PATH TABLEAUX AND COMBINATORIAL INTERPRETATIONS FOR S_n CLASS FUNCTIONS

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Outline

(1) $\mathfrak{S}_n$ class functions $\chi : \mathfrak{S}_n\mathbb{Z}$.
(2) Interpretations for $\chi(w)$.
(3) Interpretations for $\chi(\sum c_w w)$.

$\mathfrak{S}_n$ class functions and interpretations

Call $\chi : \mathfrak{S}_n \to \mathbb{Z}$ a *class function* if $\chi(w) = \chi(v^{-1}wv)$.

Interpret $\chi(w) \in \mathbb{N}$ as $|\mathcal{T}(w)|$ for some set $\mathcal{T}$.
Interpret $\chi(w) \in \mathbb{Z}$ as $(-1)^{|\mathcal{S}(w)|}|\mathcal{T}(w)|$ for some sets $\mathcal{S}$, $\mathcal{T}$.

$$\mathrm{triv}(w) = 1, \qquad \mathrm{sgn}(w) = (-1)^{\ell(w)}.$$

For $\lambda \vdash n$, and $\eta^\lambda = \mathrm{triv}\!\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}$, $\epsilon^\lambda = \mathrm{sgn}\!\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}$, we have

$$\eta^\lambda(w) = |\mathcal{T}_\lambda(w)|, \quad \epsilon^\lambda(w) = (-1)^{\ell(w)}|\mathcal{T}_{\lambda^\top}(w)|,$$

where $\mathcal{T}_\lambda(w) = \#$ colorings: $\{\text{cycles of } w\} \to [r] = \{1, \ldots, r\}$ with λ_i letters having color i.

For irreducible characters χ^λ, Murnaghan-Nakayama rule gives

$$\chi^\lambda(w) = \sum_{T \in \mathcal{U}(w)} (-1)^{\mathcal{S}(T)},$$

where $\mathcal{U}(w) =$ (something) and $\mathcal{S}(T) =$ (something else).

η^λ, ϵ^λ related to irr. characters χ^μ as h_λ, e_λ to Schur fns. s_μ:

$$\eta^\lambda = \sum_\mu K_{\mu,\lambda} \chi^\mu, \quad h_\lambda = \sum_\mu K_{\mu,\lambda} s_\mu,$$

$$\epsilon_\lambda = \sum_\mu K_{\mu^\top,\lambda} \chi^\mu, \quad e_\lambda = \sum_\mu K_{\mu^\top,\lambda} s_\mu.$$

Define *monomial class functions* (virtual characters) ϕ^λ by

$$\phi^\lambda = \sum_\mu K^{-1}_{\lambda,\mu}\chi^\mu, \quad \text{where} \quad m_\lambda = \sum_\mu K^{-1}_{\lambda,\mu} s_\mu.$$

Then for $\lambda = (\lambda_1, \ldots, \lambda_r)$ we have

$$\phi^\lambda(w) = (-1)^{n+r+\ell(w)}|\mathcal{V}_\lambda(w)|,$$

where $\mathcal{V}_\lambda(w) = \#$ ways to cut cycles of w into paths of cardinalities $\lambda_1, \ldots, \lambda_r$.

Linear extension of $\chi : \mathfrak{S}_n \to \mathbb{Z}$ to $\chi : \mathbb{Z}[\mathfrak{S}_n] \to \mathbb{Z}$

Idea (G-J, G): Define $\chi(v + cw) = \chi(v) + c\chi(w)$ and study evaluations $\chi(X_{I_1} \cdots X_{I_r})$ for $X_{[a,b]} \in \mathbb{Z}[\mathfrak{S}_n]$ defined by

$$X_{[a,b]} = \sum_{w \in S_{[a,b]}} w, \qquad X_{\emptyset} = e,$$

where $S_{[a,b]} = \langle s_a, \ldots, s_{b-1} \rangle$.

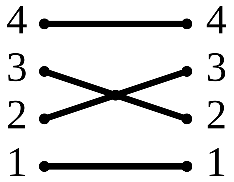
Example: In $\mathfrak{S}_5$,

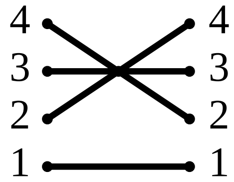
$$\begin{aligned} X_{[2,4]} &= 12345 + 13245 + 12435 + 14235 + 13425 + 14325 \\ &= e + s_2 + s_3 + s_2 s_3 + s_3 s_2 + s_2 s_3 s_2. \end{aligned}$$

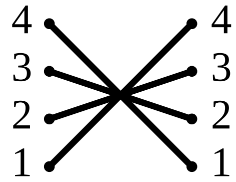
Conj (G-J, G, S): $\chi(X_{I_1} \cdots X_{I_r}) \in \mathbb{N}$ for $\chi \in \{\chi^\lambda, \epsilon^\lambda, \eta^\lambda, \phi^\lambda\}$.

