RELATIONS BETWEEN THE CLAUSEN AND KAZHDAN-LUSZTIG REPRESENTATIONS OF $\mathfrak{S}_n$

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Abstract. We use Kazhdan-Lusztig polynomials and subspaces of the polynomial ring $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ to construct irreducible $\mathfrak{S}_n$-modules. This construction produces exactly the same matrices as the Kazhdan-Lusztig construction [*Invent. Math* **53** (1979)], but does not employ the Kazhdan-Lusztig preorders. It also produces exactly the same modules as those which Clausen constructed using a different basis in [*J. Symbolic Comput.* **11** (1991)]. We show that the two resulting matrix representations are related by a unitriangular transition matrix. This provides a $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$-analog of results due to Garsia and McLarnan [*Adv. Math.* **69** (1988)], and McDonough and Pallikaros [*J. Pure Appl. Alg.* **203** (2005)].

1. Introduction

In 1979, Kazhdan and Lusztig [15] introduced a family of irreducible modules for Coxeter groups and related Hecke algebras. These modules, which have many fascinating properties, also aid in the understanding of modules for quantum groups and other algebras. Important steps in the construction of the Kazhdan-Lusztig modules are the computation of certain polynomials in $\mathbb{Z}[q]$ known as *Kazhdan-Lusztig polynomials*, and the description of preorders on Coxeter group elements known as the *Kazhdan-Lusztig preorders.* These two tasks have become fascinating research topics in their own right. For even the simplest case of a Coxeter group and corresponding Hecke algebra, the symmetric group $\mathfrak{S}_n$ and type-A Hecke algebra $H_n(q)$, the Kazhdan-Lusztig polynomials and preorders are somewhat poorly understood. (See, e.g., [1], [28] and references listed there.) These difficulties have led authors to study irreducible $\mathfrak{S}_n$-representations indexed by partitions λ of n and to search for a connection between the matrices $\{X_1^\lambda(w) \,|\, w \in \mathfrak{S}_n\}$ arising from the Kazhdan-Lusztig representation and those arising from other more elementary represention s.

One elementary $\mathfrak{S}_n$-representation, Young's *Natural* representation (see [25]), may be defined in terms of combinatorial objects called polytabloids. Garsia and McLarnan [12] and McDonough and Pallikaros [21] described the connection between Young's Natural matrices $\{X_2^\lambda(w) \,|\, w \in \mathfrak{S}_n\}$ and the Kazhdan-Lusztig matrices as conjugation by a unitriangular matrix $B = B(\lambda)$,

$$X_1^\lambda(w) = B^{-1} X_2^\lambda(w) B, \quad \text{for all } w \in \mathfrak{S}_n. \tag{1.1}$$

The former authors used properties of the Kazhdan-Lusztig and Natural modules to solve the equations (1.1) for B. The latter authors proved (1.1) by realizing the two modules as a single subspace of $\mathbb{C}[\mathfrak{S}_n]$ having two distinguished bases. (More precisely, they proved a q-analog by realizing the two modules as a single subspace of $H_n(q)$.) Thus B is a transition

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matrix which allows one to express Kazhdan-Lusztig basis elements as linear combinations of Young's basis elements. Moreover, this realization of the Kazhdan-Lusztig module eliminates the need for preorders. (See also [10, Rmk. 2.3(i)], [20, Sec. 5] for earlier constructions of preorder-avoiding modules.)

Another elementary $\mathfrak{S}_n$-representation, Clausen's *Bideterminant* representation [5], may be defined in terms of subspaces of the polynomial ring $\mathbb{C}[x] = \mathbb{C}[x_{1,1}, \dots, x_{n,n}]$ and polynomials called *bideterminants* which appeared earlier in the work of Mead [22], Désarménien-Kung-Rota [6], and others. Proving results analogous to those above, we will describe the connection between the Clausen matrices $\{X_3^\lambda(w) \,|\, w \in \mathfrak{S}_n\}$ and the Kazhdan-Lusztig matrices as conjugation by a unitriangular matrix. We will accomplish this by realizing the two modules as a single subspace of $\mathbb{C}[x]$ having two distinguished bases: the bideterminant basis, which produces the Clausen representations and the *dual canonical basis*, which produces the Kazhdan-Lusztig representations. Our unitriangular transition matrix is a submatrix of that studied in [24] to express dual-canonical basis elements as linear combinations of bideterminants. Like the McDonough-Pallikaros construction of the Kazhdan-Lusztig module, our module also eliminates the need for preorders.

In Sections 2-3, we review basic definitions related to the symmetric group, Hecke algebra, and Kazhdan-Lusztig modules. In Section 4 we review definitions related to the polynomial ring $\mathbb{C}[x]$ and a particular $n!$-dimensional subspace of $\mathbb{C}[x]$ called the *immanant space*. We recall the definition of the bideterminant basis of the immanant space and Clausen's use of this basis to construct irreducible $\mathfrak{S}_n$-modules [5]. In Section 5, we use the basis of Kazhdan-Lusztig immanants introduced in [7] to transfer the traditional Kazhdan-Lusztig representations from $\mathbb{C}[\mathfrak{S}_n]$ to the immanant space of $\mathbb{C}[x]$. Borrowing ideas from Clausen, and applying vanishing properties of Kazhdan-Lusztig immanants obtained in [24], we modify the above representations in Section 6. This leads to our main result that the resulting modified representations, which do not rely upon the Kazhdan-Lusztig preorders, have matrices equal to those corresonding to the original Kazhdan-Lusztig representations in [15]. We finish by showing that the relationship between the bideterminant and Kazhdan-Lusztig immanant bases studied in [24, Sec. 5] leads to unitriangular transition matrices relating Clausen's irreducible representations of $\mathfrak{S}_n$ to those of Kazhdan and Lusztig.

2. The symmetric group, tableaux, and partial orders

The standard presentation of the symmetric group $\mathfrak{S}_n$ is given by generators $s_1, \dots, s_{n-1}$ and relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1, \dots, n-1, \\ s_i s_j s_i &= s_j s_i s_j, && \text{if } |i-j| = 1, \\ s_i s_j &= s_j s_i, && \text{if } |i-j| \geq 2. \end{aligned} \tag{2.1}$$

We let $\mathfrak{S}_n$ act on rearrangements of the letters $[n] = \{1, \dots, n\}$ by

$$s_i \circ v_1 \cdots v_n \underset{\text{def}}{=} v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_n, \tag{2.2}$$

and we define the *one-line notation* of a permutation $w = s_{i_1} \cdots s_{i_\ell} \in \mathfrak{S}_n$ by

$$w_1 \cdots w_n \underset{\text{def}}{=} s_{i_1} \circ (\cdots (s_{i_\ell} \circ (1 \cdots n)) \cdots). \tag{2.3}$$

It is well known that this one-line notation does not depend upon the particular expression $s_{i_1} \cdots s_{i_\ell}$ for w. We say that such an expression is *reduced* if ℓ is as small as possible. We then call $\ell = \ell(w)$ the *length* of w.

