INEQUALITIES IN PRODUCTS OF MINORS OF TOTALLY NONNEGATIVE MATRICES

Mark Skandera

(U. Michigan)

Outline

1. Totally nonnegative matrices
2. Interpretation in terms of path families
3. Inequalities in products of minors
4. Characterization theorems
5. Open questions

Total nonnegativity

Given an $n \times n$ matrix A and two subsets I, I' of $[n] = \{1, \ldots, n\}$, let $\Delta_{I,I'}$ be the (I, I')-*minor* of A: the determinant of the submatrix of A corresponding to rows I and columns I'.

$$A = \begin{bmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix},$$

$$\Delta_{\{1,3\},\{2,3\}} = \det \begin{bmatrix} 6 & 3 \\ 4 & 4 \end{bmatrix} = 12.$$

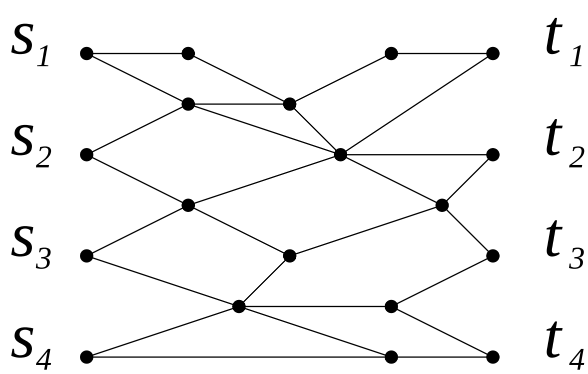
Define a matrix to be *totally nonnegative* (TNN) if each of its minors is nonnegative.

Such matrices arise in various mathematical settings:

1. representation theory
2. algebraic geometry
3. stochastic processes
4. matroids
5. electrical networks
6. roots of polynomials
7. posets
8. differential equations

Planar networks

Define a *planar network of order n* to be a directed acyclic planar graph, in which $2n$ boundary vertices are labeled counterclockwise as $s_1, \ldots, s_n, t_n, \ldots, t_1$ as below.



Define the *path matrix* of a planar network by $A = [a_{ij}]$, where a_{ij} counts paths from s_i to t_j.

Theorem: (Karlin-McGregor '59, Lindström '73) The path matrix of a planar network is always TNN.

Proof idea: $\Delta_{I,I'}$ counts families of nonintersecting paths from sources

$$S_I = \{s_i \,|\, i \in I\}$$

to sinks

$$T_{I'} = \{t_i \,|\, i \in I'\}.$$

More generally, if we *weight* edges in the network by positive real numbers, and define the weight of a path to be the product of its edge weights, then the corresponding *weighted* path matrix is TNN.

Theorem: (Whitney '52, Loewner '55, Cryer '76, Brenti '95) A TNN matrix is always the weighted path matrix of a planar network.

Proof idea: A TNN matrix A can be factored into elementary TNN matrices each of which is easily representable by a planar network. The concatenation of these networks represents A.

Question: Is an integer TNN matrix always the (unweighted) path matrix of a planar network?

TNN polynomials

Call a polynomial $f(x_{1,1}, \ldots, x_{n,n})$ in n^2 variables *totally nonnegative* if for any TNN matrix $A = [a_{i,j}]$, we have

$$f(a_{1,1}, \ldots, a_{n,n}) \geq 0.$$

Theorem: (Lusztig '94) The elements of Zelevinsky and Berenstein's dual canonical basis for the coordinate ring of GL_n are TNN.

Problem: Find a simple description of the dual canonical basis for GL_n.

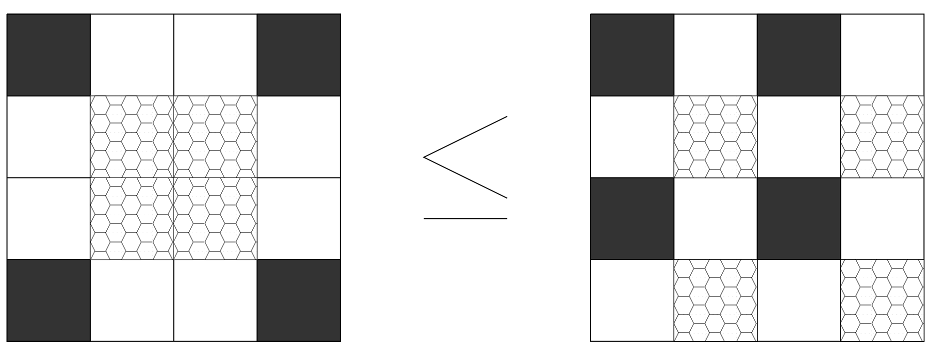
Problem: Find a simple description of any family of TNN polynomials.

