PERMANENTAL INEQUALITIES FOR TOTALLY POSITIVE MATRICES

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Outline

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(3) A multigrading of $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$ and the Bruhat order
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Totally positive matrices

Given $n \times n$ matrix $A = (a_{i,j})$, subsets $I, J \subseteq [n] := \{1, \ldots, n\}$, $|I| = |J|$, define square submatrix $A_{I,J} = (a_{i,j})_{i \in I, j \in J}$.

Call A *totally positive* if $\det(A_{I,J}) > 0$ whenever $|I| = |J|$.
Let $\mathcal{M}_n^{\mathrm{TP}}$ be the set of $n \times n$ totally positive matrices.

Koteljanskii's inequality: For $A \in \mathcal{M}_n^{\mathrm{TP}}$, $I, J \subseteq [n]$, we have

$$\det(A_{I \cup J, I \cup J}) \det(A_{I \cap J, I \cap J}) \leq \det(A_{I,I}) \det(A_{J,J}).$$

Question: For all $A \in \mathcal{M}_n^{\mathrm{TP}}$, $I, J \subseteq [n]$, we do not have

$$\mathrm{per}(A_{I \cup J, I \cup J})\mathrm{per}(A_{I \cap J, I \cap J}) \geq \mathrm{per}(A_{I,I})\mathrm{per}(A_{J,J}).$$

Are these products related somehow?

Inequalities

Question: For which pairs of products

$$\mathrm{prod}_1(A) := \mathrm{per}(A_{I_1,J_1}) \cdots \mathrm{per}(A_{I_r,J_r}),$$
$$\mathrm{prod}_2(A) := \mathrm{per}(A_{I'_1,J'_1}) \cdots \mathrm{per}(A_{I'_r,J'_r})$$

do we have $\mathrm{prod}_1(A) \leq \mathrm{prod}_2(A)$ for all $A \in \mathcal{M}_n^{\mathrm{TP}}$?

Question: For which pairs are there constants k_1, k_2 satisfying $k_1\,\mathrm{prod}_1(A) \leq \mathrm{prod}_2(A) \leq k_2\,\mathrm{prod}_1(A)$ for all $A \in \mathcal{M}_n^{\mathrm{TP}}$?

Question: Which rational functions

$$\frac{\mathrm{per}(x_{I_1,J_1}) \cdots \mathrm{per}(x_{I_r,J_r})}{\mathrm{per}(x_{I'_1,J'_1}) \cdots \mathrm{per}(x_{I'_q,J'_q})}$$

in variables $x = (x_{i,j})_{i,j \in [n]}$ are bounded above and nontrivially bounded below (not just by 0) on $\mathcal{M}_n^{\mathrm{TP}}$?

Multigrading of $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$

Given r-element multisets $M = 1^{\alpha_1} \cdots n^{\alpha_n}$, $O = 1^{\beta_1} \cdots n^{\beta_n}$, define the set of matrices and the $\mathbb{Z}$-submodule

$$\mathcal{C}_{M,O} = \{C \in \mathrm{Mat}_{n\times n}(\mathbb{N}) \,|\, \mathrm{rowsums}(C) = \alpha,\ \mathrm{colsums}(C) = \beta\},$$

$$\mathcal{A}_{M,O} := \mathrm{span}_{\mathbb{Z}}\Big\{ \prod_{i,j\in[n]} x_{i,j}^{c_{i,j}} \,|\, C \in \mathcal{C}_{M,O} \Big\}.$$

We have $\mathbb{Z}[x] = \bigoplus_{r\geq 0} \bigoplus_{M,O} \mathcal{A}_{M,O}$.

Define the *Bruhat order* on $\mathcal{C}_{M,O}$ by mapping

$$C \mapsto C^* = (c^*_{i,j}), \qquad c^*_{i,j} = c_{1,1} + \cdots + c_{i,j},$$

and declaring $C \leq D$ if $C^* \geq D^*$ in the componentwise order, i.e., $c^*_{i,j} \geq d^*_{i,j}$ for all i, j.

