AN EULERIAN PARTNER FOR INVERSIONS

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Outline

1. Permutations
2. Permutation Statistics
3. Bijections
4. A New Statistic
5. Equidistribution Results

Permutations

RICHARD P STANLEY
SHIPYARD CENTRAL
DRY SPINACH ALERT
HIP TRENDY RASCAL
SPRAY AT CHILDREN

DANA
ANDA

MROWKA
AKWORM (A. Perlin, '98)

RELAXED NAN PILER
ALEXANDER PERLIN (T. Mrowka, '99)

Definition. Let S_n be the symmetric group on n letters. A *permutation statistic* is a function $\phi : S_n \to \mathbb{N}$.

Two important examples are "des" (descents) and "exc" (excedances).

Descents

Let $\pi = \pi_1, \ldots, \pi_n$ be a permutation in S_n and define position i to be a *descent* in π if $\pi_i > \pi_{i+1}$.

Example: S_4 with marked descents.

4|3|2|1	3|14|2	124|3	23|14
34|2|1	3|24|1	134|2	13|24
24|3|1	2|14|3	234|1	4|123
14|3|2	4|3|12	34|12	3|124
4|13|2	4|2|13	24|13	2|134
4|23|1	3|2|14	14|23	1234

Excedances

Let $\pi = \pi_1, \ldots, \pi_n$ be a permutation in S_n and define position i to be an *excedance* in π if $\pi_i > i$.

Example: S_4 with marked excedances.

$\bar{2}\bar{3}\bar{4}1$	$1\bar{3}\bar{4}2$	$12\bar{4}3$	$\bar{3}214$
$\bar{4}\bar{3}21$	$\bar{3}1\bar{4}2$	$1\bar{3}42$	$\bar{4}123$
$\bar{4}\bar{3}12$	$\bar{2}\bar{4}31$	$1\bar{4}23$	$\bar{4}132$
$\bar{3}\bar{4}21$	$\bar{2}\bar{4}13$	$1\bar{4}32$	$\bar{4}213$
$\bar{3}\bar{4}12$	$\bar{2}\bar{3}14$	$\bar{2}134$	$\bar{4}231$
$\bar{3}2\bar{4}1$	$\bar{2}1\bar{4}3$	$\bar{3}124$	1234

Distribution of des and exc on $\mathbf{S_n}$

The permutation statistics des and exc are called *Eulerian* because the *Eulerian numbers* count permutations π in S_n with

$$\mathrm{des}(\pi) = k$$
$$(\text{ or } \mathrm{exc}(\pi) = k).$$

$n \backslash k$	0	1	2	3	4	5
1	1					
2	1	1				
3	1	2	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

Descent set and major index

To each permutation with k descents, we associate a *descent set* $D(\pi)$.

$$D(\pi) = \{i | i \text{ is a descent in } \pi\}.$$

We define the statistic MAJ (major index) to be the sum of the descents of π.

$$\mathrm{MAJ}(\pi) = \sum_{i \in D(\pi)} i.$$

Descent set and major index of some permutations

π	$D(\pi)$	$\mathrm{MAJ}(\pi)$
$4\|3\|2\|1$	$\{1,2,3\}$	6
$34\|2\|1$	$\{2,3\}$	5
$4\|13\|2$	$\{1,3\}$	4
$4\|3\|12$	$\{1,2\}$	3
$124\|3$	$\{3\}$	3
$34\|12$	$\{2\}$	2
$4\|123$	$\{1\}$	1
1234	$\emptyset$	0

Inversions

Definition. An *inversion* in a permutation is a pair $(\pi_i > \pi_j)$ such that $i < j$.

$\textsc{inv}(\pi)$ counts the number of inversions in π.

Example. $\pi = 45123$ has six inversions:

$$(4,1),(4,2),(4,3),(5,1),(5,2),(5,3).$$

Distribution of MAJ and INV on $\mathbf{S_n}$

The statistics MAJ and INV are called *Mahonian*, after MacMahon. Numbers of permutations π in $S_1, \ldots, S_5$ with

$$\text{MAJ}(\pi) = k$$
$$(\text{ or } \text{INV}(\pi) = k).$$

are shown below.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
1	1										
2	1	1									
3	1	2	2	1							
4	1	3	5	6	5	3	1				
5	1	4	9	15	20	22	20	15	9	4	1

Substair vectors

Definition. Let E_n be the set of all vectors v in $\mathbb{N}^n$ which are less than or equal to the *stair vector* of length n

$$(n-1, n-2, \dots, 1, 0).$$

Note that $|E_n| = n!$.