We define the *Bruhat order* on $\mathfrak{S}_n$ by $v \leq w$ if some (equivalently every) reduced expression for w contains a reduced expression for v as a subword. (See [2] for more information). We call a generator s a *left ascent* for a permutation v if we have $sv > v$, and a *left descent* otherwise. Right ascents and descents are defined analogously. We denote the unique maximal element in the Bruhat order by w_0. This permutation has one-line notation $n(n-1)\cdots 21$. It is well known that the maps $v \mapsto w_0 v w_0$ and $v \mapsto v^{-1}$ induce automorphisms of the Bruhat order, while the maps $v \mapsto v w_0$ and $v \mapsto w_0 v$ induce antiautomorphisms. Thus we have

$$v \leq w \Leftrightarrow v^{-1} \leq w^{-1} \Leftrightarrow w_0 v w_0 \leq w_0 w w_0 \Leftrightarrow w w_0 \leq v w_0 \Leftrightarrow w_0 w \leq w_0 v. \tag{2.4}$$

Important in the study of $\mathfrak{S}_n$ are weakly decreasing sequences $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers which sum to n. We call such a sequence an *integer partition* of n and write $\lambda \vdash n$ or $|\lambda| = n$. The components of λ are called *parts*. A left-justified array of boxes with λ_i boxes in row i $(1 \leq i \leq k)$ is called a *Young diagram of shape* λ. Transposing this diagram as one would transpose a matrix, we obtain a diagram whose shape is another integer partition of n which we denote by $\lambda^\top$. (This is often called the *conjugate* of λ.) We define the *dominance order* on partitions of n by declaring $\lambda \preceq \mu$ if we have

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i, \tag{2.5}$$

for $i = 1, \ldots, n$ (with λ_i and μ_j defined to be zero for i, j larger than the number of parts of these partitions). It is well known that we have $\lambda \preceq \mu$ if and only if $\lambda^\top \succeq \mu^\top$.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, define a subset $J = J(\lambda)$ of the generators $\{s_1, \ldots, s_{n-1}\}$ by

$$J \underset{\text{def}}{=} \{s_1, \ldots, s_{n-1}\} \smallsetminus \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, s_{\lambda_1+\lambda_2+\lambda_3}, \ldots, s_{n-\lambda_r}\}, \tag{2.6}$$

and let W_J be the subgroup of $\mathfrak{S}_n$ generated by J. This subgroup (and any subgroup isomorphic to it) is called a *Young subgroup* of $\mathfrak{S}_n$ *of type* λ. Each coset of the form vW_J forms an interval in the Bruhat order, i.e., a subposet with a unique minimal and maximal element. The permutation v is maximal in vW_J if and only if we have $vs < v$ for all generators $s \in J$, equivalently, if and only if the one-line notation of v^{-1} satisfies

$$\begin{gathered} v^{-1}_1 > \cdots > v^{-1}_{\lambda_1}, \\ v^{-1}_{\lambda_1+1} > \cdots > v^{-1}_{\lambda_1+\lambda_2}, \\ \vdots \\ v^{-1}_{n-\lambda_r+1} > \cdots > v^{-1}_{n}. \end{gathered} \tag{2.7}$$

Let W^J_+ be the set of Bruhat-maximal representatives of the cosets $\{vW_J \mid v \in \mathfrak{S}_n\}$.

Filling a Young diagram of shape $\lambda \vdash n$ with positive integers, we obtain a *Young tableau* T of shape λ and write $\mathrm{sh}(T) = \lambda$. T is called *injective* if no number appears more than once in T, *column-(semi)strict* if entries (weakly) increase downward in columns, *row-(semi)strict* if entries (weakly) increase to the right in rows, *semistandard* if it is column-strict and row-semistrict, and *standard* if it is injective, semistandard, and has entries $1, \ldots, n$. A standard tableau is called *superstandard* if it contains $1, \ldots, n$ in reading order. We denote the superstandard tableau of shape λ by $T(\lambda)$. For example,

$$T(4,2,1) = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & & \\ 7 & & & \end{matrix}. \tag{2.8}$$

Analogous to transposition of a partition is transposition $T \mapsto T^{\top}$ of a tableau. Analogous to the dominance order on partitions of n is an *iterated dominance order* on standard tableaux of size n, which we define by declaring $S \trianglelefteq_I T$ if we have

$$\mathrm{sh}(T_{[i]}) \preceq \mathrm{sh}(U_{[i]}) \tag{2.9}$$

for $i = 1, \ldots, n$, where $T_{[i]}$ is the subtableau of T consisting of all entries less than or equal to i. It is easy to see that the superstandard tableau $T(\lambda)$ is maximal among tableaux of shapes $\{\mu \,|\, \mu \preceq \lambda\}$ in iterated dominance.

We define a *bitableau* to be a pair of tableaux of the same shape, and say that it posesses a certain tableau property of both of its tableaux posess this property. We extend the iterated dominance order on standard tableaux of size n to an *iterated dominance order* on standard bitableaux of size n by declaring $(T, U) \trianglelefteq_I (T', U')$ if we have

$$T \trianglelefteq_I T', \quad U \trianglelefteq_I U'. \tag{2.10}$$

For each permutation $v \in \mathfrak{S}_n$, we define the bitableau $(P(v), Q(v))$ by applying the *Robinson-Schensted column insertion* algorithm to $v_1 \cdots v_n$. (See, e.g., [25, Sec. 3.1-3.2].) We define the *shape* of v to be the partition $\mathrm{sh}(v) = \mathrm{sh}(P(v)) = \mathrm{sh}(Q(v))$ of n. Following [24], we use the map $v \mapsto (P(v), Q(v))$ to transfer the iterated dominance order on standard bitableaux of size n to $\mathfrak{S}_n$. To be precise, we define the *iterated dominance order* on $\mathfrak{S}_n$ by declaring $v \leq_I w$ if we have

$$(P(v), Q(v)) \trianglelefteq_I (P(w), Q(w)). \tag{2.11}$$

It is well known that the Robinson-Schensted bitableaux satisfy

$$\begin{aligned} (P(v^{-1}), Q(v^{-1})) &= (Q(v), P(v)), \\ (P(w_0 v), Q(w_0 v)) &= (P(v)^{\top}, \mathrm{evac}(Q(v))^{\top}), \\ (P(v w_0), Q(v w_0)) &= (\mathrm{evac}(P(v))^{\top}, Q(v)^{\top}), \\ (P(w_0 v w_0), Q(w_0 v w_0)) &= (\mathrm{evac}(P(v)), \mathrm{evac}(Q(v))), \end{aligned} \tag{2.12}$$

where evac is Schützenberger's evacuation algorithm. (See [2, Sec. A3.9], where left and right multiplication by w_0 correspond to our right and left multiplication by w_0, respectively.) It is therefore easy to see that we have

$$\mathrm{sh}(v) = \mathrm{sh}(v^{-1}) = \mathrm{sh}(w_0 v w_0) = \mathrm{sh}(w_0 v)^{\top} = \mathrm{sh}(v w_0)^{\top}. \tag{2.13}$$

It is also easy to see that the map $v \mapsto v^{-1}$ induces an automorphism of the iterated dominance order on $\mathfrak{S}_n$,

$$v \leq_I w \Leftrightarrow v^{-1} \leq_I w^{-1}. \tag{2.14}$$

On the other hand, none of the maps $v \mapsto v w_0$, $v \mapsto w_0 v$, $v \mapsto w_0 v w_0$ induces an automorphism or an antiautomorphism in general.

Superstandard tableaux can be used to prove the maximality of certain permutations within cosets of the form vW_J.

Lemma 2.1. *Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ and define the Young subgroup W_J as in (2.6). Then each permutation v satisfying $P(v) = T(\lambda)$ is maximal in the coset vW_J.*

Proof. For each index $j = 1, \ldots, r$, the letters

$$(\lambda_1 + \cdots + \lambda_{j-1} + 1), (\lambda_1 + \cdots + \lambda_{j-1} + 2), \ldots, (\lambda_1 + \cdots + \lambda_j) \tag{2.15}$$

appear in the jth row of $T(\lambda) = P(v) = Q(v^{-1})$. Properties of the Robinson-Schensted column correspondence then imply that the one-line notation $v_1^{-1} \cdots v_n^{-1}$ of v^{-1} satisfies

$$v^{-1}_{\lambda_1+\lambda_2+\cdots+\lambda_{j-1}+1} > v^{-1}_{\lambda_1+\lambda_2+\cdots+\lambda_{j-1}+2} > \cdots > v^{-1}_{\lambda_1+\lambda_2+\cdots+\lambda_j}. \tag{2.16}$$

□

More general results in the literature relate subsequences of the one-line notation of an arbitrary permutation v to subtableaux of $P(v)$ and $Q(v)$. (See, e.g., [25, Sec. 3.3-3.5].) In particular, let us define for each permutation $v \in \mathfrak{S}_n$ and each index $j \leq n$ the permutation $v_{[j]} \in S_j$ by arranging $1, \ldots, j$ in the same relative order as the letters in the subword $v_1 \cdots v_j$ of the one-line notation of v. Schützenberger showed the following relationship between $v_{[j]}$ and the standard tableau $Q(v)_{[j]}$ of size j. (See [16, Thm. 5.1.4 C].)

