A QUANTIZATION OF A THEOREM OF GOULDEN AND JACKSON

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- Outline

(1) S_n character immanants
(2) Merris-Watson and Goulden-Jackson identities
(3) The MacMahon Master Theorem
(4) The Hecke algebra and its characters
(5) The quantum polynomial ring and quantum character immanants
(6) Quantized Merris-Watson and Goulden-Jackson identities
(7) A quantized MacMahon Master Theorem

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S_n character immanants

Let $x = (x_{1,1}, \ldots, x_{n,n})$.

Given character $\chi : S_n \to \mathbb{C}$, define $\mathrm{Imm}_\chi(x) \in \mathbb{C}[x]$ by

$$\mathrm{Imm}_\chi(x) \underset{\mathrm{def}}{=} \sum_{w \in S_n} \chi(w) x_{1,w_1} \cdots x_{n,w_n}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, abbreviate $\mathrm{Imm}_\lambda(x) \underset{\mathrm{def}}{=} \mathrm{Imm}_{\chi^\lambda}(x)$.

$$\mathrm{Imm}_{1^n}(x) = \det(x), \qquad \mathrm{Imm}_{(n)}(x) = \mathrm{per}(x).$$

Example: $n = 3$, $\lambda = (2, 1)$.

$$\chi^{21}(123) = 2, \qquad \chi^{21}(231) = \chi^{21}(312) = -1,$$

$$\chi^{21}(213) = \chi^{21}(132) = \chi^{21}(321) = 0,$$

$$\mathrm{Imm}_{21}(x) = 2x_{1,1}x_{2,2}x_{3,3} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2}.$$

Given Young subgroup $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_\ell} \subset S_n$, define

$$\epsilon^\lambda = \mathrm{sgn}\uparrow_{S_\lambda}^{S_n}, \qquad \eta^\lambda = \mathrm{triv}\uparrow_{S_\lambda}^{S_n}.$$

Induced characters are related to irreducible characters by

$$\chi^\lambda = \sum_\mu K^{-1}_{\mu,\lambda^\top} \epsilon^\mu = \sum_\mu K^{-1}_{\mu,\lambda} \eta^\mu,$$

where $\{K^{-1}_{\mu,\lambda} \mid \lambda, \mu \vdash n\}$ are the *inverse Kostka numbers*, defined by

$$\det(\xi_{\lambda_i+j-i})_{i,j=1}^{\ell} = \sum_{\mu \vdash n} K^{-1}_{\mu,\lambda} \xi_{\mu_1} \cdots \xi_{\mu_\ell},$$

where $\xi_0 = 1$, $\xi_k = 0$ if $k < 0$.

Littlewood-Merris-Watkins identities

Given subset I of $[n]$, define $x_{I,I} = I, I$ *submatrix* of x.

Theorem: (L '40, M-W '85) For $\lambda = (\lambda_1, \ldots, \lambda_\ell)$,

$$\mathrm{Imm}_{\epsilon^\lambda}(x) = \sum_{(I_1,\ldots,I_\ell)} \det(x_{I_1,I_1}) \cdots \det(x_{I_\ell,I_\ell}),$$

$$\mathrm{Imm}_{\eta^\lambda}(x) = \sum_{(I_1,\ldots,I_\ell)} \mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_\ell,I_\ell}),$$

summed over all ordered set partitions with $|I_j| = \lambda_j$.

Example: $\mathrm{Imm}_{\epsilon^{21}}(x) =$

$$\det\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} x_{3,3} + \det\begin{bmatrix} x_{1,1} & x_{1,3} \\ x_{3,1} & x_{3,3} \end{bmatrix} x_{2,2} + \det\begin{bmatrix} x_{2,2} & x_{2,3} \\ x_{3,2} & x_{3,3} \end{bmatrix} x_{1,1}.$$