The code of a permutation

A well known bijection maps a permutation to its *code*.

$$\begin{aligned} \gamma : S_n &\to E_n \\ \pi &\mapsto \mathrm{code}(\pi). \end{aligned}$$

We define $\mathrm{code}(\pi) = c_1, \ldots, c_n$, where c_i counts letters to the *right* of position i and *smaller* than π_i.

Example.

$$\begin{aligned} \pi \quad &= 3\ 5\ 4\ 2\ 6\ 1 \\ \mathrm{code}(\pi) &= 2\ 3\ 2\ 1\ 1\ 0. \end{aligned}$$

Note that $\sum_i^n c_i = \textsc{inv}(\pi)$.

The major index table

Let $\pi^{(i)}$ be the restriction of π to the letters $i, \ldots, n$.

Construct the sequence $\pi^{(n)}, \ldots, \pi^{(1)}$, and set

$$m_i = \begin{cases} \mathrm{MAJ}(\pi^{(i)}) - \mathrm{MAJ}(\pi^{(i+1)}) & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases}$$

Define $\mathrm{majtable}(\pi) = (m_1, \ldots, m_n)$.

Note that $\sum_i^n m_i = \mathrm{MAJ}(\pi)$.

Example. Let $\pi = 413265$. Inserting the letters in the order $6, 5, \ldots, 1$, we obtain

$\pi^{(i)}$	$\text{MAJ}(\pi^{(i)})$	m_i
6	0	0
$6\|5$	1	1
$46\|5$	2	1
$4\|36\|5$	4	2
$4\|3\|26\|5$	7	3
$4\|13\|26\|5$	9	2

Thus, $\text{majtable}(413265) = 232110$.

Theorem. (Carlitz, 1975) *The map*

$$\mu : S_n \to E_n,$$

taking a permutation to its major index table, is a bijection.

Proof. We invert μ by writing the partial permutations $\pi^{(n)} = n, \ldots, \pi^{(1)} = \pi$ such that the insertion of each letter i increases MAJ by m_i. This is possible by the following lemma. $\square$

Let π be a word on the letters $\{i+1, \ldots, n\}$, and suppose π has k descents. Call the $n-i-k$ ascent positions

$$a_1 < \cdots < a_{n-i-k},$$

call the k descent positions

$$d_{k-1} < \cdots < d_0,$$

and define $d_k = 0$.

Lemma.

1. The insertion of i into position $d_\ell + 1$ of π creates no new descent, and increases MAJ by ℓ, for $\ell = 0, \ldots, k$.
2. The insertion of i into position $a_\ell + 1$ of π creates one new descent and increases MAJ by $k + \ell$, for $\ell = 1, \ldots, n-i-k$.

Example. We insert the letter 1 into 43265 and calculate MAJ.

π	$\text{MAJ}(\pi)$
$4\|3\|26\|5$	7
$4\|3\|26\|15$	7
$4\|3\|126\|5$	8
$4\|13\|26\|5$	9
$14\|3\|26\|5$	10
$4\|3\|2\|16\|5$	11
$4\|3\|26\|5\|1$	12

Proof. (1.) Insertion of i into position $d_\ell + 1$ increments ℓ descents.

(2.) Suppose there are p descents before a_ℓ, and $k - p$ descents after. Then, $a_\ell = p + \ell$.

Insertion of i into position $a_\ell + 1$ creates a new descent at a_ℓ and increments $k - p$ descents. The total increase is

$$p + \ell + k - p = k + \ell.$$

□

Corollary. *The permutation statistics* INV *and* MAJ *are equally distributed on* S_n.

Proof. The map $\phi : S_n \to S_n$ defined by $\phi = \gamma^{-1}\mu$ is a bijection satisfying

$$\mathrm{MAJ}(\pi) = \mathrm{INV}(\phi(\pi)).$$

$\square$

$$\begin{tikzcd} \pi & & \rho \\ S_n \arrow[rr, "\phi"] \arrow[dr, "\mu"'] & & S_n \\ & M_n \arrow[ur, "\gamma^{-1}"'] & \end{tikzcd}$$

$$\mathrm{majtable}(\pi) = \mathrm{code}(\rho)$$

Example. $\phi(413265) = 354261$, since

$$\mathrm{majtable}(413265) = 232110 = \mathrm{code}(354261).$$

Eulerian - Mahonian pairs

The *joint* distribution on S_n of $(\mathrm{des}, \textsc{maj})$ has nice symmetries which the pairs $(\mathrm{des}, \textsc{inv})$, $(\mathrm{exc}, \textsc{maj})$, and $(\mathrm{exc}, \textsc{inv})$ lack.

Consider the distribution of $(\mathrm{des}, \textsc{maj})$ on S_5.

des\MAJ	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		4	9	9	4						
2				6	16	22	16	6			
3							4	9	9	4	
4											1

Question: For what natural Mahonian statistic X is $(\mathrm{exc}, \mathrm{X})$ distributed on S_n like $(\mathrm{des}, \mathrm{MAJ})$?