Inequalities in principal minors

Observation: (FGJ '01) The inequality

$$\Delta_{14,14}\Delta_{23,23} \le \Delta_{13,13}\Delta_{24,24}$$

holds for all TNN matrices.



Equivalently, the polynomial

$$\Delta_{13,13}\Delta_{24,24} - \Delta_{14,14}\Delta_{23,23}$$

is TNN.

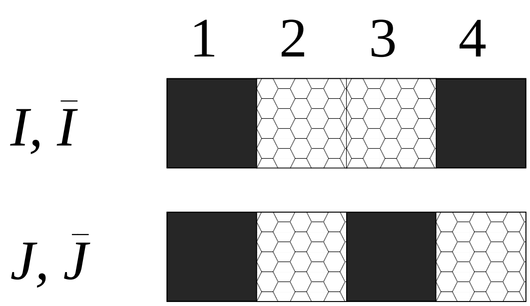
Theorem: (FGJ '01) The inequality

$$\Delta_{I,I}\Delta_{\overline{I},\overline{I}} \leq \Delta_{J,J}\Delta_{\overline{J},\overline{J}}$$

holds for all TNN matrices if and only if the set partition $(J, \overline{J})$ of $[n]$ is at least as *sparse* as $(I, \overline{I})$.

i.e., if for each subinterval B of $[n]$, we have

$$\max\{|B \cap I|, |B \cap \overline{I}|\} \geq \max\{|B \cap J|, |B \cap \overline{J}|\}.$$

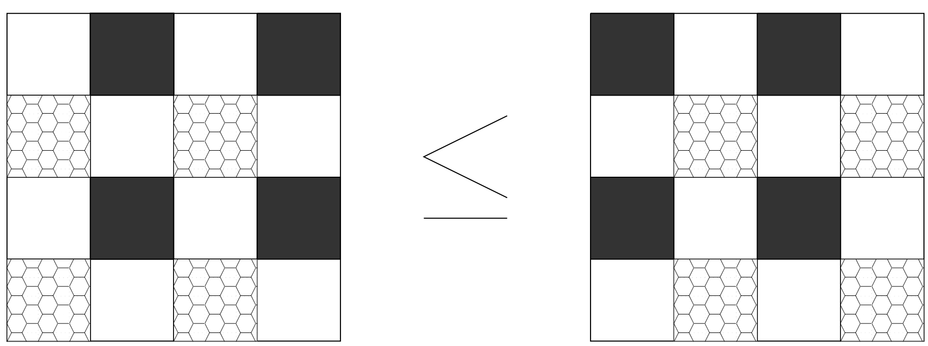


Inequalities in nonprincipal minors

Observation: The inequality

$$\Delta_{13,24}\Delta_{24,13} \leq \Delta_{13,13}\Delta_{24,24}$$

holds for all TNN matrices.



