AN EULERIAN PARTNER FOR INVERSIONS

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ABSTRACT. A number of authors studying permutation statistics on the symmetric group S_n have considered pairs (x, Y), where x is an Eulerian statistic and Y is a Mahonian statistic. Of special interest are pairs involving the statistics des, exc, MAJ, and INV, which arise often in combinatorics. One pair of statistics which has a particularly nice joint distribution on S_n is (des, MAJ). A second pair, (exc, DEN), was shown to be equidistributed with (des, MAJ) by Foata and Zeilberger in 1989. This left open the problem of finding a natural Eulerian statistic z such that (z, INV) is equidistributed with (des, MAJ) and (exc, DEN). We present such a statistic z, along with a simple bijective proof that the pairs of statistics are equidistributed.

1. Introduction

Let S_n be the symmetric group on n letters. A *permutation statistic* is a function $\phi : S_n \to \mathbb{N}$ which maps permutations to nonnegative integers. The *distribution* of a permutation statistic on S_n is simply a count of the number of permutations π in S_n for which $\phi(\pi) = k$, for all possible values of k.

For example, "des" is a statistic counting the number of descents of a permutation. Writing $\pi = \pi_1, \ldots, \pi_n$, we call i a *descent* in π if $\pi_i > \pi_{i+1}$. Therefore,

$$\mathrm{des}(\pi) = \#\{i|\pi_i > \pi_{i+1}; i = 1, \ldots, n-1\}. \tag{1.1}$$

The number of permutations in S_n with $k-1$ descents is equal to the *Eulerian number* $A(n,k)$. The numbers $A(n,k)$ are often written as the coefficients of the *Eulerian polynomial*

$$A_n(x) = \sum_{\pi \in S_n} A(n,k)x^k = \sum_{\pi \in S_n} x^{1+\mathrm{des}(x)}. \tag{1.2}$$

While no closed formula is known for the Eulerian polynomials, their generating function is

$$1 + \sum_{n \geq 1} \frac{A_n(x)u^n}{n!} = \frac{1}{1 - \sum_{n\geq 1} \frac{(x-1)^{n-1}u^n}{n!}}, \tag{1.3}$$

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and the coefficients $A(n,k)$ satisfy the recurrence

$$A(n,k) = kA(n-1,k) + (n-k-1)A(n-1,k-1), \tag{1.4}$$

subject to the initial conditions

$$A(1,k) = \begin{cases} 1 & \text{for } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

A permutation statistic, like des, whose distribution on S_n is given by the nth Eulerian polynomial $A_n(x)$ is known as an *Eulerian statistic*. A second Eulerian statistic is "exc", the number of excedances of a permutation.

$$\text{exc}(\pi) = \#\{i|\pi_i > i\}. \tag{1.5}$$

Thus,

$$\sum_{\pi \in S_n} x^{1+\text{des}(x)} = \sum_{\pi \in S_n} x^{1+\text{exc}(x)} = A_n(x). \tag{1.6}$$

Another important class of permutation statistics, distributed differently than the class of Eulerian statistics, is the class of *Mahonian statistics*.

An important example is MAJ, the major index of a permutation. If we define the *descent set* of a permutation to be

$$D(\pi) = \{i|\pi_i > \pi_{i+1}\}, \tag{1.7}$$

then MAJ is defined by

$$\text{MAJ}(\pi) = \sum_{i \in D(\pi)} i. \tag{1.8}$$

Another Mahonian statistic is INV, the number of inversions of a permutation.

$$\text{INV}(\pi) = \#\{(i,j)|i < j \text{ and } \pi_i > \pi_j\}. \tag{1.9}$$

The distribution on S_n of any Mahonian statistic is given by the q-analog of n factorial. Thus,

$$\sum_{\pi \in S_n} q^{\text{MAJ}(x)} = \sum_{\pi \in S_n} q^{\text{INV}(x)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}). \tag{1.10}$$

Following Clark and Steingrimmson [?], we will write Eulerian statistics with lowercase letters and Mahonian statistics with capitals.