Lemma 2.2. *For $v \in \mathfrak{S}_n$ and $1 \leq j \leq n$ we have* $\mathrm{sh}(v_{[j]}) = \mathrm{sh}(Q(v)_{[j]})$.

3. Kazhdan and Lusztig's $H_n(q)$-modules

A quantum analog of the symmetric group algebra $\mathbb{C}[\mathfrak{S}_n]$ is known as the *Hecke algebra* $H_n(q)$. This noncommutative ring with multiplicative identity $\widetilde{T}_e = 1$ is generated as a $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra by elements $\{\widetilde{T}_{s_i} \,|\, 1 \leq i \leq n-1\}$, subject to the relations

$$\begin{aligned} \widetilde{T}_{s_i}^2 &= (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{s_i} + \widetilde{T}_e, && \text{for } i = 1, \ldots, n-1, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j}, && \text{if } |i-j| = 1, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_j} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i}, && \text{if } |i-j| \geq 2. \end{aligned} \tag{3.1}$$

It is easy to see that the specialization of $H_n(q)$ at $q^{\frac{1}{2}} = 1$ is simply the group algebra $\mathbb{C}[\mathfrak{S}_n]$.

Inverses of these generators are given by $\widetilde{T}_s^{-1} = \widetilde{T}_s - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_e$ and a multiplication rule is given by

$$\widetilde{T}_s\widetilde{T}_v = \begin{cases} \widetilde{T}_{sv} & \text{if } sv > v, \\ \widetilde{T}_{sv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_v & \text{if } sv < v. \end{cases} \tag{3.2}$$

If $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for $v \in \mathfrak{S}_n$, we define the element $\widetilde{T}_v \in H_n(q)$ by

$$\widetilde{T}_v = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_\ell}}. \tag{3.3}$$

It is known that this definition does not depend upon the particular reduced expression for v and that the natural collection of elements $\{\widetilde{T}_v \,|\, v \in \mathfrak{S}_n\}$ forms a basis of $H_n(q)$ as a $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module.

An involutive automorphism of $H_n(q)$ called the *bar involution* is defined by

$$\sum_v a_v \widetilde{T}_v \mapsto \overline{\sum_v a_v \widetilde{T}_v} = \sum_v \overline{a_v}\, \widetilde{T}_{v^{-1}}^{-1}, \tag{3.4}$$

where $\overline{q^{\frac{1}{2}}} = q^{\frac{-1}{2}}$. We call an element g of $H_n(q)$ *bar-invariant* if it satisfies $\overline{g} = g$. Kazhdan and Lusztig showed [15] that $H_n(q)$ has a unique basis of bar-invariant elements $\{C_v(q) \,|\, v \in \mathfrak{S}_n\}$ satisfying

$$C_v(q) \in \widetilde{T}_v + \sum_{u<v} q^{\frac{1}{2}} \mathbb{Z}[q^{\frac{1}{2}}] \widetilde{T}_u. \tag{3.5}$$

Expanding $C_v(q)$ in terms of the natural basis and defining

$$\epsilon_{u,v} \underset{\text{def}}{=} (-1)^{\ell(v)-\ell(u)}, \qquad q_{u,v} \underset{\text{def}}{=} (q^{\frac{1}{2}})^{\ell(v)-\ell(u)}, \tag{3.6}$$

we have

$$C_v(q) = \sum_{u \leq v} \epsilon_{u,v} q_{u,v} P_{u,v}(q^{-1}) \widetilde{T}_u, \tag{3.7}$$

where $\{P_{u,v}(q) \,|\, u, v \in \mathfrak{S}_n\}$ are polynomials belonging to $\mathbb{N}[q]$. This basis is called the *Kazhdan-Lusztig basis* for $H_n(q)$, and the polynomials $\{P_{u,v}(q) \,|\, u, v \in \mathfrak{S}_n\}$ are called the *Kazhdan-Lusztig polynomials*.

The proof in [15, Sec. 2.2] of the existence of this basis relies upon the function

$$\mu(u,v) \underset{\text{def}}{=} \begin{cases} \text{coefficient of } q^{(\ell(v)-\ell(u)-1)/2} \text{ in } P_{u,v}(q), & \text{if } u < v, \\ 0 & \text{otherwise,} \end{cases} \tag{3.8}$$

and leads to the formula

$$\widetilde{T}_s C_v(q) = \begin{cases} q^{\frac{1}{2}} C_v(q) + C_{sv}(q) + \sum_{\substack{u<v \\ su<u}} \mu(u,v) C_u(q) & \text{if } sv > v, \\ -q^{\frac{-1}{2}} C_w(q) & \text{if } sv < v, \end{cases} \tag{3.9}$$

describing the action of $\widetilde{T}_s$ on the basis element $C_v(q)$. Observe that the function μ satisfies $\mu(u,v) = 0$ if $\ell(v) - \ell(u)$ is even, since $P_{u,v}(q)$ belongs to $\mathbb{N}(q)$. Since the Kazhdan-Lusztig polynomials satisfy $P_{u,v}(q) = P_{w_0uw_0,w_0vw_0}(q)$ (see, e.g., [3, Cor. 4.3]), we also have the identity $\mu(u,v) = \mu(w_0uw_0, w_0vw_0)$. Furthermore, Kazhdan and Lusztig showed [15, Cor. 3.2] that we have $\mu(u,v) = \mu(w_0v, w_0u)$, even though $P_{u,v}(q)$ and $P_{w_0v,w_0u}(q)$ are not equal in general.

In order to construct irreducible representations of $H_n(q)$, Kazhdan and Lusztig defined several preorders on $\mathfrak{S}_n$. The *left preorder* $\leq_L$ is defined to be the transitive closure of the relation $\lessdot_L$ on $\mathfrak{S}_n$, where we declare $v \lessdot_L u$ if $C_v(q)$ appears with nonzero coefficient in the expansion of $\widetilde{T}_w C_u(q)$ for some $w \in \mathfrak{S}_n$. It is known that we have

$$\begin{aligned} w \leq_L v \qquad &\Rightarrow \quad \mathrm{sh}(v) \preceq \mathrm{sh}(w), \\ w \leq_L v \leq_L w \quad &\Rightarrow \quad P(v) = P(w). \end{aligned} \tag{3.10}$$

It is also known that the maps $v \mapsto w_0v$, $v \mapsto vw_0$ reverse the left preorder, while the map $v \mapsto w_0vw_0$ preserves it. (See [2, Prop. 6.2.9]). On the other hand, the map $v \mapsto v^{-1}$ does not in general preserve or reverse the left preorder. Thus we have

$$w \leq_L v \Leftrightarrow w_0ww_0 \leq_L w_0vw_0 \Leftrightarrow vw_0 \leq_L ww_0 \Leftrightarrow w_0v \leq w_0w. \tag{3.11}$$

The Kazhdan-Lusztig construction (as described in [13, Appendix]) of the irreducible $H_n(q)$-module indexed by a partition $\lambda \vdash n$ requires one to fix a standard tableau T of shape λ and a permutation v satisfying $P(v)^\top = T$. One then lets $H_n(q)$ act by left multiplication on the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module

$$K^\lambda \underset{\mathrm{def}}{=} \mathrm{span}\{C_w(q) \,|\, P(w)^\top = T\}, \tag{3.12}$$

regarded as the quotient

$$\mathrm{span}\{C_w(q) \,|\, w \leq_L v\}/\mathrm{span}\{C_w(q) \,|\, w \leq_L v, w \not\geq_L v\}. \tag{3.13}$$

The quotient is necessary because K^λ is not in general closed under the action of $H_n(q)$. The specialization $K^\lambda(1)$ of K^λ at $q^{\frac{1}{2}} = 1$ is an irreducible $\mathfrak{S}_n$-module indexed by λ. Even in this simpler setting, the quotient (3.13) is necessary.