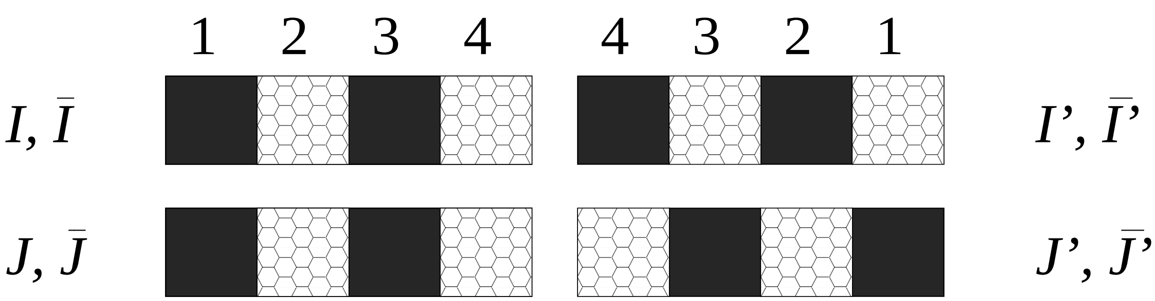
Equivalently, the polynomial

$$\Delta_{13,13}\Delta_{24,24} - \Delta_{13,24}\Delta_{24,13}$$

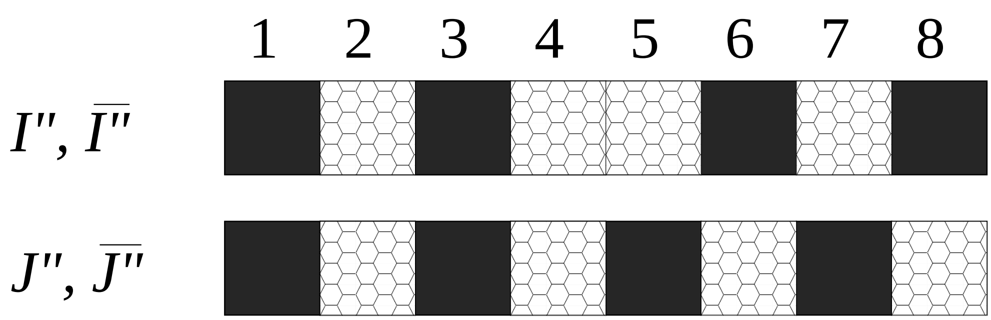
is TNN.

Question: When does $\Delta_{I,I'}\Delta_{\overline{I},\overline{I'}} \leq \Delta_{J,J'}\Delta_{\overline{J},\overline{J'}}$ hold for all TNN matrices?

Write $(I', \overline{I'})$ and $(J', \overline{J'})$ backwards.



Swap $(I', \overline{I'})$, swap $(J', \overline{J'})$, renumber.



Answer: When the set partition $(J'', \overline{J''})$ of $[2n]$ is at least as sparse as $(I'', \overline{I''})$.

Theorem: (MS '01) The inequality

$$\Delta_{I,I'}\Delta_{\overline{I},\overline{I'}} \leq \Delta_{J,J'}\Delta_{\overline{J},\overline{J'}}$$

holds for all TNN matrices if and only if the set partition $(J'', \overline{J''})$ of $[2n]$ is at least as sparse as $(I'', \overline{I''})$, where

$$I'' = I \cup \{2n+1-i \,|\, i \in \overline{I'}\},$$

$$J'' = J \cup \{2n+1-j \,|\, j \in \overline{J'}\}.$$

Corollary: The above inequality holds for all TNN matrices if and only if the inequality

$$\Delta_{I'',I''}\Delta_{\overline{I''},\overline{I''}} \leq \Delta_{J'',J''}\Delta_{\overline{J''},\overline{J''}}$$

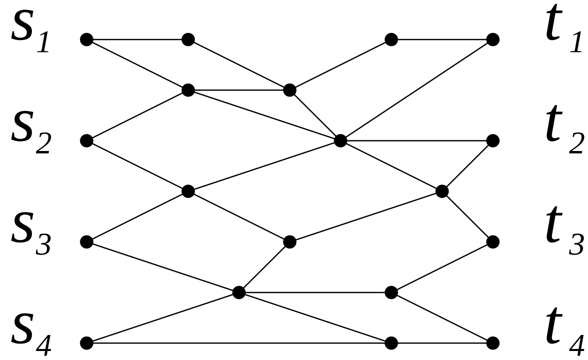
holds for all TNN matrices.

Combinatorial interpretation of the products of minors

Let A be the path matrix of a planar network G of order n. Then $\Delta_{I,I'}\Delta_{\overline{I},\overline{I'}}$ counts path families $(\pi_1, \ldots, \pi_n)$ in G such that

1. Each path begins in S_I and ends in $T_{I'}$ or a begins in $S_{\overline{I}}$ and ends in $T_{\overline{I'}}$.
2. The paths from S_I to $T_{I'}$ are pairwise disjoint, as are the paths from $S_{\overline{I}}$ to $T_{\overline{I'}}$.