Naturally extending the study of single permutation statistics and their distributions on S_n, one may consider *pairs* of permutation statistics and their *joint* distributions on S_n. For example, the generating function for (des, MAJ) on S_n is given by Carlitz's q-Eulerian polynomial [?] $B_n(t,q)$. A second pair of statistics, (exc, DEN) was conjectured by Denert [?] and shown by Foata and Zeilberger [?] and Han [?] To have the same joint distribution as (des, MAJ).

$$\sum_{\pi\in S_n} t^{\mathrm{des}(\pi)} q^{\mathrm{MAJ}(x)} = \sum_{\pi\in S_n} t^{\mathrm{exc}(\pi)} q^{\mathrm{DEN}(x)} = B_n(t,q). \tag{1.11}$$

Writing $B_n(t,q)$ as

$$B_n(t) = \sum_{k=0}^{n} B_{n,k}(q) t^k \tag{1.12}$$

Carlitz [?] showed that the coefficients $B_{n,k}(q)$ satisfy the recurrence relation

$$B_{n,k}(q) = [k+1]_q B_{n-1,k}(q) + q^k B_{n-1,k-1}(q), \tag{1.13}$$

with initial conditions $B_{0,k}(q) = \delta_{0,k}$.

Wishing for a formula analogous to ? and containing the statistic INV, Foata [?] asked (more or less) for a natural Eulerian statistic z such that (z, INV) has this same joint distribution. There is, in fact, a natural Eulerian statistic with this property. We will call it "zc", and will define it in Section **??**

Theorem 1.1. *The statistic* zc *is Eulerian, and the pair* (zc, INV) *is distributed on* S_n *like* (des, MAJ)*.*

In Section 2, we will define the *code* and *major index table* of a permutation, and note several properties of these. We will also define a bijection ϕ which maps MAJ to INV. In Section 3, we will define the statistic "zc" and give a simple bijective proof of Theorem 3.1. This statistic is easily seen to be Eulerian, (and show that the map ϕ from section x ...) In Section **??** we will refine Theorem 3.1 by associating to a permutation π with $\mathrm{z}(\pi) = k$ a set of k numbers. Analogous to the *descent set* of a permutation which sums to $\mathrm{MAJ}(\pi)$, the *zc set* sums to $\mathrm{INV}(\pi)$. To prove this, we again use the bijection ϕ. We conclude in Section **??** by showing that ϕ is not the only bijection which proves all of the results.

2. Codes and major index tables

Let us define M_n to be the set of vectors v of length n, whose components are non-negative integers, which are componentwise less than or equal to the "stair vector" of length $n : (n-1, n-2, \ldots, 1, 0)$.

$$M_n = \{v \in \mathbb{N}^n \,|\, v \leq (n-1, n-1, \ldots, 1, 0)\}. \tag{2.1}$$

Throughout, we will follow the conventional notation for vector inequalities. That is, if v and w have the same dimension, n, then we write

$$\begin{aligned} v \leq w \text{ if } v_i \leq w_i \text{ for } i = 1, \dots , n. \\ v < w \text{ if } v_i < w_i \text{ for } i = 1, \dots , n. \end{aligned}$$

Informally, we will refer to the elements of M_n as *substair vectors of length n*. Clearly, the cardinality of the set M_n is $n!$, and bijections between M_n and S_n are widely used in combinatorics.

One such bijection sends a permutation to its *code*. We will denote this bijection by γ.

$$\begin{aligned} \gamma : S_n &\to M_n \\ \pi &\mapsto \text{code}(\pi). \end{aligned}$$

The code of $\pi = \pi_1, \dots , \pi_n$ is defined to be the vector $\text{code}(\pi) = c_1, \dots , c_n$, where c_i counts the number of letters in π which are to the *right* of position i and *smaller* than π_i.

Example 2.1.

$$\begin{array}{rcccccccccc} \pi & = & 2 & 8 & 4 & 3 & 6 & 7 & 9 & 5 & 1 \\ \text{code}(\pi) & = & 1 & 6 & 2 & 1 & 2 & 2 & 2 & 1 & 0 \end{array} \tag{2.2}$$

The map γ is well known to be a bijection. (See, for example [Gupta] and [EC1].) It is not hard to see that the sum of the components of $\text{code}(\pi)$ is $\text{INV}(\pi)$.

A second bijection between S_n and M_n which is less well known sends a permutation to its *major index table*. We will denote this bijection by μ.

$$\begin{aligned} \mu : S_n &\to M_n \\ \pi &\mapsto \text{majtable}(\pi). \end{aligned}$$

The major index table is a substair vector whose entries sum to $\text{MAJ}(\pi)$. The major index table of $\pi = \pi_1, \dots , \pi_n$ is defined to be the vector $\text{majtable}(\pi) = m_1, \dots , m_n$, where m_i roughly counts the contribution of the letter i to $\text{MAJ}(\pi)$.