Answer: $\mathrm{X} = \mathrm{DEN}$ (FZ '89, H '95).

Question: For what natural Eulerian statistic z is $(\mathrm{z}, \mathrm{INV})$ distributed on S_n like $(\mathrm{des}, \mathrm{MAJ})$?

Calculation of stc

Begin at the right of code(π), and moving left, circle the first letter which is at least 1, the next which is at least 2, etc., until this is no longer possible. Set

$$\mathrm{stc}(\pi) = \text{ number of circles.}$$

Example.

$$\pi \quad = 4\ 6\ 2\ 3\ 5\ 1$$

$$\mathrm{code}(\pi) = 3\ 4\ 1\ 1\ 1\ 0.$$

$$\mathrm{stc}(462351) = 3.$$

Definitions of st, stc, stm

Let $v = v_1, \ldots, v_n$ be a sub-stair vector of length n.
Define the function $st : E_n \to \mathbb{N}$ by mapping v to the length ℓ of the longest super-stair subsequence of v.

Define the permutation statistics stc and stm by

$$\begin{aligned}\mathrm{stc}(\pi) &= st(\mathrm{code}(\pi)),\\ \mathrm{stm}(\pi) &= st(\mathrm{majtable}(\pi)).\end{aligned}$$

Theorem. *The pairs of permutation statistics* (des, MAJ) *and* (stc, INV) *are equally distributed on* S_n.

Proof. Let $\phi : S_n \to S_n$ be the bijection proving the equidistribution of MAJ and INV.

We show that $\mathrm{des}(\pi) = \mathrm{stc}(\phi(\pi))$.

Since $\mathrm{majtable}(\pi) = \mathrm{code}(\phi(\pi))$, it suffices to show that $\mathrm{des}(\pi) = \mathrm{stm}(\pi)$. $\square$

Let π be a permutation in S_n with major index table $m = m_1, \ldots, m_n$. Suppose that we have circled positions of m in calculating stm.

Fix i and assume that the number of circles in positions $i+1, \ldots, n$ is precisely $\mathrm{des}(\pi^{(i+1)})$. Call this number k_i.

By the definition of stc, position i is circled if and only if $m_i > k_i$.

By the majtable bijection, $m_i > k_i$ if and only if the insertion of i into $\pi^{(i+1)}$ creates a new descent.

Proceeding by induction, we prove the theorem.

A descent set analog for stc

To a permutation with k descents, we associate a descent set of k numbers which sum to MAJ.

Similarly, to a permutation with $\mathrm{stc} = k$ we will associate a *stc-set* of k numbers which sum to INV.

$$
\begin{array}{rcllllll}
m & = & 2 & 3 & 2 & 1 & 1 & 0 \\
\omega_4 m & = & 2 & 3 & 2 & 2 & 0 & 0 \\
\eta_3\omega_4 m & = & 2 & 3 & 3 & 1 & 0 & 0 \\
\omega_3\omega_4 m & = & 2 & 3 & 3 & 1 & 0 & 0 \\
\eta_2\omega_3\omega_4 m & = & 2 & 4 & 2 & 1 & 0 & 0 \\
\eta_3\eta_2\omega_3\omega_4 m & = & 2 & 4 & 2 & 1 & 0 & 0 \\
\omega_2\omega_3\omega_4 m & = & 2 & 4 & 2 & 1 & 0 & 0 \\
\eta_1\omega_2\omega_3\omega_4 m & = & 5 & 1 & 2 & 1 & 0 & 0 \\
\eta_2\eta_1\omega_2\omega_3\omega_4 m & = & 5 & 3 & 0 & 1 & 0 & 0 \\
\eta_3\eta_2\eta_1\omega_2\omega_3\omega_4 m & = & 5 & 3 & 1 & 0 & 0 & 0 \\
\omega_4 m & = & 5 & 3 & 1 & 0 & 0 & 0
\end{array}
$$

Let $H = \{\eta_1, \ldots, \eta_{n-2}\}$ be a set of operators on E_n.