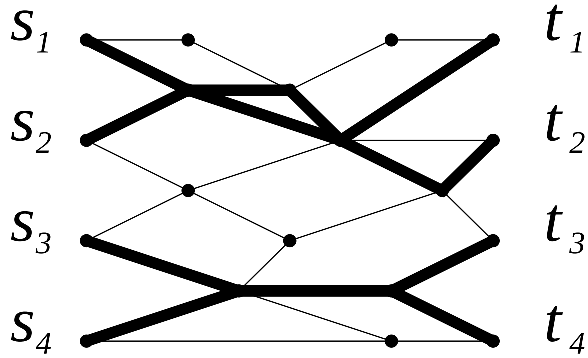
We will say that such a path family *obeys the* (I, I') *crossing rule.*



Path families which cover a planar network

Question: When do path families which obey the (J, J') crossing rule outnumber those which obey the (I, I') crossing rule in every planar network?

Observation: It suffices to consider only those planar networks which can be covered by n paths.



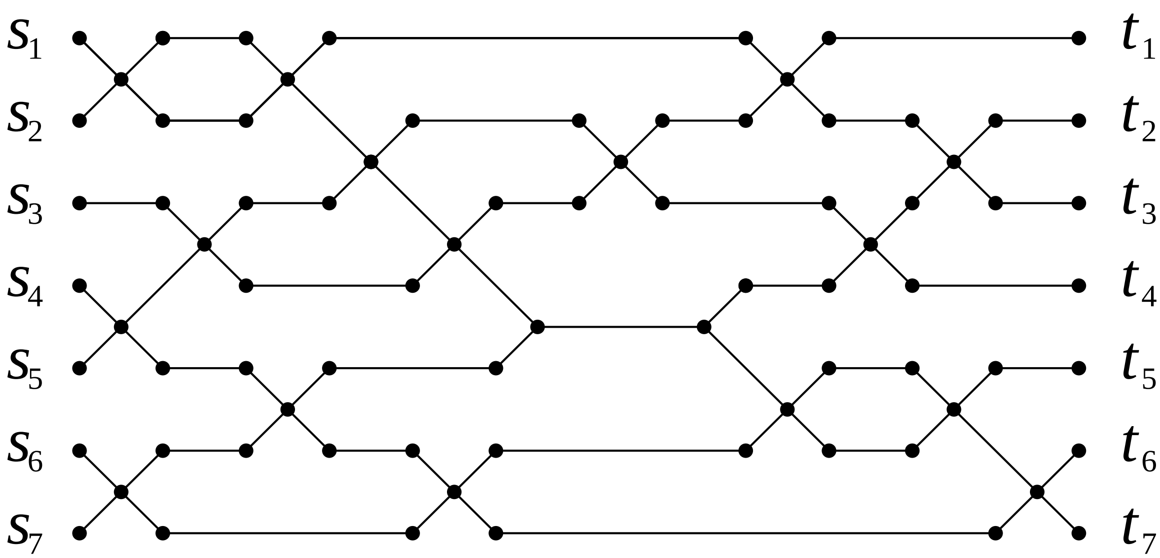
Comparison reduces to existence

Proposition: Let G be a planar network. Path families which cover G and obey the (J, J') crossing rule cannot outnumber those which cover G and obey the (I, I') crossing rule, unless there are none of the latter.