More precisely, we define the major index table as follows. Given a permutation π, denote by $\pi^{(k)}$ be the restriction of π to the letters k through n. Building π one letter at a time in the order $n, n-1, \dots , 1$, we construct the sequence of permutations $\pi^{(n)} = n, \pi^{(n-1)}, \dots , \pi^{(1)} = \pi$. We define $m_n = 0$ and $m_i(\pi) = \text{MAJ}(\pi^{(i)}) - \text{MAJ}(\pi^{(i+1)})$. That is, m_i is the amount by which the major index increases with the insertion of i.

Example 2.2. Let $\pi = 413265$. Inserting the letters in the order $6, 5, \ldots, 1$, we obtain the following sequence of permutations. Descents are marked by bars.

$\pi^{(i)}$	$\text{MAJ}(\pi^{(i)})$	m_i
6	0	0
6|5	1	1
46|5	2	1
4|36|5	4	2
4|3|26|5	7	3
4|13|26|5	9	2

Calculating MAJ for each of these, and subtracting, we have

$$\text{majtable}(413265) = m = 232110.$$

The map $\mu : S_n \to M_s$, taking a permutation to its major index table, is known to be a bijection. While the result is difficult to find stated explicitly in the literature, it is nearly stated in [Gupta], [Carlitz], [Rawling].

Proposition 2.1. *The map $\mu : S_n \to M_n$, taking a permutation to its major index table, is a bijection.*

Proof. One inverts μ by writing partial permutations $n = \pi^{(n)}, \ldots, \pi^{(1)}$ in such a way that for $i = n-1, \ldots, 1$, the permutation $\pi^{(i)}$ is obtained from $\pi^{(i+1)}$ by inserting the letter i into the unique position such that $\text{MAJ}\pi^{(i)} - \text{MAJ}\pi^{(i+1)} = m_i$. This is possible by the following lemma. □

Lemma 2.2. *Let π be a word on the letters $\{i+1, \ldots, n\}$, and suppose π has k descents. Let $a_1 < \cdots < a_{n-i-k} = n$ be the positions of the $n-i-k$ ascents, let $d_{k-1} < \cdots < d_0$ be the positions of the k descents, and define $d_k = 0$. The insertion of the letter i into one of the $n-i+1$ possible positions in π creates a new word π' with at most $k+1$ descents and $\text{MAJ}(\pi')$ at most $\text{MAJ}(\pi') + n - i$. In particular,*

1. *The insertion of i into position $d_\ell + 1$ of π creates no new descent, and increases MAJ by ℓ, for $\ell = 0, \ldots, k$.*
2. *The insertion of i into position $a_\ell + 1$ of π creates one new descent and increases MAJ by $k + \ell$, for $\ell = 1, \ldots, n-i-k$.*

Proof. (1.) Insertion of i causes each of the ℓ descent positions to the right of position d_ℓ to increase by one.

(2.) Suppose that there are p descents before position a_ℓ, and $k-p$ after. Thus, $a_\ell = p + \ell$. The insertion of i creates a new descent at position a_ℓ, and increases $k-p$ descents by one each. Thus, MAJ increases by $k - p + a_\ell = k + \ell$. □

Combining the bijections γ and μ, we have a bijection from S_n to itself. Using this bijection, we prove the equidistribution on S_n of MAJ and INV as a corollary of Lemma 2.2.

Corollary 2.3. *The permutation statistics* INV *and* MAJ *are equally distributed on* S_n.

Proof. The map $\phi : S_n \to S_n$ defined by $\phi = \gamma^{-1}\mu$ is a bijection satisfying

$$\text{MAJ}(\pi) = \text{INV}(\phi(\pi)). \tag{2.3}$$

□

As another corollary of Lemma 2.2, we can show that des is an Eulerian statistic.

Corollary 2.4. *The number of permutations in* S_n *with* $k-1$ *descents is equal to the Eulerian number* $A(n,k)$

Proof. Each permutation π in S_n, ... Using the recursive definition of the Eulerian numbers (Equation 1.4), there is a simple way to see that these numbers count permutations by descents. (See [?].)

Namely, if we think of each permutation π in S_n as being built from a permutation on the letters $[n-1]$, with the letter n inserted somewhere. That is, each permutation on n letters may be built uniquely by inserting the letter n into a permutation on $n-1$ letters. In every permutation on $n-1$ letters having k descents, there are k insertions of n which result in a permutation on n letters with k descents, and $n-k$ insertions which result in a permutation on n letters with $k+1$ descents.

