Symmetric group + Hecke algebra

S_n = symmetric group
= permutations $w = w_1 \cdots w_n$ of $\underbrace{1 \cdots n}_{\text{"e"}}$

$|S_n| = n!$

Special elements called reversals are
$s_{[a,b]} = 1 \cdots a\text{-}1 \;\underbrace{b \cdots a}_{\text{reversed}}\; b\text{+}1 \cdots n.$

Mult: $s_{[i,i+1]}\, w = w_1 \cdots w_{a-1}\, w_b \cdots w_a\, w_{b+1} \cdots w_n.$

$H_n(q)$ = Hecke algebra
$= \mathrm{span}_{\mathbb{Z}[q]} \{ T_w \mid w \in S_n \}$

$\dim(H_n(q)) = n!$

Mult: $$T_{s_{[i,i+1]}} T_w = \begin{cases} T_{s_{[i,i+1]} w} & \text{if } w_i < w_{i+1} \\ q\, T_{s_{[i,i+1]} w} + (q-1) T_w & \text{if } w_i > w_{i+1} \end{cases}$$

Ex:

$$T_{123} - q\, T_{213} - q\, T_{132} + (q^2+1) T_{312}$$

Two bases of $H_n(q)$

1. Natural basis $\{T_w \mid w \in S_n\}$
2. Kazhdan-Lusztig basis $\{\tilde{C}_w \mid w \in S_n\}$
   defined existentially or recursively.

(There exists a unique basis satisfying ...)
There is no concise explicit definition of K-L basis.

The change-of-basis matrix relating the two bases contains the Kazhdan-Lusztig polynomials $\{P_{v,w}(q) \mid v, w \in S_n\} \subseteq \mathbb{N}[q]$:

$$\tilde{C}_w = \sum_{v \in S_n} P_{v,w}(q)\, T_v .$$

Q: Can we interpret coefficients of $P_{vw}(q)$ as set cardinalities?
At least for some permutations $w \in S_n$?

Factorization answer

Theorem (Clearwater-Skandera '21)
If we can factor $\tilde{C}_w = \tilde{C}_{s_{[a_1,b_1]}} \cdots \tilde{C}_{s_{[a_r,b_r]}}$
then we can

1. draw a certain directed planar network
   $F = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_r,b_r]}$,

2. examine a certain set $\Pi_v(F)$ of "path families of type v" covering F,

3. for each path family $\pi \in \Pi_v(F)$, count it "defects", $\mathrm{dfct}(\pi)$,

4. write $P_{v,w}(q) = \sum_{\pi \in \Pi_v(F)} q^{\mathrm{dfct}(\pi)}$.

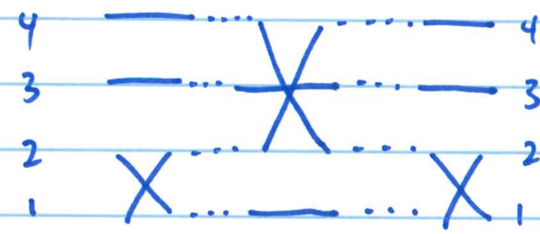
So coefficient of q^d in $P_{vw}(q)$ is cardinality of $\Pi_{v,d}(F) := \{\pi \in \Pi_v(F) \mid \mathrm{dfct}(\pi) = d\}$, i.e., the number of path families of type v having d defects.

Let's define $F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_r,b_r]}$, Π_v, $\mathrm{dfct}(\pi)$ by looking at an example.

Factorization + interpretation

Ex: Let $n=4$, $w = 4231$.
It is possible to check (omitted) that
$$\tilde{C}_{4231} = \tilde{C}_{s_{[1,2]}} \cdot \tilde{C}_{s_{[2,4]}} \cdot \tilde{C}_{s_{[1,2]}} .$$

We draw

$F =$ 

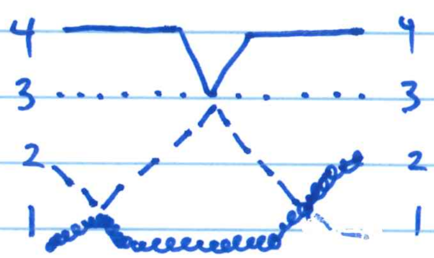
(all edges directed $\rightarrow$)

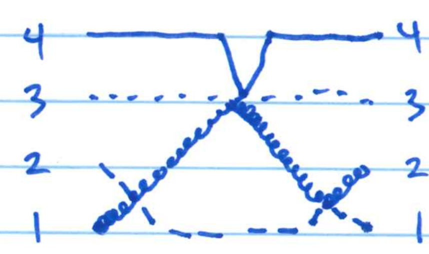
$$F_{[1,2]} \cdot F_{[2,4]} \cdot F_{[1,2]} .$$

Path families $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ of type $v \in S_n$ are those that start at $1, 2, 3, 4$ on left and terminate at v_1, v_2, v_3, v_4 on right.