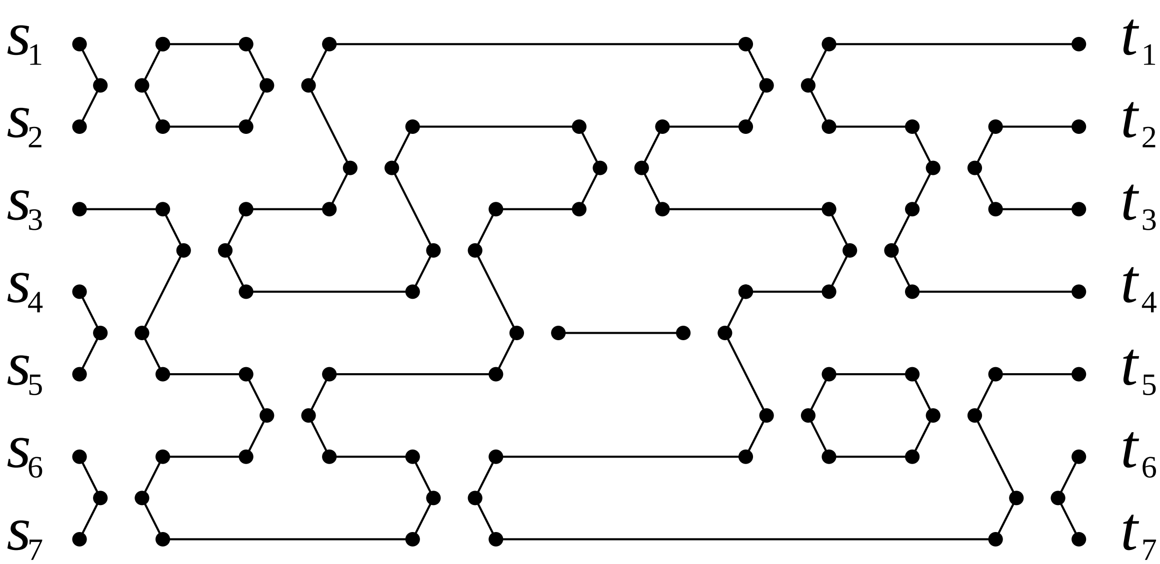
Proof idea: Suppose that G can be covered by at least one path family which obeys each crossing rule. Then the number of such path families is 2^k, where k is the number of cyclic components of the graph $\phi(G)$ defined on the next page.

Counting covering path families

A planar network G.



The graph $\phi(G)$.

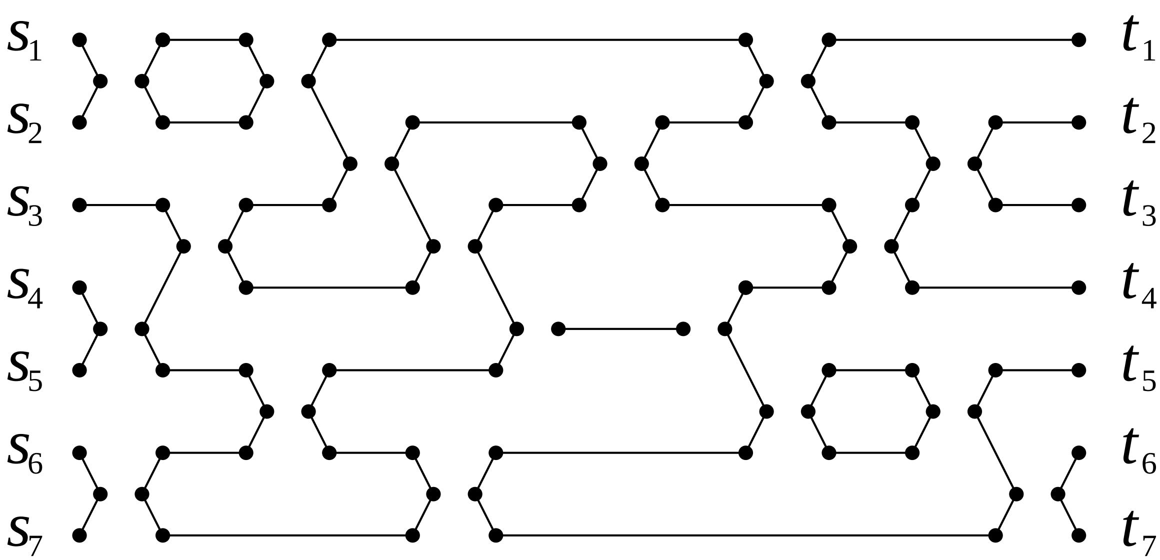


Existence reduces to matching

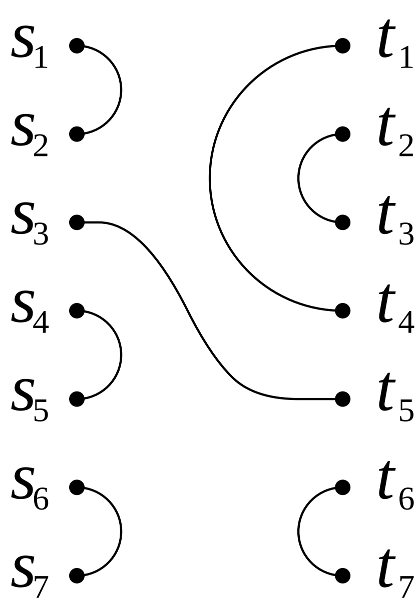
Corollary: G can be covered by a path family which obeys the (I, I') crossing rule if and only if n path components of $\phi(G)$ induce a perfect matching of $S_I \cup T_{\overline{I'}}$ with $S_{\overline{I}} \cup T_{I'}$.