Thus, the number of permutations on n letters with k descents is given by the recursion 1.4. □

3. Main Result

Before restating and proving the main result, we introduce a function z on M_n and define two permutation statistics *zc*, and *zm*.

Definition 3.1. Let $v = v_1, \dots, v_n$ be a sub-stair vector of length n. Define $z : M_n \to \mathbb{N}$ to be the fuction which maps v to the length ℓ of the longest subsequence $b = v_{i_\ell}, v_{i_{\ell-1}}, \dots, v_{i_1}$ of v, which is strictly greater than the stair vector of size ℓ. That is, such that $b > (\ell-1, \ell-2, \dots, 0)$.

Define the permutation statistics zc and zm by

$$\text{zc}(\pi) = z(\text{code}(\pi)),$$
$$\text{zm}(\pi) = z(\text{majtable}(\pi)).$$

While there may be several such subsequences of maximum length, identifying one and calculating this maximum length is quite easy. Let v be a given vector. Starting from the rightmost component of v and reading left, we circle the first letter which is at least one, the next which is at least two, etc., until we cannot continue. The number of circled components of v is then $z(v)$. Let us refer to this method of circling letters as the *standard* calculation of $z(v)$.

An equivalent description of the standard calculation of $z(v)$ is to say that the rightmost nonzero position is circled, and that each letter which is greater than the number of circles to its right is circled as well.

Example 3.2. Let $\pi = 368459172$. Then $\mathrm{code}(\pi) = 245223010$. Starting from the right, we circle the 1, 3, 5, and 4. Thus,

$$\mathrm{zc}(\pi) = z(\mathrm{code}(\pi)) = 4. \tag{3.1}$$

It is easy to see from the recursive definition of Eulerian numbers (Equation 1.4) that the statistic zc is Eulerian. Noting that each code vector $c = (c_1, \dots, c_n)$ in M_n is constructed uniquely from a code vector $(c_2, \dots, c_n)$ in M_{n-1} and a letter $0, \dots, n-1$ inserted in front of the code. Temporarily letting $\alpha(n,k)$ be the number of permutations π in S_n with $\mathrm{zc}(\pi) = k-1$, we see that one obtains a permutation π in S_n with $\mathrm{zc}(\pi) = k-1$ by taking one of the $\alpha(n-1,k)$ permutations in S_{n-1} with $\mathrm{zc} = k-1$ and inserting a letter $0, \dots, k-1$ in front of its code, or by taking one of the $\alpha(n-1,k-1)$ permutations in S_{n-1} with $\mathrm{zc} = k-2$ and inserting a letter $k-1, \dots, n-1$ in front of its code. Thus, the numbers $\alpha(n,k)$ are in fact the numbers $A(n,k)$.

Analogously, we can show that the statistic z is Eulerian directly from the recurrence relations 1.4.

Identifying a permutation with its code, note that each code in M_n can be built uniquely from a code in M_{n-1} by inserting a letter $1, \dots, n-1$ in front of the code. If $\mathrm{z}(\mathrm{code}(\pi)) = k$, then inserting a letter greater than k increases zc, while inserting a letter less than or equal to k leaves it unchanged.

Theorem 3.1. *The pairs of permutation statistics* (des, MAJ) *and* (zc, INV) *are equally distributed on* S_n.

Proof. Let π be a permutation in S_n. As we have seen, the bijection $\phi : S_n \to S_n$ in the proof of Corollary 2.4 satisfies $\mathrm{INV}(\pi) = \mathrm{MAJ}(\phi(\pi))$. We now show that in addition, $\mathrm{des}(\pi) = \mathrm{zc}(\phi(\pi))$.

Since $\mathrm{majtable}(\pi) = \mathrm{code}(\phi(\pi))$, it will suffice to show that $\mathrm{des}(\pi) = \mathrm{zm}(\pi)$. Let $m = \mathrm{majtable}(\pi) = m_1, \dots, m_n$ be the major index table of π, and let $\pi^{(n)}, \dots, \pi^{(1)}$ be the partial permutations as in the calculation of m. Suppose that we have circled positions of m as in the standard calculation of zm.