Goulden-Jackson identities

Let $z = (z_1, \dots, z_n)$ and $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$. Define polynomials $(\alpha_k)_{k\in\mathbb{Z}}$, $(\beta_k)_{k\in\mathbb{Z}}$ and matrices A, B by

$$\det(I + t\mathrm{diag}(z)x) = \sum_k \alpha_k t^k, \qquad A = (\alpha_{\lambda_i^\top + j - i})_{i,j=1}^{\lambda_1},$$

$$\det(I - t\mathrm{diag}(z)x)^{-1} = \sum_k \beta_k t^k, \qquad B = (\beta_{\lambda_i + j - i})_{i,j=1}^{\ell}.$$

Theorem: (G-J '92)

$$\det(A) \equiv \det(B) \equiv \mathrm{Imm}_\lambda(x), \quad \mathrm{mod}\ (z_1^2, \dots, z_n^2).$$

Proof idea: (K-S '08)

$$\mathrm{Imm}_\lambda(x) = \sum_\mu K^{-1}_{\mu,\lambda^\top} \mathrm{Imm}_{\epsilon^\mu}(x) = \sum_\mu K^{-1}_{\mu,\lambda} \mathrm{Imm}_{\eta^\mu}(x).$$

MacMahon Master Theorem

Given multiset K of $[n]$, let $x_{K,K}$ be the K, K *generalized submatrix* of x.

Example: For $K = 113$,

$$x_{113,113} = \begin{bmatrix} x_{1,1} & x_{1,1} & x_{1,3} \\ x_{1,1} & x_{1,1} & x_{1,3} \\ x_{3,1} & x_{3,1} & x_{3,3} \end{bmatrix}.$$

Theorem: (M '93, G-J '92) Coefficients of $z_1^{k_1} \cdots z_n^{k_n}$ in

$$\Big(\sum_{j=1}^{n} x_{1,j} z_j\Big)^{k_1} \cdots \Big(\sum_{j=1}^{n} x_{n,j} z_j\Big)^{k_n} \text{ and } \frac{1}{\det(I - \mathrm{diag}(z)x)}$$

are both equal to $\mathrm{per}(x_{K,K})/(k_1! \cdots k_n!)$.

The Hecke algebra $H_n(q)$

Generators over $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$: $\widetilde{T}_{s_1}, \ldots, \widetilde{T}_{s_{n-1}}$.

Relations:

$$\widetilde{T}_{s_i}^2 = (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{s_i} + 1 \qquad \text{for } i = 1, \ldots, n-1,$$

$$\widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} = \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j} \qquad \text{for } |i-j| = 1,$$

$$\widetilde{T}_{s_i}\widetilde{T}_{s_j} = \widetilde{T}_{s_j}\widetilde{T}_{s_i} \qquad \text{for } |i-j| \geq 2.$$

Natural basis: $\{\widetilde{T}_w \,|\, w \in S_n\}$,

$$\widetilde{T}_w = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_\ell}}, \qquad (w = s_{i_1} \cdots s_{i_\ell} \text{ reduced}),$$

$$\widetilde{T}_e = 1.$$

Characters are functions $\chi_q : H_n(q) \to \mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$.

Quantum polynomial ring

Define $\mathcal{A}(n;q) \cong \mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]\langle x_{1,1}, \ldots, x_{n,n}\rangle$, modulo

$$
\begin{aligned}
x_{i,\ell}x_{j,k} &= x_{j,k}x_{i,\ell} && \text{if } i < j,\ k < \ell,\\
x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell} && \text{if } k < \ell,\\
x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k} && \text{if } i < j,\\
x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k} && \text{if } i < j,\ k < \ell.
\end{aligned}
$$

A central element is the quantum determinant,

$$\det(x;q) = \sum_{w \in S_n} (-q^{\frac{-1}{2}})^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}.$$

We have $\mathcal{O}_q(SL_n\mathbb{C}) \cong \mathcal{A}(n;q)/(\det(x;q) - 1)$.

$H_n(q)$ character immanants

Given character $\chi_q : H_n(q) \to \mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$, define

$$\mathrm{Imm}_{\chi_q}(x;q) \underset{\mathrm{def}}{=} \sum_{w \in S_n} \chi_q(\widetilde{T}_w) x_{1,w_1} \cdots x_{n,w_n} \in \mathcal{A}(n;q)$$

Abbreviate $\mathrm{Imm}_\lambda(x;q) \underset{\mathrm{def}}{=} \mathrm{Imm}_{\chi_q^\lambda}(x;q)$.