4. The polynomial ring and Clausen's $\mathfrak{S}_n$-modules

Many $\mathfrak{S}_n$-modules in the literature are subspaces of the group algebra $\mathbb{C}[\mathfrak{S}_n]$. An alternative construction due to Clausen [5] uses subspaces of the polynomial ring in n^2 variables instead.

Let $x = (x_{i,j})$ be an $n \times n$ matrix of variables. The polynomial ring $\mathbb{C}[x] = \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ has a natural grading

$$\mathbb{C}[x] = \bigoplus_{r \geq 0} \mathcal{A}_r(x), \tag{4.1}$$

where $\mathcal{A}_r(x)$ is the span of all monomials of total degree r. Further decomposing each space $\mathcal{A}_r(x)$, we define a multigrading

$$\mathbb{C}[x] = \bigoplus_{r \geq 0} \mathcal{A}_r(x) = \bigoplus_{r \geq 0} \bigoplus_{L,M} \mathcal{A}_{L,M}(x), \tag{4.2}$$

where $L = \{\ell(1) \leq \ldots \leq \ell(r)\}$ and $M = \{m(1) \leq \ldots \leq m(r)\}$ are r-element multisets of $[n]$, written as weakly increasing sequences, and where $\mathcal{A}_{L,M}(x)$ is the span of monomials whose row and column indices are given by L and M, respectively. We refer to the space

$$\mathcal{A}_{[n],[n]}(x) = \mathrm{span}\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in \mathfrak{S}_n\}, \tag{4.3}$$

as the *immanant space* of $\mathbb{C}[x]$, and define the notation

$$x^{u,v} \underset{\mathrm{def}}{=} x_{u_1,v_1} \cdots x_{u_n,v_n} \tag{4.4}$$

for permutations $u, v \in \mathfrak{S}_n$. We define the (L, M) *generalized submatrix* of x by

$$x_{L,M} = \begin{bmatrix} x_{\ell(1),m(1)} & \cdots & x_{\ell(1),m(r)} \\ x_{\ell(2),m(1)} & \cdots & x_{\ell(2),m(r)} \\ \vdots & & \vdots \\ x_{\ell(r),m(1)} & \cdots & x_{\ell(r),m(r)} \end{bmatrix}. \tag{4.5}$$

It is clear that for each pair (u, v) of permutations in $\mathfrak{S}_r$, the monomial $(x_{L,M})^{u,v}$ belongs to $\mathcal{A}_{L,M}(x)$.

Given subsets $I, J \subset [n]$ we define the I, J *minor* of x to be the determinant

$$\Delta_{I,J}(x) \underset{\mathrm{def}}{=} \det(x_{I,J}), \tag{4.6}$$

and given a semistandard bitableau (S,T), we define the *bideterminant* $(S\,|\,T)(x)$, to be the polynomial

$$(S\,|\,T)(x) = \Delta_{I_1,J_1}(x)\cdots\Delta_{I_k,J_k}(x), \tag{4.7}$$

where $I_1,\ldots,I_k$ are the sets of entries in columns $1,\ldots,k$ of S and $J_1,\ldots,J_k$ are the sets of entries in columns $1,\ldots,k$ of T. For example, we have

$$\left(\begin{matrix}1 & 2 & 4\\ 3 & & \end{matrix}\,\middle|\,\begin{matrix}1 & 3 & 4\\ 2 & & \end{matrix}\right)(x) = \Delta_{13,12}(x)x_{2,3}x_{4,4} = x_{1,1}x_{3,2}x_{2,3}x_{4,4} - x_{1,2}x_{3,1}x_{2,3}x_{4,4}. \tag{4.8}$$

For each permutation v in $\mathfrak{S}_n$, we follow [24] in defining the bideterminant $R_v(x)$ by

$$R_v(x) \underset{\text{def}}{=} (Q(v)\,|\,P(v))(x), \tag{4.9}$$

where $(P(v),Q(v))$ is the bitableau obtained by applying the Robinson-Schensted column insertion algorithm to v. (Note the reversal of the tableaux.) With little effort one can see that each semistandard bideterminant can be viewed as a standard bideterminant of a generalized submatrix. Similarly, each standard bideterminant evaluated at generalized submatrix of x is either zero or is equal to a semistandard bideterminant. It follows that for multisets L,M of $[n]$ with $|L|=|M|=r$, we may describe the space $\mathcal{A}_{L,M}(x)$ as

$$\mathcal{A}_{L,M}(x) = \mathrm{span}\{R_w(x_{L,M})\,|\,w\in\mathfrak{S}_r\}. \tag{4.10}$$

A natural $\mathfrak{S}_n$-action on $\mathbb{C}[x]$ is given by

$$s\circ g(x) \underset{\text{def}}{=} g(sx), \tag{4.11}$$

where g belongs to $\mathbb{C}[x]$ and sx is interpreted as the product of the permutation matrix of the standard generator s and the matrix x. Clausen [5, Thm. 8.1] showed that one can construct an irreducible $\mathfrak{S}_n$-module indexed by $\lambda\vdash n$ by defining the multiset $M = 1^{\lambda_1}\cdots n^{\lambda_n}$ and by letting $\mathfrak{S}_n$ act on the space

$$B^\lambda \underset{\text{def}}{=} \mathrm{span}\{R_w(x_{[n],M})\,|\,P(w)=T(\lambda)\}. \tag{4.12}$$

5. The quantum polynomial ring and Kazhdan-Lusztig immanants

A quantum analog of the polynomial ring $\mathbb{C}[x]$ is known as the *quantum polynomial ring* $\mathcal{A}(x;q)$. This noncommutative ring with multiplicative identity 1 is generated as a $\mathbb{C}[q^{\frac{1}{2}},q^{\frac{-1}{2}}]$-algebra by the n^2 variables $x=(x_{1,1}\ldots,x_{n,n})$, with relations

$$\begin{aligned}
x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell},\\
x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k},\\
x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k},\\
x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k},
\end{aligned} \tag{5.1}$$

for all pairs of variables with indices satisfying $i<j$ and $k<\ell$. A natural basis for $\mathcal{A}(x;q)$ as a $\mathbb{C}[q^{\frac{1}{2}},q^{\frac{-1}{2}}]$-module consists of the set of monomials in which variables appear in lexicographic order. We can use the relations above to convert any other monomial to this standard form.

Analogous to the multigrading (4.2) of $\mathbb{C}[x]$ is the multigrading

$$\mathcal{A}(x;q) = \bigoplus_{r\geq 0} \mathcal{A}_r(x;q) = \bigoplus_{r\geq 0} \bigoplus_{L,M} \mathcal{A}_{L,M}(x;q) \tag{5.2}$$

of $\mathcal{A}(x;q)$, where $\mathcal{A}_r(x;q)$ is the span of all monomials of total degree r, and where $\mathcal{A}_{L,M}(x;q)$ is the span of monomials whose row and column indices are given by r-element multisets L and M of $[n]$. We again call the space $\mathcal{A}_{[n],[n]}(x;q)$ the (quantum) *immanant space* of $\mathcal{A}(x;q)$, and we call $\{x^{e,w} \,|\, w \in \mathfrak{S}_n\}$ the *natural basis* of $\mathcal{A}_{[n],[n]}(x;q)$.