Combinatorial interpretation of $X_{I_1} \cdots X_{I_r}$

Fix n and define planar networks $\{F_{[a,b]} \,|\, 1 \le a < b \le n\}$ by

$F_{[2,3]} =$ 

, $F_{[2,4]} =$ 

, $F_{[1,4]} =$ 

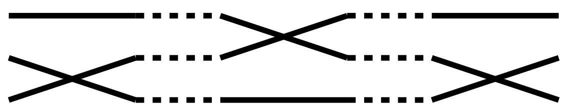
, etc.

For the concatenation $F = F_{I_1} \cdots F_{I_r}$, define $\beta(F) \in \mathbb{Z}[\mathfrak{S}_n]$ by

$$\beta(F) = \sum_{w \in \mathfrak{S}_n} \gamma(w, F) w,$$

where $\gamma(w, F) = \#$ path families $(\pi_1, \ldots, \pi_n)$ covering F, with π_i a path from source i to sink w_i for all i.

Fact: (G-J, G) $X_{I_1} \cdots X_{I_r} = \beta(F_{I_1} \cdots F_{I_r})$.

Ex: $F_{[1,2]} F_{[2,3]} F_{[1,2]} =$  .

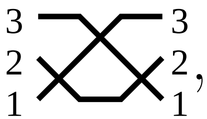
$$\beta(F_{[1,2]} F_{[2,3]} F_{[1,2]}) = (e + s_1)(e + s_2)(e + s_1)$$
$$= 2e + 2s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1.$$

For $F = F_{I_1} \cdots F_{I_r}$, define an *$F$-tableau* to be a (French) Young tableau holding paths $(\pi_1, \ldots, \pi_n)$ which cover F, with π_i a path from source i to sink i. Call the tableau *column-strict* if

$$\begin{array}{|c|}\hline \pi_j \\ \hline \pi_i \\ \hline \end{array} \Rightarrow \pi_i \text{ lies entirely below } \pi_j.$$

Fact: (K-McG '59, L '72, L '50, M-W '85)

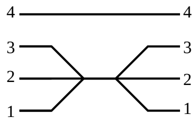
$$\operatorname{sgn}(X_{I_1} \cdots X_{I_r}) = \#\text{column-strict } F\text{-tableaux of shape } 1^n;$$
$$\epsilon^{\lambda}(X_{I_1} \cdots X_{I_r}) = \#\text{column-strict } F\text{-tableaux of shape } \lambda^{\top}.$$

Ex: For $F =$ 

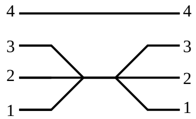
, we have

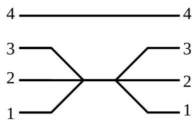
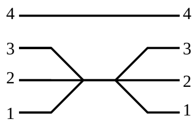
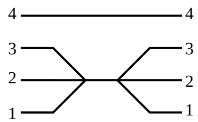
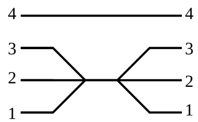
$\epsilon^{21}(\beta(F)) = 2$ column-strict F-tableaux of shape $21^{\top} = 21$:

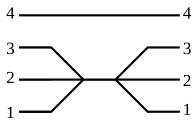
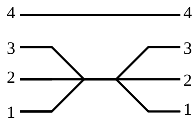
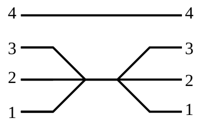
Call an F-tableau *row-semistrict* if

 $\Rightarrow \pi_j$ intersecting or entirely above π_i.

Fact: $\text{triv}(X_{I_1} \cdots X_{I_r}) = \#$row-semistrict F-tableaux of shape n;
$\eta^\lambda(X_{I_1} \cdots X_{I_r}) = \#$row-semistrict F-tableaux of shape λ.

Ex: For $F =$ , we have $\eta^{21}(\beta(F)) = 9$ row-semistrict F-tableaux of shape 21:

, , , ,

, , .

Irreducible characters

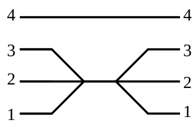
Thm: (JS '91) $\chi^\lambda(X_{I_1} \cdots X_{I_r}) \geq 0$.