Fix $i < n$ and assume that the number of circled positions in $m_{i+1}, \dots, m_n$ is equal to the number of descents in $\pi^{(i+1)}$. Certainly this is true if there are no circled positions of m which are strictly to the right of position i. By Lemma 2.2, the insertion of the letter i into $\pi^{(i+1)}$ creates a new descent if and only if m_i is greater than the number of descents in $\pi^{(i+1)}$. In this case, position i must be circled. Thus, the assumption is true for any i between 1 and n. □

4. The zc-set of a permutation

Theorem 3.1 states that the relationship of the statistics zc and INV is analogous to that of des and MAJ. The analogy is deeper, in fact. For just as a permutation with k descents has a descent set of k numbers which sum to MAJ, a permutation with zc $= k$ has a *zc-set* of k numbers which sums to INV. In order to give a concise description of the zc-set, we define a set of $n-2$ operators on M_n. As we will see, these operators constitute an action of the 0-Hecke algebra on M_n.

Let H be a set of $n-2$ operators $\eta_1, \dots, \eta_{n-2}$. Applied to a sub-stair vector $v = v_1, \dots, v_n$ in M_n, the operator η_i will modify the ith and $(i+1)$st components of v and will fix all other components. Letting $v' = \eta_i v$, we define the operator η_i as follows.

$$(v'_i, v'_{i+1}) = \begin{cases} (v_i, v_{i+1}) & \text{if } v_i > v_{i+1} \text{ or } v_i = v_{i+1} = 0, \\ (v_{i+1}+1, v_i - 1) & \text{if } v_i \leq v_{i+1} \text{ and } v_i \neq 0, \\ (v_{i+1}, v_i) & \text{if } v_i = 0 \text{ and } v_{i+1} > 0. \end{cases} \tag{4.1}$$

Note that $v'_i + v'_{i+1} = v_i + v_{i+1}$, and that $v'_i > v'_{i+1}$, unless both are zero.

Example 4.1. Consider the action of η_1 on five different vectors in M_4.

$$\begin{aligned} \eta_1(3200) &= 3200, \\ \eta_1(0010) &= 0010, \\ \eta_1(1200) &= 3000, \\ \eta_1(2200) &= 3100, \\ \eta_1(0110) &= 1010. \end{aligned}$$

It is not difficult to see that the operators in the set H satisfy the relations of $H_{n-2}(0)$, the 0-*Hecke algebra* on $n-2$ generators:

$$\begin{aligned} \eta_i\eta_j &= \eta_j\eta_i \text{ for } |i-j| \geq 2; \\ \eta_i\eta_{i+1}\eta_i &= \eta_{i+1}\eta_i\eta_{i+1}; \\ \eta_i^2 &= \eta_i. \end{aligned}$$

To simplify the arguments that follow, let us introduce notation for several other elements of $H_{n-2}(0)$. For $i = 1, \ldots, n-2$, we will denote by ω_i the element $\eta_{n-2}\eta_{n-3}\cdots\eta_i$. Now, let ω be the product of these $n-2$ elements.

$$\omega = \omega_1 \cdots \omega_{n-2}. \tag{4.2}$$

Analogously to the definitions of the functions z, zc, and zm, we we will define a *z-set* for any substair vector, and will define the *zc-set* and *zm-set* of a permutation to be the z-sets of its code vector and major index table, respectively.

Definition 4.2. The z-set of a substair vector $v = v_1, \ldots, v_n$, denoted $Z(v)$, is the set of distinct, nonzero letters in $\omega(v)$. The zc-set of a permutation $\pi = \pi_1, \ldots, \pi_n$, denoted $ZC(\pi)$, is $Z(\text{code}(\pi))$, the set of distinct nonzero letters in $\omega(\text{code}(\pi))$. Similarly, we define zm-set of a permutation, denoted $ZM(\pi)$, to be $Z(\text{majtable}(\pi))$, the set of distinct nonzero letters in $\omega(\text{majtable}(\pi))$.

In Example 4.1, consider the action of the element ω in $H_4(0)$ on the substair vector 332110 in M_6. We compute

$$\begin{aligned}
\omega_1 &= \eta_4\eta_3\eta_2\eta_1, \\
\omega_2 &= \eta_4\eta_3\eta_2, \\
\omega_3 &= \eta_4\eta_3, \\
\omega_4 &= \eta_4, \\
\omega &= \omega_1\omega_2\omega_3\omega_4 \\
&= (\eta_4\eta_3\eta_2\eta_1)(\eta_4\eta_3\eta_2)(\eta_4\eta_3)(\eta_4).
\end{aligned}$$

An “X” below entries i and $i+1$ of a row in the figure signifies that the action of η_i on this row switches (and possibly changes) the two entries, while “)(” signifies that the action of η_i fixes the two entries.

Theorem 4.1. *1. Given a permutation π on n letters with* $\text{zc}(\pi) = k$*, then the* zc-set *of π, as defined above, is a set of k distinct letters in $[n-1]$ which sum to* $\text{INV}(\pi)$*.*

2. For any subset S of $[n-1]$, the number of permutations ρ with zc-set S equals the number of permutations π with descent set S.