Example:

$$M = (1,1,2,3) = 1^2 2^1 3^1, \qquad O = (1,2,2,2) = 1^1 2^3 3^0,$$

$$\mathcal{C}_{M,O} = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} < \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} < \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\},$$

$$\mathcal{A}_{M,O} = \operatorname{span}_{\mathbb{Z}}\{x_{1,1}x_{1,2}x_{2,2}x_{3,2},\ x_{1,2}^2 x_{2,1}x_{3,2},\ x_{1,2}^2 x_{2,2}x_{3,1}\},$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 4 \end{bmatrix} \geq_{\text{comp}} \begin{bmatrix} 0 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 4 \end{bmatrix} \geq_{\text{comp}} \begin{bmatrix} 0 & 2 & 2 \\ 0 & 3 & 3 \\ 1 & 4 & 4 \end{bmatrix}.$$

Bounded Ratio Theorem

Theorem: The rational function

$$R(x) = \frac{\mathrm{per}(x_{I_1,J_1}) \cdots \mathrm{per}(x_{I_r,J_r})}{\mathrm{per}(x_{I'_1,J'_1}) \cdots \mathrm{per}(x_{I'_q,J'_q})},$$

with associated matrices C, D defined by

$$c_{i,j} = \#\{k \,|\, x_{i,j} \text{ on diagonal of submatrix } x_{I_k,J_k}\},$$
$$d_{i,j} = \#\{k \,|\, x_{i,j} \text{ on diagonal of submatrix } x_{I'_k,J'_k}\}$$

is bounded above on $\mathcal{M}_n^{\mathrm{TP}}$ if and only if $C \geq D$ in $\mathcal{C}_{M,O}$ for some M, O. Specifically, $R(x) \leq |I_1|! \cdots |I_r|!$.

Corollary: If $C = D$, then $R(x)$ is bounded above and below:

$$\frac{1}{|I'_1|! \cdots |I'_q|!} \leq R(x) \leq |I_1|! \cdots |I_r|!.$$

Example: Consider products in $\mathcal{A}_{1123,1222}$ and matrices

$$\mathrm{prod}_1 = \mathrm{per}(x_{12,12})x_{1,2}x_{3,2}, \quad \begin{array}{|c|c|c|}\hline \bullet & \circ & \\ \hline & \bullet & \\ \hline & \star & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 3 & 3 \\ \hline 1 & 4 & 4 \\ \hline \end{array},$$

$$\mathrm{prod}_2 = \mathrm{per}(x_{13,12})x_{1,2}x_{2,2}, \quad \begin{array}{|c|c|c|}\hline \bullet & \circ & \\ \hline & \star & \\ \hline & \bullet & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 3 & 3 \\ \hline 1 & 4 & 4 \\ \hline \end{array},$$

$$\mathrm{prod}_3 = x_{1,1}x_{1,2}x_{2,2}x_{3,2}, \quad \begin{array}{|c|c|c|}\hline \bullet & \circ & \\ \hline & \star & \\ \hline & \wr & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 3 & 3 \\ \hline 1 & 4 & 4 \\ \hline \end{array},$$

$$\mathrm{prod}_4 = x_{1,2}^2\ \mathrm{per}(x_{23,12}), \quad \begin{array}{|c|c|c|}\hline & \bullet\bullet & \\ \hline \circ & & \\ \hline & \circ & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 2 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 2 & 2 \\ \hline 1 & 3 & 3 \\ \hline 1 & 4 & 4 \\ \hline \end{array},$$

$$\mathrm{prod}_5 = x_{1,2}^2x_{2,1}x_{3,2}, \quad \begin{array}{|c|c|c|}\hline & \bullet\bullet & \\ \hline \circ & & \\ \hline & \star & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 2 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 2 & 2 \\ \hline 1 & 3 & 3 \\ \hline 1 & 4 & 4 \\ \hline \end{array},$$

$$\mathrm{prod}_6 = x_{1,2}^2x_{2,2}x_{3,1}, \quad \begin{array}{|c|c|c|}\hline & \bullet\bullet & \\ \hline & \circ & \\ \hline \star & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 2 & 0 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 2 & 2 \\ \hline 0 & 3 & 3 \\ \hline 1 & 4 & 4 \\ \hline \end{array}.$$