It is easy to see that the monomials $\{x^{u,v} \,|\, u,v \in \mathfrak{S}_n\}$ belong to the immanant space and satisfy

$$x^{su,v} = \begin{cases} x^{u,sv} & \text{if } su > u \text{ and } sv > v, \text{ or if } su < u \text{ and } sv < v, \\ x^{u,sv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{u,v} & \text{if } su > u \text{ and } sv < v, \\ x^{u,sv} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{u,v} & \text{if } su < u \text{ and } sv > v. \end{cases} \tag{5.3}$$

Define a left action of the Hecke algebra on $\mathcal{A}_{[n],[n]}(x;q)$ by

$$\widetilde{T}_s \circ x^{e,v} \underset{\text{def}}{=} x^{s,v} = \begin{cases} x^{e,sv} & \text{if } sv > v, \\ x^{e,sv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,v} & \text{if } sv < v. \end{cases} \tag{5.4}$$

With a bit more work, we obtain the following formulae describing the action on monomials of the form $x^{u,v}$ not necessarily belonging to the natural basis.

Lemma 5.1. *We have*

$$\widetilde{T}_{s_j} \circ x^{u,v} = \begin{cases} x^{us_j,v} & \textit{if } us_j > u, \\ x^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{u,v} & \textit{if } us_j < u, \end{cases} \tag{5.5}$$

Proof. Assume the formula (5.5) to hold for all monomials $x^{u,v}$ with $\ell(u) < k$. Certainly it holds for $\ell(u) = 0$, i.e., $u = e$. Now fix one permutation u of length k, and let s_i be a left descent for u. By (5.3) we have

$$\widetilde{T}_{s_j} \circ x^{u,v} = \begin{cases} \widetilde{T}_{s_j} \circ x^{s_iu,s_iv} & \text{if } s_iv > v, \\ \widetilde{T}_{s_j} \circ x^{s_iu,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{s_j} \circ x^{s_iu,v} & \text{if } s_iv < v, \end{cases} \tag{5.6}$$

which by induction is equal to

$$\begin{cases} x^{s_ius_j,s_iv} & \text{if } s_iv > v \text{ and } s_ius_j > s_iu, \\ x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_iu,s_iv} & \text{if } s_iv > v \text{ and } s_ius_j < s_iu, \\ x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_ius_j,v} & \text{if } s_iv < v \text{ and } s_ius_j > s_iu, \\ x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x^{s_iu,s_iv} + x^{s_ius_j,v}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2 x^{s_iu,v} & \text{if } s_iv < v \text{ and } s_ius_j < s_iu. \end{cases} \tag{5.7}$$

Now we return to the right-hand side of the claimed formula. Suppose first that $us_j > u$. This implies that $s_iu < s_ius_j < us_j$. By (5.3) we then have

$$x^{us_j,v} = \begin{cases} x^{s_ius_j,s_iv} & \text{if } s_iv > v, \\ x^{s_ius_j,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_ius_j,v} & \text{if } s_iv < v, \end{cases} \tag{5.8}$$

which is equal to $\widetilde{T}_{s_i} \circ x_{u,v}$ by the first and third cases of (5.7). Now suppose that $us_j < u$. Then we have $u > s_i us_j$ or $u = s_i us_j$. If $u = s_i us_j$, then $us_j = s_i u < u = s_i us_j$. Applying (5.3) to (just the first monomial in)

$$x^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{u,v} = x^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_i us_j,v}, \tag{5.9}$$

we again obtain the expressions on the right-hand side of (5.8). If $u > s_i us_j$, then $s_i u < u$ and $s_i us_j < us_j$. By (5.3) we then have

$$x^{us_j,v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{u,v} = \begin{cases} x^{s_i us_j, s_i v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x^{s_i us_j,v} + x^{s_i u, s_i v}) & \\ \quad +(q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2 x^{s_i u, v} & \text{if } s_i v < v, \\ x^{s_i us_j, s_i v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{s_i u, s_i v} & \text{if } s_i v > v, \end{cases} \tag{5.10}$$

which is equal to $\widetilde{T}_{s_i} \circ x_{u,v}$ by the second and fourth cases of (5.7).

□

Similar to the bar involution on $H_n(q)$ is another *bar involution* on $\mathcal{A}_{[n],[n]}(x;q)$ defined by

$$\sum_v a_v x^{e,v} \mapsto \overline{\sum_v a_v x^{e,v}} = \sum_v \overline{a_v} x^{w_0, w_0 v} \tag{5.11}$$

where $\overline{q^{\frac{1}{2}}} = q^{\frac{-1}{2}}$. Expressing these elements in terms of the natural basis, we have

$$\overline{x^{e,v}} = \sum_{w \geq v} \epsilon_{v,w} q_{v,w} S_{v,w}(q^{-1}) x^{e,w}, \tag{5.12}$$

where $\{S_{v,w}(q) \,|\, v, w \in \mathfrak{S}_n\}$ belong to $\mathbb{Z}[q]$. It is possible, but not essential for our purposes, to show that these polynomials are equal to the *R-polynomials* defined in [15]. Details will appear in [26]. As before, we call an element g of $\mathcal{A}_{[n],[n]}(x;q)$ *bar-invariant* if it satisfies $\overline{g} = g$. This bar involution and that defined in (3.4) are compatible with the action of $H_n(q)$ on $\mathcal{A}_{[n],[n]}(x;q)$ in the following sense.

Proposition 5.2. *For all $v \in \mathfrak{S}_n$ we have $\overline{\widetilde{T}_{s_i}} \circ \overline{x^{e,v}} = \overline{\widetilde{T}_{s_i} \circ x^{e,v}}$.*

Proof. By the definitions we have

$$\overline{\widetilde{T}_{s_i} \circ x^{e,v}} = \overline{x^{s_i,v}} = x^{w_0 s_i, w_0 v}. \tag{5.13}$$

On the other hand, we have

$$\begin{aligned} \overline{\widetilde{T}_{s_i}} \circ \overline{x^{e,v}} &= (\widetilde{T}_{s_i} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_e) \circ x^{w_0, w_0 v} \\ &= \widetilde{T}_{s_i} \circ x^{w_0, w_0 v} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{w_0, w_0 v} \\ &= x^{w_0 s_i, w_0 v} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{w_0, w_0 v} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{w_0, w_0 v} \end{aligned} \tag{5.14}$$

by Lemma 5.1. □

Du showed [7], [8] that the immanant space $\mathcal{A}_{[n],[n]}(x;q)$ has a unique basis of bar-invariant elements $\{\mathrm{Imm}_v(x;q) \,|\, v \in \mathfrak{S}_n\}$ satisfying

$$\mathrm{Imm}_v(x;q) \in x^{e,v} + \sum_{w > v} q^{\frac{-1}{2}} \mathbb{Z}[q^{\frac{-1}{2}}] x^{e,w}. \tag{5.15}$$

(We follow the notation of [23].) Expanding $\mathrm{Imm}_v(x;q)$ in terms of the natural basis, we have

$$\mathrm{Imm}_v(x;q) = \sum_{w \geq v} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}(q) x^{e,w}, \tag{5.16}$$

where $\{Q_{v,w}(q) \,|\, v, w \in \mathfrak{S}_n\}$ are polynomials belonging to $\mathbb{N}[q]$. This basis is called the *Kazhdan-Lusztig immanant basis* or *dual canonical basis* of $\mathcal{A}_{[n],[n]}(x;q)$. Similar to Lusztig's *D-bases* in [17, Sec. 5], the dual canonical basis of $\mathcal{A}_{[n],[n]}(x;q)$ arose naturally from Lusztig's and Kashiwara's work on canonical bases [14], [18]. (See also [4], [7, Sec. 2.3], [9, Sec. 2], [11], [19, Sec. 29.5], [29].)