Corollary: More path families which cover G obey the (J, J') crossing than the (I, I') crossing rule if and only if $\phi(G)$ induces a matching of $S_J \cup T_{\overline{J'}}$ with $S_{\overline{J}} \cup T_{J'}$, and does not induce a matching of $S_I \cup T_{\overline{I'}}$ with $S_{\overline{I}} \cup T_{I'}$.

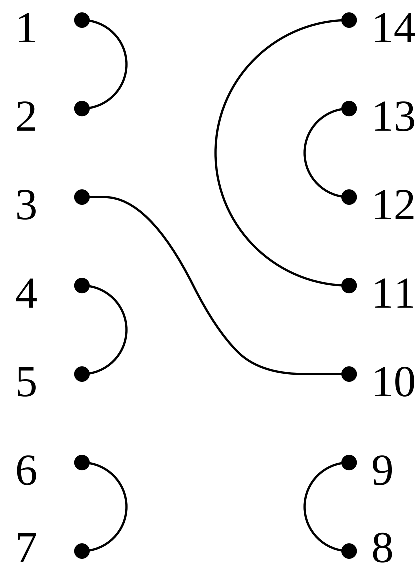
The graph $\phi(G)$.

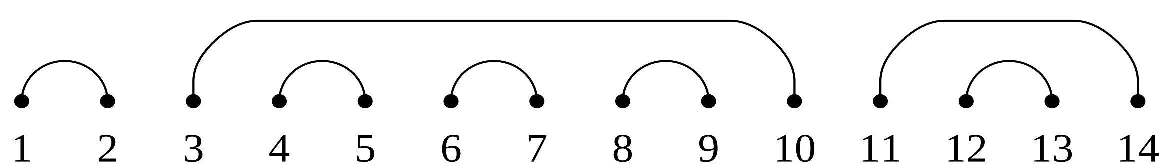


The matching $\psi(G)$ induced by $\phi(G)$.



Two embeddings of $\psi(G)$

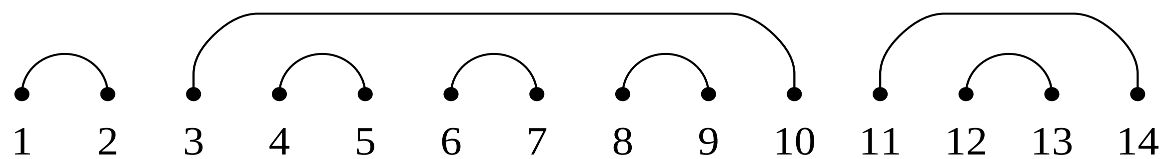




Define $I'' = I \cup \{2n+1-i \,|\, i \in \overline{I'}\}$,

$$J'' = J \cup \{2n+1-j \,|\, j \in \overline{J'}\}.$$

Matching reduces to examining intervals



Suppose that $\psi(G)$ is a matching of J'' with $\overline{J''}$ and not a matching of I'' with $\overline{I''}$. Then some arc (b_1, b_2) has both endpoints in I'' (or $\overline{I''}$) and exactly one endpoint in each of J'' and $\overline{J''}$. Choosing this arc to minimize $b_2 - b_1$, we have

$$\max\{|B \cap I''|, |B \cap \overline{I''}|\} = \tfrac{1}{2}|B| + 1,$$
$$|B \cap J''| = |B \cap \overline{J''}| = \tfrac{1}{2}|B|.$$

Conversely, suppose we are given subsets I, J of $[n]$ such that the partition $(I'', \overline{I''})$ of $[2n]$ is not less sparse than $(J'', \overline{J''})$. Then by definition there is an interval B such that