No combinatorial interpretation has been conjectured.

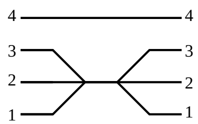
Call an F-tableau *semistandard* if it is column-strict and row-semistrict.

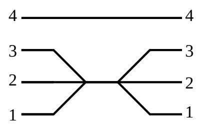
Thm: For λ a hook shape,

$$\chi^\lambda(X_{I_1} \cdots X_{I_r}) = \#\text{semistandard } F\text{-tableaux of shape } \lambda.$$

Ex: For $F =$ 

, we have $\chi^{21}(\beta(F)) = 2$ semistandard F-tableaux of shape 21:



, 

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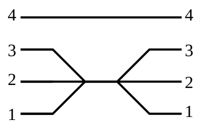
Monomial class functions

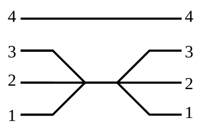
Conj: (JS '91) $\phi^\lambda(X_{I_1} \cdots X_{I_r}) \geq 0$.

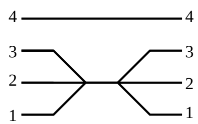
No combinatorial interpretation has been conjectured.

Thm: (CSS '10) For λ having at most 2 columns, $\phi^\lambda(X_{I_1} \cdots X_{I_r})$ $= \#$ column-strict F-tableaux of shape λ, if no column-strict F-tableaux of shape μ exists for $\mu \prec \lambda$ (0 otherwise).

Ex: For $F =$ 

, we have $\phi^{21}(\beta(F)) = 2$ column-strict F-tableaux of shape 21 (and none of shape 3):



, 

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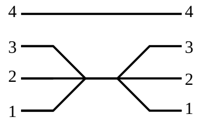
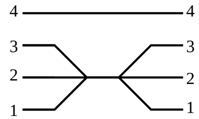
For $F =$ 

, we have $\epsilon^3(\beta(F)) = \phi^{111}(\beta(F)) = \chi^{111}(\beta(F)) =$ 1 column-strict (semistandard) F-tableau of shape 111:



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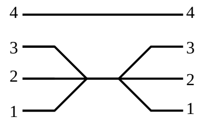
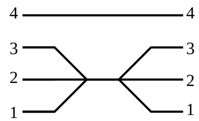
We have $\epsilon^{21}(\beta(F)) = 3$ column-strict F-tableaux of shape $21^\top =$ 21:



,

$\chi^{21}(\beta(F)) = 2$ of which are semistandard.

We have $\phi^{21}(\beta(F)) = 0$, since there is a column-strict F-tableau of shape $111 \prec 21$.



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The quantum polynomial ring $\mathcal{A}(n;q)$

Let $\mathcal{A}(n;q) \cong \mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]\langle x_{1,1}, \dots, x_{n,n}\rangle$, modulo

$$x_{i,\ell}x_{j,k} = x_{j,k}x_{i,\ell} \qquad \text{if } i < j,\ k < \ell,$$

$$x_{i,\ell}x_{i,k} = q^{\frac{1}{2}}x_{i,k}x_{i,\ell} \qquad \text{if } k < \ell,$$

$$x_{j,k}x_{i,k} = q^{\frac{1}{2}}x_{i,k}x_{j,k} \qquad \text{if } i < j,$$

$$x_{j,\ell}x_{i,k} = x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k} \qquad \text{if } i < j,\ k < \ell.$$

We have $\mathcal{O}_q(SL(n,\mathbb{C})) \cong \mathcal{A}(n;q)/(\det_q(x) - 1)$, where

$$\det_q(x) = \sum_{v \in \mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(v)} x_{1,v_1} \cdots x_{n,v_n} = \sum_{v \in \mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(v)} x^{e,v}.$$

$\mathrm{span}\{x^{e,v} \,|\, v \in \mathfrak{S}_n\}$ = (quantum) *immanant space.*

Multigrading of $\mathcal{A}(n;q)$ and immanants

$$\mathcal{A}(n;q) = \bigoplus_{r \geq 0} \bigoplus_{(L,M)} \mathcal{A}_{L,M}(n;q),$$

over r-element multisets L, M of $[n]$.

Ex: $x_{1,2}^2 x_{3,1} x_{3,2} - q^{\frac{1}{2}} x_{1,1} x_{1,2} x_{3,2}^2 \in \mathcal{A}_{1133,1222}(3;q)$.