Products 1, 4, 6 have different exponent matrices, and we have

$$0 \le \frac{\mathrm{prod}_4}{\mathrm{prod}_1} = \frac{x_{1,2}^2 \ \mathrm{per}(x_{23,12})}{\mathrm{per}(x_{12,12}) x_{1,2} x_{3,2}} \le 2!1!1!,$$

$$0 \le \frac{\mathrm{prod}_6}{\mathrm{prod}_4} = \frac{x_{1,2}^2 x_{2,2} x_{3,1}}{x_{1,2}^2 \ \mathrm{per}(x_{23,12})} \le 1!1!1!1!,$$

$$0 \le \frac{\mathrm{prod}_6}{\mathrm{prod}_1} = \frac{x_{1,2}^2 x_{2,2} x_{3,1}}{\mathrm{per}(x_{12,12}) x_{1,2} x_{3,2}} \le 1!1!1!1!.$$

Products 1, 2, 3 (and 4, 5) have the same exponent matrices, and we have

$$\frac{1}{2!1!1!} \leq \frac{\mathrm{prod}_2}{\mathrm{prod}_1} = \frac{\mathrm{per}(x_{13,12})x_{1,2}x_{2,2}}{\mathrm{per}(x_{12,12})x_{1,2}x_{3,2}} \leq 2!1!1!,$$

$$\frac{1}{2!1!1!} \leq \frac{\mathrm{prod}_3}{\mathrm{prod}_1} = \frac{x_{1,1}x_{1,2}x_{2,2}x_{3,2}}{\mathrm{per}(x_{12,12})x_{1,2}x_{3,2}} \leq 1!1!1!1!,$$

$$\frac{1}{2!1!1!} \leq \frac{\mathrm{prod}_3}{\mathrm{prod}_2} = \frac{x_{1,1}x_{1,2}x_{2,2}x_{3,2}}{\mathrm{per}(x_{13,12})x_{1,2}x_{2,2}} \leq 1!1!1!1!,$$

$$\frac{1}{2!1!1!} \leq \frac{\mathrm{prod}_5}{\mathrm{prod}_4} = \frac{x_{1,2}^2x_{2,1}x_{3,2}}{x_{1,2}^2\,\mathrm{per}(x_{23,12})} \leq 1!1!1!1!.$$

Open Problems

Weak bounds for permanental Koteljanskii ratio on $\mathcal{M}_3^{\mathrm{TP}}$:

$$\frac{1}{3!} \leq \frac{\mathrm{per}(x_{12,12})\ \mathrm{per}(x_{23,23})}{\mathrm{per}(x_{123,123})\mathrm{per}(x_{2,2})} \leq 2!2!.$$

Problem: Find tight bounds in terms of $I, J \subseteq [n]$ for ratios

$$\frac{\mathrm{per}(x_{I,I})\ \mathrm{per}(x_{J,J})}{\mathrm{per}(x_{I\cup J,I\cup J})\ \mathrm{per}(x_{I\cap J,I\cap J})} \qquad \text{on } \mathcal{M}_n^{\mathrm{TP}}.$$

Problem: Fix n, $I = \{1, 3, \ldots, 2n-1\}$, $J = \{2, 4, \ldots, 2n\}$. Show that we have

$$\frac{\mathrm{per}(x_{I,I})\ \mathrm{per}(x_{J,J})}{\mathrm{per}(x_{[n],[n]})\ \mathrm{per}(x_{[n+1,2n],[n+1,2n]})} \leq 1 \qquad \text{on } \mathcal{M}_n^{\mathrm{TP}}.$$