Proof. Let $\phi : S_n \to S_n$ be the bijection in the proofs of Corollary 2.4 and of Theorem 3.1, and let π be any permutation in S_n. In Lemma **??**, we will show that the descent set of π is equal to the zm-set of π. Using this identity, and the fact that

$$ZM(\pi) = Z(\text{majtable}(\pi)) = Z(\text{code}(\phi(\pi))) = ZC(\phi(\pi)), \tag{4.3}$$

we have that the zc-set of any permutation is the descent set of another. Thus, we prove both statements (1) and (2). ∎

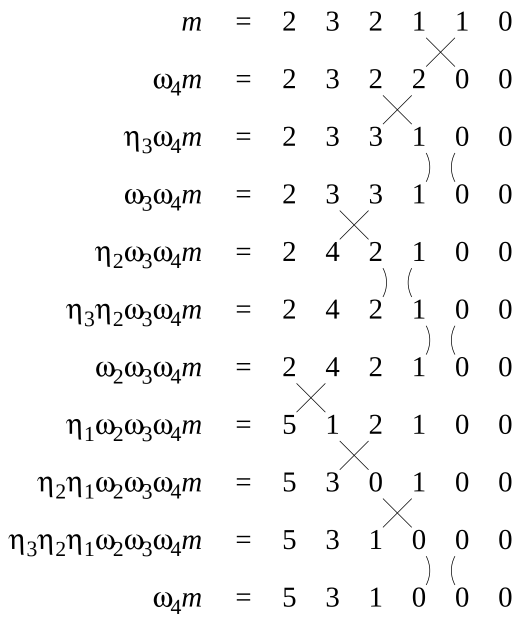

FIGURE 4.1

The following lemma relates the "partial" permutations $\{\pi^{(1)}, \dots, \pi^{(n-1)}\}$, from the definition of the major index table, to the vectors $\{m, \omega_{n-2}m, \omega_{n-3}\omega_{n-2}m, \dots, \omega m\}$, which result from action of $H_{n-2}(0)$ on $m = \text{majtable}(\pi)$.

Lemma 4.2. *Let m be the major index table of a permutation π in S_n, and let the 0-Hecke algebra $H_{n-2}(0)$ act on M_n as above. Then, for $i = 1, \dots, n-2$, the last $n-i+1$ components of the vector $\omega_i \cdots \omega_{n-2}m$ are the descent set of $\pi^{(i)}$, arranged in decreasing order, and followed by zeros.*

Note that the analogous statements for n and $n-1$ are trivially true: The set of non-zero letters in $\{m_n\}$ is empty always, as is the descent set of $\pi^{(n)} = n$. Similarly, the set of nonzero letters in $\{m_{n-1}, m_n\}$ is $\{1\}$ if $\pi^{(n-1)}$ is $(n, n-1)$, and empty otherwise, as is the descent set of $\pi^{(n-1)}$.

In Figure 4.2, we show that the statement of the lemma is true for the permutation 413265, which has major index table 232110. (For $i = 5, \dots, 1$, the last $7-i$ components of $\omega_i \cdots \omega_4$ are underlined.)

Proof. We will prove this by induction. To begin the induction, we claim that this is true for $i = n-1$. Namely, the last two $(= n-(n-1)+1)$ components of m are themselves the descent set of the partial permutation $\pi^{(n-1)}$, followed by zeros.