By relations, immanant space is

$$\begin{aligned}\mathcal{A}_{[n],[n]}(n;q) &= \mathrm{span}\{x_{u_1,v_1} \cdots x_{u_n,v_n} \,|\, u, v \in \mathfrak{S}_n\} \\ &= \mathrm{span}\{x_{1,v_1} \cdots x_{n,v_n} \,|\, v \in \mathfrak{S}_n\}.\end{aligned}$$

Natural basis of $\mathcal{A}_{L,M}(n;q)$

Let $x_{L,M}$ be the L, M generalized submatrix of x.
Let generators I, J of $\mathfrak{S}_r$ stabilize $x_{L,M}$.

Ex: $$x_{1133,1222} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \end{bmatrix}, \qquad \begin{aligned} I &= \{s_1, s_3\}, \\ J &= \{s_2, s_3\}, \\ \mathfrak{S}_r &= \mathfrak{S}_4. \end{aligned}$$

Natural basis of $\mathcal{A}_{L,M}(n;q)$ is $\{(x_{L,M})^{e,v} \,|\, v \in W_+^{I,J}\}$, where $W_+^{I,J} = \{v \in \mathfrak{S}_r \,|\, v \text{ maximal in } W_I v W_J\}$.

$$W_+^{I,J} = \{4132, 4321\},$$

Ex: $$(x_{1133,1222})^{1234,4132} = x_{1,2}x_{1,1}x_{3,2}x_{3,2} = q^{\frac{1}{2}} x_{1,1}x_{1,2}x_{3,2}^2,$$
$$(x_{1133,1222})^{1234,4321} = x_{1,2}x_{1,2}x_{3,2}x_{3,1} = q^{\frac{1}{2}} x_{1,2}^2 x_{3,1}x_{3,2}.$$

Canonical bases

Modification $\dot{U}$ of $U_q(\mathfrak{sl}(n,\mathbb{C}))$ has *canonical basis*.
This aids in construction of $U_q(\mathfrak{sl}(n,\mathbb{C}))$-modules [L 90].

Modification $\mathcal{A}(n;q)$ of $\mathcal{O}_q(SL(n,\mathbb{C}))$ has *dual canonical basis*.
This aids in construction of $U_q(\mathfrak{sl}(n,\mathbb{C}))$-modules [T 91, D 92].

$U_q(\mathfrak{sl}(n,\mathbb{C}))$, $\mathcal{O}_q(SL(n,\mathbb{C}))$ are dual Hopf algebras.
$\dot{U}$, $\mathcal{A}(n;q)$ are not Hopf algebras.
Explicit duality of bases not published [D, G-L].
Some choices are involved.

Dual canonical basis of $\mathcal{A}_{L,M}(n;q)$

Define the *bar involution* on $\mathcal{A}(n;q)$ by $\overline{q} = q^{-1}$ and

$$\overline{x_{a_1,b_1} \cdots x_{a_r,b_r}} = (q^{\frac{1}{2}})^{\alpha(a)-\alpha(b)} x_{a_r,b_r} \cdots x_{a_1,b_1},$$

where $\alpha(a) = \#\{(i,j) \,|\, i < j, a_i = a_j\}$.

Theorem: (L) $\mathcal{A}_{L,M}(n;q)$ has a unique bar-invariant basis $\{B_w^{L,M}(x;q) \,|\, w \in W_+^{I,J}\}$ satisfying

$$B_v^{L,M}(x;q) \in (x_{L,M})^{e,v} + \sum_{w>v} q^{\frac{-1}{2}}\mathbb{Z}[q^{\frac{-1}{2}}](x_{L,M})^{e,w}.$$

Call this the *dual canonical basis*.

Specializations at $q = 1$ have important nonnegativity properties [L, H, R-S, S] and applications [L-P-P].

Dual canonical basis of $\mathcal{A}_{[n],[n]}(n;q)$

Immanants in DCB are $\{\mathrm{Imm}_v(x;q) \,|\, v \in \mathfrak{S}_n\}$, where

$$\mathrm{Imm}_v(x;q) = \sum_{w \geq v} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}(q) x_{1,w_1} \cdots x_{n,w_n},$$

$$\epsilon_{v,w} = (-1)^{\ell(w)-\ell(v)},$$

$$q_{v,w} = (q^{\frac{1}{2}})^{\ell(w)-\ell(v)},$$

$$Q_{v,w}(q) = P_{w_0 w, w_0 v}(q).$$

Nonquantum ($q = 1$) analogs in $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ are

$$\mathrm{Imm}_v(x) = \sum_{w \geq v} \epsilon_{v,w} Q_{v,w}(1) x_{1,w_1} \cdots x_{n,w_n}.$$