There are $2 \cdot 6 \cdot 2^{= 24}$ path families that cover (use all edges of) F. Consider those of type $v = 2134$. It is known (omitted) that $P_{2134, 4231}(q) = 1 + q$.

Draw path families having type 2134:

$\pi =$ 

$\sigma =$ 

A <u>defect</u> is a meeting point of two paths that have previously crossed an odd number of times.

Above, $\mathrm{dfct}(\pi) = 0$, $\mathrm{dfct}(\sigma) = 1$.

So
$$P_{2134,\,4231}(q) = \underset{q^0}{\underset{\uparrow}{1}} + \underset{q^1}{\underset{\uparrow}{q}} .$$

Factorization question and pattern avoidance

Q: For which $w \in S_n$ does the K-L basis element $\tilde{C}_w$ have a reversal factorization $\tilde{C}_{s_{[a_1,b_1]}} \cdots \tilde{C}_{s_{[a_r,b_r]}}$?

2 theorems give answers in terms of pattern avoidance in w.

Ex: 45312 contains the pattern 321 four times: 431, 432, 531, 532

greatest middle smallest

It contains the pattern 3412: 4512

It avoids the pattern 4231: no subword $d\,b\,c\,a$ satisfies $a < b < c < d$.

Factorization theorems

1. (Billey - Warrington, 2001) : If w avoids the patterns 321, 56781234, 46781235, 56718234, 46718235, and if $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for w, then
$$\tilde{C}_w = \tilde{C}_{s_{i_1}} \cdots \tilde{C}_{s_{i_\ell}} .$$
$(s_j := s_{[j, j+1]})$

2. (Skandera, 2008) : If w avoids the patterns 3412, 4231, then there are intervals $[a_1, b_1], \ldots, [a_r, b_r]$ s.t.
$$\tilde{C}_w = \tilde{C}_{s_{[a_1, b_1]}} \cdots \tilde{C}_{s_{[a_r, b_r]}} .$$

Observation: Factorization of $\tilde{C}_{4231}$ is not guaranteed by either of these results. It contains 321.
It contains (is) 4231.

There must be a better theorem, waiting to be found.

Negative factorization question

Q: For which $w \in S_n$ can we prove that $\tilde{C}_w$ does not factor as
$\tilde{C}_{s_{[a_1,b_1]}} \cdots \tilde{C}_{s_{[a_r,b_r]}}$?

Our new theorem (Parisi, Skandera, Spahiu, Wang, 2025) gives a partial answer, using pattern avoidance and a little more.

A little more: For w containing the pattern 3412, define $gap_{3412}(w) = \min\{w_{i_1} - w_{i_4} \mid w_{i_1}w_{i_2}w_{i_3}w_{i_4}$ matches pattern $3412\}$

Ex:

w	$gap_{3412}(w)$
34512	$3-2=1$
45312	$4-2=2$
563124	$5-4=1$
563412	$5-4=1$
564312	$5-2=3$

Gaetz - Gao theorem

Gaetz - Gao studied internal zeros in coefficient sequence of $P_{e,w}(q)$.

Ex: $P_{e,45312}(q) = 1 + q^2$ coeff seq: $(1,0,1)$.
$P_{e,4231}(q) = 1+q$ coeff seq: $(1,1)$.

Define $\mathrm{singdeg}(w)$ = 1^{st} power of q after const. term to have positive coefficient.

So $\mathrm{singdeg}(45312) = 2$,
$\mathrm{singdeg}(4231) = 1$.

Reason for name: $\mathrm{singdeg}(w)$ is a lower bound on degrees for which Poincaré duality fails in Schubert variety X_w.

Thm (Gaetz - Gao, 2024): For $w \in S_n$ with $P_{e,w}(q)$ nonconstant, we have

$$\mathrm{singdeg}(w) = \begin{cases} \mathrm{gap}_{3412}(w) & \text{if } w \text{ avoids } 4231 \\ 1 & \text{otherwise} \end{cases}.$$

Diminishing defects and consequences

Prop: (Parisi, Skandera, Spahiu, Wang, 2025)
For $F = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_r,b_r]}$, $v \in S_n$,
if $|\Pi_{v,d}(F)| > 0$, then
$|\Pi_{v,d-1}(F)|, \ldots, |\Pi_{v,0}(F)| > 0$ too.

Cor: If $\mathrm{singdeg}(w) > 1$ then $\tilde{C}_w$ has no reversal factorization.

Thm: If w avoids 4231, contains 3412, and has $\mathrm{gap}_{3412}(w) > 1$, then $\tilde{C}_w$ cannot be graphically represented by a planar network.

Ex: Datko-Skandera factored $\tilde{C}_w$ for all $w \in S_5$ except for $w = 45312$. That's ok: 45312 avoids 4231, contains 3412, has gap_{3412} of $2 > 1$. It doesn't factor.

Ex: $\tilde{C}_{453129786}$ doesn't factor either, but it contains 4231, so isn't covered by theorem.

Conj: If w contains the pattern 45312, then $\tilde{C}_w$ has no reversal factorization.