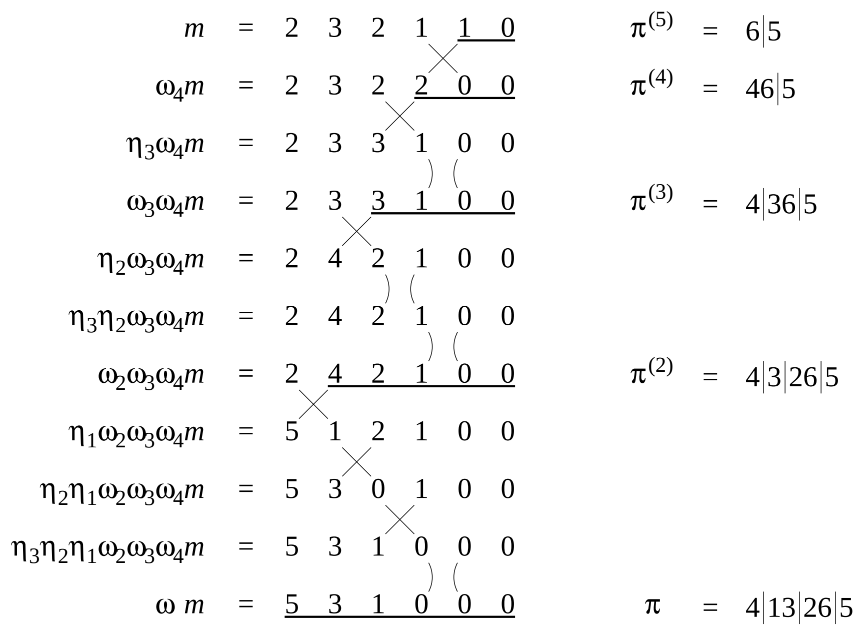

FIGURE 4.2

If the insertion of the letter $n-1$ into $\pi^{(n)}$ causes no descent, then $m_{n-1} = m_n = 0$, and the set of nonzero letters is empty. If it does cause a descent, then that descent is in position 1, and $m_{n-1} = 1$. Thus, $D(\pi^{(n-1)}) = \{1\}$.

Now fix $i < n$ and assume that the permutation $\pi^{(i+1)}$ on the letters $\{i+1, \ldots, n\}$ has k descents at positions $d_0 > \cdots > d_{k-1}$, and that the result of the action of $\omega_{i+1} \cdots \omega_{n-2}$ on m is

$$\omega_{i+1} \cdots \omega_{n-2} m = (m_1, \ldots, m_i, d_0, d_1, \ldots, d_{k-1}, 0, \ldots, 0). \tag{4.4}$$

That is, the nonzero letters in the last $n-i+1$ positions of the vector $\omega_{i+1} \cdots \omega_{n-2} m$ are the descent set of the partial permutation $\pi^{(i+1)}$, arranged in decreasing order, and followed by zeros. Note that the descent set, viewed as a sequence, is componentwise greater than the stair vector of length k:

$$(d_0, \ldots, d_{k-1}) > (k-1, \ldots, 0). \tag{4.5}$$

We consider two cases.

Case 1: $(m_i \leq k)$. If m_i is less than or equal to k, then the insertion of i into $\pi^{(i+1)}$ creates no new descents, and causes the greatest m_i descent positions to increase by one. Thus, $\pi^{(i)}$ has descent set

$$D(\pi^{(i)}) = \{d_0 + 1, \dots, d_{m_i} + 1, d_{m_i+1}, \dots, d_{k-1}, \}. \tag{4.6}$$

Now consider the action of $\omega_i = \eta_{n-2} \cdots \eta_i$ on $\omega_{i+1} \cdots \omega_{n-2} m$. As we apply the generators $\eta_i, \eta_{i+1}, \dots,$ the number m_i moves to the right and decreases, while the numbers $d_0, d_1, \dots$ move to the left and increase. By Equation 4.8, m_i will move right until it becomes zero, increasing the first m_i numbers $d_0, \dots, d_{m_i-1}$. It will then continue to move right without increasing the numbers $d_{m_i}, \dots, d_{k-1}$.

(4.7)

$$\begin{aligned}
\eta_i \omega_{i+1} \cdots \omega_{n-2} m &= (\dots, m_{i-1}, d_0 + 1, m_i - 1, d_1, \dots, d_{k-1}, 0, \dots), \\
\eta_{i+1} \eta_i \omega_{i+1} \cdots \omega_{n-2} m &= (\dots, m_{i-1}, d_0 + 1, d_1 + 1, m_i - 2, d_2, \dots, d_{k-1}, 0, \dots), \\
&\vdots \\
\eta_{m_i} \cdots \eta_i \omega_{i+1} \cdots \omega_{n-2} m &= (\dots, m_{i-1}, d_0 + 1, \dots, d_{m_i-1} + 1, 0, d_{m_i}, \dots, d_{k-1}, 0, \dots), \\
&\vdots \\
\omega_i \cdots \omega_{n-2} m &= (\dots, m_{i-1}, d_0 + 1, \dots, d_{m_i-1} + 1, d_{m_i}, \dots, d_{k-1}, 0, \dots).
\end{aligned}$$