For the benefit of the reader we provide a proof of the existence and uniqueness of the Kazhdan-Lusztig immanant basis which is analogous to the proof of [15, Thm. 1.1].

Theorem 5.3. *For each $v \in \mathfrak{S}_n$, there is a unique bar-invariant element $\mathrm{Imm}_v(x;q)$ in $\mathcal{A}_{[n],[n]}(x;q)$ satisfying*

$$\mathrm{Imm}_v(x;q) = \sum_{w \geq v} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}(q) x^{e,w}, \tag{5.17}$$

where $Q_{v,w}(q)$ is a polynomial in q of degree at most $\frac{1}{2}(\ell(w) - \ell(v) - 1)$ if $v < w$, and where $Q_{v,v}(q) = 1$.

Proof. Uniqueness of this basis follows from rewriting the condition $\overline{\mathrm{Imm}_v(x;q)} = \mathrm{Imm}_v(x;q)$ as

$$q_{u,w} Q_{u,w}(q^{-1}) - q_{u,w}^{-1} Q_{u,w}(q) = \sum_{u < v \leq w} q_{u,v}^{-1} S_{u,v}(q) q_{v,w}^{-1} Q_{v,w}(q) \quad \text{for all } u \leq w. \tag{5.18}$$

In particular, our assumed degree conditions imply that there can be no cancellation of terms on the left-hand side of Equation (5.18). Thus there is at most one polynomial $Q_{u,w}(q)$ satisfying this equation when all other polynomials appearing are known.

To prove the existence of this basis we define the function

$$\nu(u,v) \underset{\text{def}}{=} \begin{cases} \text{coefficient of } q^{(\ell(v)-\ell(u)-1)/2} \text{ in } Q_{u,v}(q) & \text{if } u < v, \\ 0 & \text{otherwise.} \end{cases} \tag{5.19}$$

Note that $\mathrm{Imm}_{w_0}(x;q) = x^{e,w_0}$, and assume that for some $v \in \mathfrak{S}_n$ we have already defined $\{\mathrm{Imm}_w(x;q) \,|\, w > v\}$. Now we choose a generator s of $\mathfrak{S}_n$ so that $sv < v$ and define

$$\mathrm{Imm}_{sv}(x;q) = C_s(q) \circ \mathrm{Imm}_v(x;q) - \sum_{\substack{w > v \\ sw > w}} \nu(v,w) \mathrm{Imm}_w(x;q). \tag{5.20}$$

By Proposition 5.2, we see that this element is bar-invariant. To see that its coefficients satisfy the degree condition, observe that the coefficient of $x^{e,w}$ in $C_s(q) \circ \mathrm{Imm}_v(x;q)$ is

(5.21)

$$\begin{cases} \epsilon_{v,sw} q_{v,sw}^{-1} Q_{v,sw}(q) - \epsilon_{v,w} q_{v,w}^{-1} q^{\frac{1}{2}} Q_{v,w}(q) = \epsilon_{sv,w} q_{sv,w}^{-1} (Q_{v,sw}(q) + q Q_{v,w}(q)) & \text{if } sw > w \\ \epsilon_{v,sw} q_{v,sw}^{-1} Q_{v,sw}(q) - \epsilon_{v,w} q_{v,w}^{-1} q^{\frac{-1}{2}} Q_{v,w}(q) = \epsilon_{sv,w} q_{sv,w}^{-1} (q Q_{v,sw}(q) + Q_{v,w}(q)) & \text{if } sw < w, \end{cases}$$

which is $\epsilon_{sv,w}q_{sv,w}^{-1}$ times a polynomial in q of degree at most

$$\begin{cases} \frac{1}{2}(\ell(w) - \ell(sv)) & \text{if } sw > w \\ \frac{1}{2}(\ell(w) - \ell(sv) - 1) & \text{if } sw < w. \end{cases} \tag{5.22}$$

If $sw > w$, the leading coefficient of this polynomial is $\nu(v, w)$. Thus, the subtraction from $C_s(q) \circ \mathrm{Imm}_v(x; q)$ of $\nu(v, w)\mathrm{Imm}_w(x; q)$ for each permutation w satisfying $v < w < sw$ gives an element of $\mathcal{A}_{[n],[n]}(x; q)$ which satisfies the required degree conditions. □

Du showed [7] the polynomials $\{Q_{u,v}(q) \mid u, v \in \mathfrak{S}_n\}$ in the above proof to be equal to the *inverse Kazhdan-Lusztig polynomials* introduced in [15, Sec. 3],

$$Q_{u,v}(q) = P_{w_0v,w_0u}(q) = P_{vw_0,uw_0}(q). \tag{5.23}$$

Thus we have

$$\nu(u, v) = \mu(w_0v, w_0u) = \mu(vw_0, uw_0) = \mu(u, v), \tag{5.24}$$

and Equation (5.20) implies the following formula for the action of natural basis elements of $H_n(q)$ on the Kazhdan-Lusztig immanants.

Corollary 5.4. *For all $v \in \mathfrak{S}_n$ we have*

$$\widetilde{T}_s \circ \mathrm{Imm}_v(x; q) = \begin{cases} q^{\frac{1}{2}}\mathrm{Imm}_v(x; q) + \mathrm{Imm}_{sv}(x; q) + \sum\limits_{\substack{w>v \\ sw>w}} \mu(v, w)\mathrm{Imm}_w(x; q) & \text{if } sv < v, \\ -q^{\frac{-1}{2}}\mathrm{Imm}_v(x; q) & \text{if } sv > v. \end{cases} \tag{5.25}$$

This formula, analogous to (3.9) allows us to relate the left preorder to the Kazhdan-Lusztig immanant basis of $\mathcal{A}_{[n],[n]}(x; q)$.

Lemma 5.5. *The relation $\lessdot_L$ defined in Section 3 satisfies $w \lessdot_L v$ if and only if* $\mathrm{Imm}_v(x; q)$ *appears with nonzero coefficient in $\widetilde{T}_u \circ \mathrm{Imm}_w(x; q)$ for some $u \in \mathfrak{S}_n$.*

Proof. Replacing v, w in Equation (5.25) by vw_0, uw_0 (respectively), we have

$$\widetilde{T}_s \circ \mathrm{Imm}_{vw_0}(x; q) = \begin{cases} q^{\frac{1}{2}}\mathrm{Imm}_{vw_0}(x; q) + \mathrm{Imm}_{svw_0}(x; q) + \sum\limits_{\substack{u<v \\ su<u}} \mu(u, v)\mathrm{Imm}_{uw_0}(x; q) & \text{if } sv > v, \\ -q^{\frac{-1}{2}}\mathrm{Imm}_{vw_0}(x; q) & \text{if } sv < v. \end{cases} \tag{5.26}$$

Comparing this formula with (3.9), we see that C_u appears with nonzero coefficient in the expansion of $\widetilde{T}_s C_v$ if and only if $\mathrm{Imm}_{uw_0}(x; q)$ appears with nonzero coefficient in the expansion of $\widetilde{T}_s\mathrm{Imm}_{vw_0}(x; q)$. The result now follows from Equation (3.11). □

We now use the Kazhdan-Lusztig immanants and left preorder to define irreducible $H_n(q)$-modules analogous to $\{K^\lambda \mid \lambda \vdash n\}$. For each partition $\lambda \vdash n$, we again fix a standard Young tableau T of shape λ, but now we fix a permutation v satisfying $P(v) = T$ (rather than $P(v)^\top = T$). Then we let $H_n(q)$ act as in (5.4) on the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module

$$V^\lambda \underset{\mathrm{def}}{=} \mathrm{span}\{\mathrm{Imm}_u(x; q) \mid P(u) = T\}, \tag{5.27}$$

regarded as the quotient

$$\text{span}\{\text{Imm}_u(x;q) \,|\, u \geq_L v\}/\text{span}\{\text{Imm}_u(x;q) \,|\, u \geq_L v, u \not\leq_L v\}. \tag{5.28}$$

The quotient is necessary because like K^λ, the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module V^λ is not in general closed under the action of $H_n(q)$. In other words, the first containment in

$$V^\lambda \subseteq H_n(q)V^\lambda \subseteq V^\lambda \oplus \text{span}\{\text{Imm}_u(x;q) \,|\, u \geq_L v, u \not\leq_L v\} \tag{5.29}$$

is often strict.