η_i acts on v by modifying only its ith and $(i+1)$st components. These become

$$\begin{cases} (v_{i+1}+1, v_i - 1) & \text{if } v_i \leq v_{i+1} \text{ and } v_i \neq 0, \\ (v_{i+1}, v_i) & \text{if } v_i = 0 \text{ and } v_{i+1} > 0, \\ (v_i, v_{i+1}) & \text{otherwise.} \end{cases}$$

Example. Consider the action of η_1 on five different vectors in M_4.

$$\begin{aligned}
\eta_1(3200) &= 3200, \\
\eta_1(0010) &= 0010, \\
\eta_1(1200) &= 3000, \\
\eta_1(2200) &= 3100, \\
\eta_1(0110) &= 1010.
\end{aligned}$$

It is not difficult to see that the operators in the set H satisfy the relations of $H_{n-2}(0)$, the 0-*Hecke algebra* on $n-2$ generators:

$$\begin{aligned} \eta_i \eta_j &= \eta_j \eta_i, \text{ for } |i-j| \geq 2; \\ \eta_i \eta_{i+1} \eta_i &= \eta_{i+1} \eta_i \eta_{i+1}; \\ \eta_i^2 &= \eta_i. \end{aligned}$$

Let us introduce notation for several elements of $H_{n-2}(0)$.

For $i = 1, \ldots, n-2$, define

$$\omega_i = \eta_{n-2}\eta_{n-3}\cdots\eta_i,$$

and let ω be the product of these $n-2$ elements,

$$\omega = \omega_1 \cdots \omega_{n-2}.$$

Example. We compute ω in $H_4(0)$.

$$\begin{aligned}
\omega_1 &= \eta_4\eta_3\eta_2\eta_1, \\
\omega_2 &= \eta_4\eta_3\eta_2, \\
\omega_3 &= \eta_4\eta_3, \\
\omega_4 &= \eta_4, \\
\omega &= \omega_1\omega_2\omega_3\omega_4 \\
&= (\eta_4\eta_3\eta_2\eta_1)(\eta_4\eta_3\eta_2)(\eta_4\eta_3)(\eta_4).
\end{aligned}$$

st sets

Definition. The *st-set* of a substair vector v, denoted $ST(v)$, is the set of distinct, nonzero letters in $\omega(v)$.

The stc-set of a permutation π, denoted $STC(\pi)$, is $ST(\text{code}(\pi))$.

The stm-set of a permutation, denoted $STM(\pi)$, is $ST(\text{majtable}(\pi))$.

Theorem.

1. Given a permutation π with $stc(\pi) = k$, then the *stc-set* of π has cardinality k, and its elements sum to $\textsc{inv}(\pi)$.

2. For any subset T of $[n-1]$, the number of permutations ρ with stc-set T equals the number of permutations π with descent set T.

Proof. By the following lemma, descent set and stm-set are equivalent. Thus, stc-sets and descent sets are in bijective correspondence.

$$
\begin{aligned}
STC(\pi) &= ST(\text{code}(\pi)) \\
&= ST(\text{majtable}(\phi^{-1}(\pi))) \\
&= STM(\phi^{-1}(\pi)).
\end{aligned}
$$

□

Let π be a permutation in S_n, let m be its major index table, and let $H_{n-2}(0)$ act on E_n as defined earlier.

Lemma. For $i = 1, \ldots, n-2$, the last $n-i+1$ components of $\omega_i \cdots \omega_{n-2} m$ are the descent set of $\pi^{(i)}$, arranged in decreasing order, and followed by zeros.

$$
\begin{array}{rcccccccccc}
m & = & 2 & 3 & 2 & 1 & \underline{1} & \underline{0} & \qquad \pi^{(5)} & = & 6|5 \\
\omega_4 m & = & 2 & 3 & 2 & \underline{2} & \underline{0} & \underline{0} & \qquad \pi^{(4)} & = & 46|5 \\
\eta_3\omega_4 m & = & 2 & 3 & 3 & 1 & 0 & 0 & & & \\
\omega_3\omega_4 m & = & 2 & 3 & \underline{3} & \underline{1} & \underline{0} & \underline{0} & \qquad \pi^{(3)} & = & 4|36|5 \\
\eta_2\omega_3\omega_4 m & = & 2 & 4 & 2 & 1 & 0 & 0 & & & \\
\eta_3\eta_2\omega_3\omega_4 m & = & 2 & 4 & 2 & 1 & 0 & 0 & & & \\
\omega_2\omega_3\omega_4 m & = & 2 & \underline{4} & \underline{2} & \underline{1} & \underline{0} & \underline{0} & \qquad \pi^{(2)} & = & 4|3|26|5 \\
\eta_1\omega_2\omega_3\omega_4 m & = & 5 & 1 & 2 & 1 & 0 & 0 & & & \\
\eta_2\eta_1\omega_2\omega_3\omega_4 m & = & 5 & 3 & 0 & 1 & 0 & 0 & & & \\
\eta_3\eta_2\eta_1\omega_2\omega_3\omega_4 m & = & 5 & 3 & 1 & 0 & 0 & 0 & & & \\
\omega\, m & = & \underline{5} & \underline{3} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \qquad \pi & = & 4|13|26|5
\end{array}
$$