Case 2: ($m_i > k$). If m_i is greater than k, then the insertion of i into $\pi^{(i+1)}$ creates one new descent. Let j be the number of descents to the right of the position into which i is inserted. The new descent must therefore occur at position $m_i - j$, and the number $m_i - j$ must satisfy

$$d_j < m_i - j \leq d_{j-1}. \tag{4.8}$$

Since the insertion of i into position $m_i - j$ of $\pi^{(i+1)}$ causes j descent positions to increase by one, the new descent set must be

$$D(\pi^{(i)}) = \{d_0 + 1 \dots, d_{j-1} + 1, m_i - j, d_j, \dots, d_{k-1}, \}. \tag{4.9}$$

Now consider the action of $\omega_i = \eta_{n-2} \cdots \eta_i$ on $\omega_{i+1} \cdots \omega_{n-2} m$. Again, as we apply the generators $\eta_i, \eta_{i+1}, \dots,$ the number m_i moves one position to the right and decreases, while the numbers $d_0, d_1, \dots$ move to the left and increase. By Equation 4.11, m_i will move right only until it becomes $m_i - j$, which is greater than d_j. Beyond d_j, no entries will change.

$$\eta_i\omega_{i+1}\cdots\omega_{n-2}m = (\dots, m_{i-1}, d_0+1, m_i-1, d_1, \dots, d_{k-1}, 0, \dots),$$
$$\eta_{i+1}\eta_i\omega_{i+1}\cdots\omega_{n-2}m = (\dots, m_{i-1}, d_0+1, d_1+1, m_i-2, d_2, \dots, d_{k-1}, 0, \dots),$$
$$\vdots$$
$$\eta_{i+j}\cdots\eta_i\omega_{i+1}\cdots\omega_{n-2}m = (\dots, m_{i-1}, d_0+1, \dots, d_{j-1}+1, m_i-j, d_j, \dots, d_{k-1}, 0, \dots),$$
$$\vdots$$
$$\omega_i\cdots\omega_{n-2}m = (\dots, m_{i-1}, d_0+1, \dots, d_{j-1}+1, m_i-j, d_j, \dots, d_{k-1}, 0, \dots).$$

□

5. More bijections

Looking again at the bijection above, we note the surprising fact that the order of insertion of the letters $1, \dots, n$ was not critical. That is, if we were to insert the letters $1, \dots, n$ in some different order $\sigma = \sigma_1, \dots, \sigma_n$, we would obtain another bijection. Let us refer to the resulting vector as the σ *major index table.*

Proposition 5.1. *The map $\mu_\sigma : S_n \to M_n$, taking a permutation to its σ major index table, is a bijection.*

Assume that the letters $\sigma_n, \dots, \sigma_{i+1}$ have been inserted already, and that the resulting partial permutation $\pi^{(i+1)}$ has k descents, factoring it into $k+1$ subwords $\pi = w_k \cdot w_{k-1} \cdots w_0$.

Let $e_k < \cdots < e_0$ be the positions in each block of the greatest letter less than or equal to σ_i. If there is no such letter, define e_k to be 0 or the position of the last letter in the previous block. Let $a_1 < \cdots < a_{n-k} = n$ be the positions of the $n-k$ other positions.

Observation 5.2. *The insertion of σ_i into $\pi^{(i+1)}$ to the right of position e_ℓ increases* MAJ *by ℓ, for $\ell = 0, \dots, k-1$. The insertion of i in front of position 1 increases* MAJ *by k. The insertion of i to the right of position a_ℓ increases* MAJ *by $k+\ell$, for $\ell = 1, \dots, n-k$.*

Again, we may prove Theorem 2.1 by induction.

In the remaining sections we will use the term *major index table* to refer to the letter order $n, n-1, \dots, 1$, and will mention an order σ only when necessary.

Of course, the map $\phi_\sigma : S_n \to S_n$ defined by $\phi_\sigma = \gamma^{-1}\mu_\sigma$ is a bijection satisfying

$$\text{MAJ}(\pi) = \text{INV}(\phi_\sigma(\pi)). \tag{5.1}$$

Perhaps not surprising at this point is the fact that we may restate Lemma 4.2 in terms of the σ major index table.

Lemma 5.3. *Let m be the σ major index table of a permutation π in S_n, and let the 0-Hecke algebra $H_{n-2}(0)$ act on M_n as above. Then, for $i = 1, \dots, n-2$, the last $n-i+1$ components of the vector $\omega_i \cdots \omega_{n-2} m$ are the descent set of $\pi^{(\sigma,i)}$, arranged in decreasing order, and followed by zeros.*

Thus, the ZC set of a permutation in no way depends upon the choice of σ in the construction of a σ major index table.

It is not hard to see that any of the bijections ϕ_σ would suffice to prove the main theorem as well.