Theorem 5.6. *V^λ is an irreducible $H_n(q)$-module indexed by λ. Furthermore, for each permutation u in $\mathfrak{S}_n$, the Kazhdan-Lusztig basis of K^λ and the Kazhdan-Lusztig immanant basis of V^λ produce the same matrix representing $\widetilde{T}_u$.*

Proof. Fix a standard Young tableau T of shape λ and define the $H_n(q)$-module K^λ as in Equation (3.12) to be the span of the basis $\{C_w(q) \,|\, P(w)^\top = T\}$. For each permutation u in $\mathfrak{S}_n$, define $X_1^\lambda(\widetilde{T}_u)$ to be the matrix of $\widetilde{T}_u$ with respect to this basis. Entries of $X_1^\lambda(\widetilde{T}_u)$ are indexed by pairs (w,v) of permutations satisfying $P(w)^\top = P(v)^\top = T$, and the (w,v) entry of $X_1^\lambda(\widetilde{T}_u)$ is equal to the coefficient $a^u_{v,w}$ appearing in the equation

$$\widetilde{T}_u C_v(q) = \sum_{w \in \mathfrak{S}_n} a^u_{v,w} C_w(q). \tag{5.30}$$

Now define the $H_n(q)$-module V^λ as in Equation (5.27) to be the span of the basis

$$\{\text{Imm}_w(x;q) \,|\, P(w) = \text{evac}(T)\} = \{\text{Imm}_{ww_0}(x;q) \,|\, P(w)^\top = T\}. \tag{5.31}$$

For each element u of $\mathfrak{S}_n$, define $Y(\widetilde{T}_u)$ to be the matrix of $\widetilde{T}_u$ with respect to this basis. Entries of $Y(\widetilde{T}_u)$ are indexed by pairs (ww_0, vw_0) satisfying $P(w)^\top = P(v)^\top = T$, and the (ww_0, vw_0) entry of $Y(\widetilde{T}_u)$ is equal to the coefficient $b^u_{vw_0,ww_0}$ appearing in the equation

$$\widetilde{T}_u \circ \text{Imm}_{vw_0}(x;q) = \sum_{w \in \mathfrak{S}_n} b^u_{vw_0,ww_0} \text{Imm}_{ww_0}(x;q). \tag{5.32}$$

By Equations (3.9) and (5.25), we have $a^s_{v,w} = b^s_{vw_0,ww_0}$ for all standard generators s, and therefore $X_1^\lambda(\widetilde{T}_s) = Y(\widetilde{T}_s)$. It follows that we have $X_1^\lambda(\widetilde{T}_u) = Y(\widetilde{T}_u)$ for all permutations u in $\mathfrak{S}_n$. □

The specialization $V^\lambda(1)$ of V^λ at $q^{\frac{1}{2}} = 1$ is an irreducible $\mathfrak{S}_n$-module indexed by λ. Even in this simpler setting, the quotient (5.28) is necessary.

6. Main results

Recall that the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-modules K^λ (3.12) and V^λ (5.27) are not in general closed under the $H_n(q)$-actions (3.2) and (5.4). To obtain $H_n(q)$-modules from these $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-modules, we must view them as the quotients (3.13) and (5.28). Similarly, the vector spaces $K^\lambda(1)$ and $V^\lambda(1)$ obtained by specializing K^λ and V^λ at $q^{\frac{1}{2}} = 1$ are not closed under the corresponding $\mathfrak{S}_n$-actions. We must specialize the quotients (3.13), (5.28) at $q^{\frac{1}{2}} = 1$ in order to obtain $\mathfrak{S}_n$-modules, and will write $\text{Imm}_w(x) = \text{Imm}_w(x;1)$ for the nonquantum Kazhdan-Lusztig immanants in $\mathcal{A}_{M,[n]}(x)$.

In contrast, Clausen's vector space B^λ (4.12) is indeed an $\mathfrak{S}_n$-module which requires no quotient. To examine this fact more closely, we recall the following definition and results from [24]. Given an $n \times n$ matrix A and an integer $j \leq n$, we define a partition $\nu_{[j]}(A)$ of j by

$$\nu_{[j]}(A) \underset{\text{def}}{=} (\nu_1, \dots, \nu_k), \tag{6.1}$$

where k is the number of distinct columns in the $n \times j$ submatrix $A_{[n],[j]}$, and $\nu_1, \dots, \nu_k$ are the multiplicities with which distinct columns appear, written in weakly decreasing order. The function $\nu_{[j]}$ facilitates the statement of sufficient conditions on an $n \times n$ matrix which imply a bideterminant or Kazhdan-Lusztig immanant to vanish on that matrix. (See [24, Sec. 4].) In particular, we have the following special case of [24, Thms. 4.10-4.11].

Proposition 6.1. *Fix a permutation $w \in \mathfrak{S}_n$ and an $n \times n$ matrix A. If* $\mathrm{sh}(w^{-1}_{[j]}) \not\preceq \nu_{[j]}(A)$ *for some $1 \leq j \leq n$, then* $\mathrm{Imm}_w(A) = R_w(A) = 0$.

This proposition has the following simple consequence.

Corollary 6.2. *Fix an integer partition $\lambda \vdash n$ and define the multiset $M = 1^{\lambda_1} \cdots n^{\lambda_n}$. For each permutation w satisfying* $\mathrm{sh}(w) \prec \lambda$ *or satisfying* $\mathrm{sh}(w) = \lambda$ *and $P(w) \neq T(\lambda)$, we have* $\mathrm{Imm}_w(x_{[n],M}) = R_w(x_{[n],M}) = 0$.

Proof. If w satisfies $\mathrm{sh}(w) \prec \lambda$, then the case $j = n$ of Proposition 6.1 implies that we have $\mathrm{Imm}_w(x_{[n],M}) = R_w(x_{[n],M}) = 0$. Suppose therefore that $\mathrm{sh}(w) = \lambda$ and $P(w) \neq T(\lambda)$. Since the tableau $T(\lambda)$ is maximal in iterated dominance among all tableaux of shape λ we have $T(\lambda) \triangleright_I P(w) = Q(w^{-1})$, and there exists an index j such that

$$\mathrm{sh}(Q(w^{-1})_{[j]}) \prec \mathrm{sh}(T(\lambda)_{[j]}) = \nu_{[j]}(x_{[n],M}). \tag{6.2}$$

By Lemma 2.2 we then have $\mathrm{sh}(w^{-1}_{[j]}) \prec \nu_{[j]}(x_{[n],M})$, which by Proposition 6.1 implies the desired result. ☐

Thus the absence of a quotient in the definition (4.12) of B^λ is explained by applying Corollary 6.2 to the result [5, Thm. 4.5]

$$\mathfrak{S}_n \circ R_v(x) \subseteq \mathrm{span}\{R_u(x) \,|\, u \leq_I v\}. \tag{6.3}$$

We now use the multiset $M = 1^{\lambda_1} \cdots n^{\lambda_n}$ to eliminate the quotient from the definition of $V^\lambda(1)$, while maintaining an $\mathfrak{S}_n$-module equivalent to that defined by Kazhdan and Lusztig in [15]. Define the space

$$W^\lambda \underset{\text{def}}{=} \mathrm{span}\{\mathrm{Imm}_w(x_{[n],M}) \,|\, P(w) = T(\lambda)\}. \tag{6.4}$$