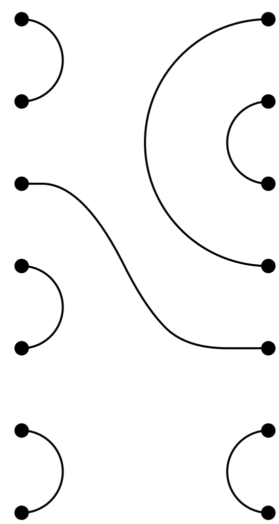
$$|B \cap I''| = |B \cap \overline{I''}| = \tfrac{1}{2}|B|,$$
$$\max\{|B \cap J''|, |B \cap \overline{J''}|\} = \tfrac{1}{2}|B| + 1.$$

It is then easy to create a planar network G in which families which obey the (I, I') crossing rule outnumber those which obey the (J, J') crossing rule.

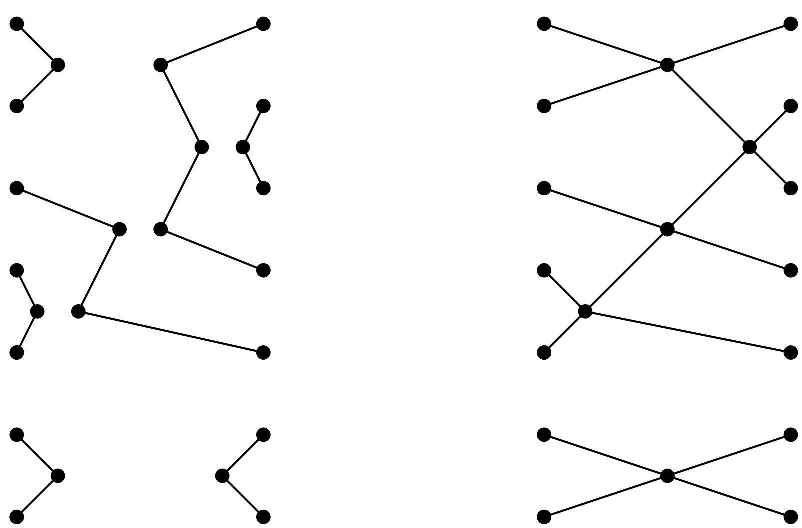
The path matrix of G then satisfies

$$\Delta_{I,I'}\Delta_{\overline{I},\overline{I'}} > \Delta_{J,J'}\Delta_{\overline{J},\overline{J'}}.$$

Create a matching H including edge (b_1, b_2).



Create a planar network G s.t. $\psi(G) = H$.



Corollary of main theorem

Corollary: (MS '01) Assume that

$$\Delta_{I,I'}\Delta_{\overline{I},\overline{I'}} \leq \Delta_{J,J'}\Delta_{\overline{J},\overline{J'}}$$

holds for all TNN matrices, and let A be a TNN matrix which is the (weighted) path matrix of the planar network G.

The nonnegative number

$$\Delta_{J,J'}\Delta_{\overline{J},\overline{J'}} - \Delta_{I,I'}\Delta_{\overline{I},\overline{I'}}$$

is equal to the weighted sum of path families in G which obey the (J, J') crossing rule and can not be covered by any path family which obeys the (I, I') crossing rule.

Theorem: (MS '01) The inequality

$$\Delta_{I,I'}\Delta_{K,K'} \leq \Delta_{J,J'}\Delta_{L,L'}$$

holds for all TNN matrices if and only if we can delete repeated indices and reduce to the previous theorem.

Corollary: All of the inequalities of the above form are consequences of inequalities of the form

$$\Delta_{I,I}\Delta_{\overline{I},\overline{I}} \leq \Delta_{J,J}\Delta_{\overline{J},\overline{J}},$$

where I, J are n-subsets of $[2n]$.

Open questions

1. Which TNN polynomials can be written as subtraction-free rational expressions (or Laurent polynomials) in matrix minors?

2. How can one characterize inequalities that hold between products of k minors of TNN matrices, for $k > 2$?

3. Which TNN polynomials when applied to Jacobi-Trudi matrices evaluate to schur-positive symmetric functions?

4. Is there a true combinatorial interpretation for the minors of integer TNN matrices?