not currently known how to determine the set $\{v^{(1)}, \ldots, v^{(k)}\}$ from w.

The evaluations in Theorem 2 have applications to the study of regular semisimple Hessenberg varieties of type A. Such a variety $\mathcal{H}$ corresponds to a unit interval order $P = P(\mathcal{H})$, and by [16], [4], [8] the cohomology of $\mathcal{H}$ has a character which corresponds via the characteristic map to a symmetric function $X_{P,q}$ describing colorings of P. I proved [5] that the coefficients appearing in various symmetric function expansions of $X_{P,q}$ are precisely the character evaluations in Theorem 2.

Theorem 3. *For each unit interval order P, there is a 312-avoiding permutation w such that the monomial, Schur, and forgotten expansions of $X_{P,q}$ are*

$$X_{P,q} = \sum_{\lambda \vdash n} \epsilon_q^\lambda(\widetilde{C}_w) m_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(\widetilde{C}_w) s_\lambda = \sum_{\lambda \vdash n} \eta_q^\lambda(\widetilde{C}_w) f_\lambda.$$

It follows that coefficients in the power sum and elementary expansions of $X_{P,q}$ are given by similar (but non-character) trace evaluations.

3. Total nonnegativity

Work of Lusztig [14] implies that in $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$, certain polynomials which are related to the dual canonical basis of the quantum group $\mathcal{O}_q(SL(n,\mathbb{C}))$ have a property called *total nonnegativity.* A matrix is called totally nonnegative (TNN) if each minor (determinant of a square submatrix) is nonnegative. A polynomial $p(x_{1,1}, x_{1,2}, \ldots, x_{n,n})$ is called totally nonnegative if we have $p(A) := p(a_{1,1}, a_{1,2}, \ldots, a_{n,n}) \geq 0$ for each TNN matrix A. While deciding if a matrix is TNN is straightforward, no known algorithm decides if a polynomial is TNN, even in the special case that the polynomial belongs to $\mathrm{span}_{\mathbb{Z}}\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in S_n\}$. Work of Stembridge [19], [20], Haiman [9], and others [1], [18] implies that TNN polynomials are connected to $H_n(q)$-characters via permutations $\{s_{[a,b]} \,|\, 1 \leq a < b \leq n\}$ called *reversals*, whose one-line notations have the form

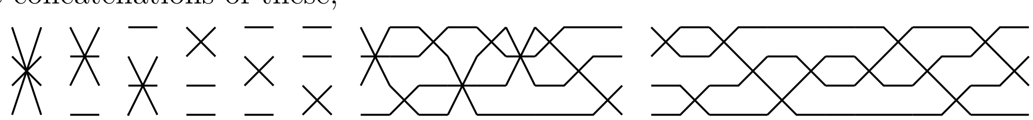
$$s_{[a,b]} = 1 \cdots (a-1) b (b-1) \cdots a (b+1) \cdots n.$$

In particular, we have the following implications.

Proposition 4. *Given a linear function $\theta_q : H_n(q) \to \mathbb{Z}[q]$ and its $q = 1$ specialization θ, each of the following statements implies the next.*

1. *For all $w \in S_n$, we have $\theta_q(\widetilde{C}_w) \in \mathbb{N}[q]$.*
2. *For all $m > 0$ and all sequences $(s_{[i_1,j_1]}, \ldots, s_{[i_m,j_m]})$ of reversals in S_n, we have $\theta_q(\widetilde{C}_{s_{[i_1,j_1]}} \cdots \widetilde{C}_{s_{[i_m,j_m]}}) \in \mathbb{N}[q]$.*
3. *The polynomial $\mathrm{Imm}_\theta(x) := \sum_{w \in S_n} \theta(w) x_{1,w_1} \cdots x_{n,w_n}$ is TNN.*

Haiman proved [9, Lem. 1.1] that for each irreducible character χ^λ and each $w \in S_n$, we have $\chi_q^\lambda(\widetilde{C}_w) \in \mathbb{N}[q]$, which implies that the polynomial $\mathrm{Imm}_{\chi^\lambda}(x)$ is TNN. However, his proof does not give a combinatorial interpretation for the coefficients of the resulting polynomials $\chi_q^\lambda(\widetilde{C}_w) \in \mathbb{N}[q]$, the coefficients of the polynomials $\chi_q^\lambda(\widetilde{C}_{s_{[i_1,j_1]}} \cdots \widetilde{C}_{s_{[i_m,j_m]}}) \in \mathbb{N}[q]$, or the nonnegative number $\mathrm{Imm}_{\chi^\lambda}(A)$, when A is a TNN matrix. To provide the first such combinatorial interpretation of statments (2) and (3) of the proposition when θ_q is an $H_n(q)$-character, I represented products of Kazhdan-Lusztig basis elements of the forms $\widetilde{C}_{s_{[i,j]}}$ by concatenations of the simplest graphs appearing in (2.1). For example, when $n = 4$, these are six simple stars corresponding to elements $\widetilde{C}_{s_{[1,4]}}, \widetilde{C}_{s_{[2,4]}}, \widetilde{C}_{s_{[1,3]}}, \widetilde{C}_{s_{[3,4]}}, \widetilde{C}_{s_{[2,3]}}, \widetilde{C}_{s_{[1,2]}}$, and all finite concatenations of these,

(3.1)  , . . .

With graduate students and a postdoc I introduced statistics CR on path families and INVNC on path tableaux to interpret induced sign character evaluations as follows [10], [6].

Theorem 5. *Let star network G represent the product $\widetilde{C}_{s_{[i_1,j_1]}} \cdots \widetilde{C}_{s_{[i_m,j_m]}} \in H_n(q)$. For all $\lambda \vdash n$ we have*
$$\epsilon_q^\lambda(\widetilde{C}_{s_{[i_1,j_1]}} \cdots \widetilde{C}_{s_{[i_m,j_m]}}) = \sum_\pi q^{\frac{\text{CR}(\pi)}{2}} \sum_U q^{\text{INVNC}(U)},$$
where the first sum is over path families π covering G, and the second sum is over column-strict π-tableaux of shape $\lambda^\top$.

This is the first combinatorial formula applying to all evaluations of the form $\theta_q(g)$ where θ_q varies over a basis of the $H_n(q)$-character space and g varies over a spanning set of $H_n(q)$.

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