Proposition 6.3. *For $\lambda \vdash n$, the space W^λ (6.4) is an $\mathfrak{S}_n$-module.*

Proof. Choose a permutation v satisfying $P(v) = T(\lambda)$ and define $M = 1^{\lambda_1} \cdots n^{\lambda_n}$. By (5.29) we have

$$\mathfrak{S}_n \circ W^\lambda \subseteq W^\lambda \oplus \mathrm{span}\{\mathrm{Imm}_u(x_{[n],M}) \,|\, u \geq_L v, u \not\leq_L v\}. \tag{6.5}$$

Consider a permutation u satisfying $v \leq_L u \not\leq_L v$. By (3.10) we have either $\mathrm{sh}(u) \prec \mathrm{sh}(v)$ or $\mathrm{sh}(u) = \mathrm{sh}(v)$ and $P(u) = P(v)$. In both cases, Corollary 6.2 implies that we have $\mathrm{Imm}_u(x_{[n],M}) = 0$, and it follows that $\mathfrak{S}_n \circ W^\lambda = W^\lambda$. ☐

Checking that the matrix specialization $x \mapsto x_{[n],M}$ introduces no linear relations among the images of the basis elements $\{\mathrm{Imm}_w(x) \,|\, P(w) = T(\lambda)\}$, we see that the module W^λ is irreducible.

Theorem 6.4. *For each partition $\lambda \vdash n$, the space W^λ defined in (6.4) is an irreducible $\mathfrak{S}_n$-module indexed by λ. Furthermore, the Kazhdan-Lusztig immanant basis of W^λ produces the same matrix representation of $\mathfrak{S}_n$ as does the Kazhdan-Lusztig basis of $K^\lambda(1)$.*

Proof. Write $\lambda = (\lambda_1, \dots, \lambda_r)$, and define the Young subgroup W_J of $\mathfrak{S}_n$ and the set W^J_+ of maximal coset representatives as in Section 2. By Lemma 2.1, the permutations w satisfying $P(w) = T(\lambda)$ belong to W^J_+. By [7, Sec. 2] and [27, Thm. 2.1], the polynomials $\{\mathrm{Imm}_v(x_{[n],M}) \,|\, v \in W^J_+\}$ form a basis of $\mathcal{A}_{[n],M}(x)$. Thus the subset $\{\mathrm{Imm}_v(x_{[n],M}) \,|\, P(v) = T(\lambda)\}$ is linearly independent and forms a basis of W^λ.

Using this basis to represent elements of $\mathfrak{S}_n$ by matrices, we clearly obtain the same matrices as those obtained by using the basis $\{\mathrm{Imm}_v(x) \,|\, P(v) = T(\lambda)\}$ for $V^\lambda(1)$. Specializing Theorem 5.6 at $q^{\frac{1}{2}} = 1$, we see that these matrices are also equal to those obtained by using the Kazhdan-Lusztig basis of $K^\lambda(1)$. □

Like Clausen's module B^λ, the module W^λ requires no quotient and therefore does not rely upon the Kazhdan-Lusztig preorders. Not only do B^λ and W^λ have this attribute in common, they are in fact equal. This follows from work in [24, Sec. 6] on partial filtrations of the immanant space.

Theorem 6.5. *For all partitions $\lambda \vdash n$, Clausen's module B^λ is equal to the Kazhdan-Lusztig immanant module W^λ.*

Proof. By [24, Thm. 6.4], we have the equality of spaces

$$\mathrm{span}\{R_v(x) \,|\, \mathrm{sh}(v) \preceq \lambda\} = \mathrm{span}\{\mathrm{Imm}_v(x) \,|\, \mathrm{sh}(v) \preceq \lambda\}. \tag{6.6}$$

Specializing at $x = x_{[n],M}$ and applying Corollary 6.2 and Theorem 6.4, we then have

$$\begin{aligned} B^\lambda = \mathrm{span}\{R_v(x_{[n],M}) \,|\, P(v) = T(\lambda)\} &= \mathrm{span}\{R_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\} \\ &= \mathrm{span}\{\mathrm{Imm}_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\} \\ &= \mathrm{span}\{\mathrm{Imm}_v(x_{[n],M}) \,|\, P(v) = T(\lambda)\} = W^\lambda. \end{aligned} \tag{6.7}$$

□

By [24, Sec. 5], the bitableau and Kazhdan-Lusztig immanant bases of any component $\mathcal{A}_{L,M}(x)$ of $\mathbb{C}[x]$ are related by a unitriangular transition matrix having nonnegative integer entries. Thus for $L = [n]$ and $M = 1^{\lambda_1} \cdots n^{\lambda_n}$ we have

$$\mathcal{A}_{[n],M}(x) = \mathrm{span}\{R_w(x_{[n],M}) \,|\, w \in W^J_+\} = \mathrm{span}\{\mathrm{Imm}_w(x_{[n],M}) \,|\, w \in W^J_+\}, \tag{6.8}$$

and there exist nonnegative integers $\{d^{[n],M}_{u,v} \,|\, u, v \in W^J_+\}$ satisfying

$$R_v(x_{[n],M}) = \mathrm{Imm}_v(x_{[n],M}) + \sum_{u <_I v} d^{[n],M}_{u,v} \mathrm{Imm}_u(x_{[n],M}), \tag{6.9}$$

where the relation $<_I$ is the iterated dominance order on $\mathfrak{S}_n$, defined in Section 2. For convenience, we define $d^{[n],M}_{u,u} = 1$ and $d^{[n],M}_{u,v} = 0$ if $u \not\leq_I v$. By Equation (6.7), the fact (6.9) restricts nicely to the bitableau and Kazhdan-Lusztig immanant bases of the subspace

$B^\lambda = W^\lambda$ of $\mathcal{A}_{[n],M}(x)$. Specifically, for v satisfying $P(v) = T(\lambda)$, the sum on the right-hand side of (6.9) may be taken over permutations u belonging to the set

$$Z(\lambda) \underset{\text{def}}{=} \{u \,|\, P(u) = T(\lambda)\}. \tag{6.10}$$

It follows that the matrix representations corresponding to the Kazhdan-Lusztig basis of $K^\lambda(1)$ and bideterminant basis of B^λ are related by a unitriangular transition matrix with integer coefficients. For fixed λ let $k = \dim W^\lambda$ be the number of standard Young tableaux of shape λ, and let the maps $X_1^\lambda, X_3^\lambda : \mathfrak{S}_n \to GL_k(\mathbb{C})$ be the matrix representations of $\mathfrak{S}_n$ corresponding to the Kazhdan-Lusztig basis of K^λ and the bideterminant basis of B^λ, each ordered by any linear extension of the iterated dominance order on $Z(\lambda)$. Then we have the following.

Corollary 6.6. *For each partition $\lambda \vdash n$, there exists a unitriangular matrix $A = A(\lambda)$ with nonnegative integer coefficients such that the matrix representations X_1^λ, X_3^λ defined above are related by*

$$X_1^\lambda(v) = A^{-1} X_3^\lambda(v) A \tag{6.11}$$

for all $v \in \mathfrak{S}_n$.

Proof. Defining $Z = Z(\lambda)$ as in (6.10), we have that $A = (a_{u,v})_{u,v \in Z} = (d_{u,v}^{[n],M})_{u,v \in Z}$ is the $k \times k$ transition matrix relating the bideterminant basis of B^λ to the Kazhdan-Lusztig immanant basis. ☐

It would be interesting to quantize all of the results in this section so that the quantizations of Theorem 6.5 and Corollary 6.6 would provide an analog in the quantum polynomial ring $\mathcal{A}(x;q)$ of the McDonough-Pallikaros result [21, Thm. 4.1] for the Hecke algebra $H_n(q)$. ($\star$ Rephrase this?)

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