Since the sign and trivial characters of $H_n(q)$ are given by

$$\mathrm{sgn} : \widetilde{T}_{s_i} \mapsto -q^{\frac{-1}{2}}, \qquad \mathrm{triv} : \widetilde{T}_{s_i} \mapsto q^{\frac{1}{2}},$$

we have

$$\mathrm{Imm}_{1^n}(x;q) = \det(x;q),$$

$$\mathrm{Imm}_{(n)}(x;q) = \mathrm{per}(x;q) \underset{\mathrm{def}}{=} \sum_{w \in S_n} (q^{\frac{1}{2}})^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}.$$

Example: $n = 3$, $\lambda = (2, 1)$.

$$\chi_q^{21}(\widetilde{T}_e) = 2, \qquad \chi_q^{21}(\widetilde{T}_{s_1 s_2}) = \chi_q^{21}(\widetilde{T}_{s_2 s_1}) = -1,$$
$$\chi_q^{21}(\widetilde{T}_{s_1}) = \chi_q^{21}(\widetilde{T}_{s_2}) = q^{\frac{1}{2}} - q^{\frac{-1}{2}}, \qquad \chi_q^{21}(\widetilde{T}_{s_1 s_2 s_1}) = 0.$$

$$\begin{aligned}\mathrm{Imm}_{21}(x; q) = {} & 2x_{1,1}x_{2,2}x_{3,3} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2} \\ & + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{1,2}x_{2,1}x_{3,3} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{1,1}x_{2,3}x_{3,2}.\end{aligned}$$

Induced sign (ϵ_q^μ) and trivial (η_q^μ) characters from Young subalgebras are related to irreducible characters by

$$\chi_q^\lambda = \sum_\mu K^{-1}_{\mu,\lambda^\top} \epsilon_q^\mu = \sum_\mu K^{-1}_{\mu,\lambda} \eta_q^\mu,$$

i.e., by the *classical* inverse Kostka numbers.

Quantum Littlewood-Merris-Watkins identities

Theorem: (K-S '08) For $\lambda = (\lambda_1, \ldots, \lambda_\ell)$,

$$\mathrm{Imm}_{\epsilon_q^\lambda}(x;q) = \sum_{(I_1,\ldots,I_\ell)} \det(x_{I_1,I_1};q) \cdots \det(x_{I_\ell,I_\ell};q),$$

$$\mathrm{Imm}_{\eta_q^\lambda}(x;q) = \sum_{(I_1,\ldots,I_\ell)} \mathrm{per}(x_{I_1,I_1};q) \cdots \mathrm{per}(x_{I_\ell,I_\ell};q),$$

summed over all ordered set partitions with $|I_j| = \lambda_j$.

Example: $\mathrm{Imm}_{\epsilon_q^{21}}(x;q) =$

$$\det(x_{12,12};q)x_{3,3} + \det(x_{13,13};q)x_{2,2} + \det(x_{23,23};q)x_{1,1}.$$

Quantum Goulden-Jackson identities

Let $z = (z_1, \dots, z_n)$ and $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$. Define $(\alpha_k)_{k\in\mathbb{Z}}$, $(\beta_k)_{k\in\mathbb{Z}} \in \mathcal{A}(n;q)[z]$ and matrices A, B by

$$\det(I + t\mathrm{diag}(z)x;q) = \sum_k \alpha_k t^k, \qquad A = (\alpha_{\lambda_i^\top + j - i})_{i,j=1}^{\lambda_1},$$

$$\det(I - t\mathrm{diag}(z)x;q)^{-1} = \sum_k \beta_k t^k, \qquad B = (\beta_{\lambda_i + j - i})_{i,j=1}^{\ell}.$$

Theorem: (K-S '08)

$$\det(A) \equiv \det(B) \equiv \mathrm{Imm}_\lambda(x;q), \quad \mathrm{mod}\ (z_1^2, \dots, z_n^2).$$

Proof idea: By Quantum L-W-M identities,

$$\alpha_j\alpha_k \equiv \alpha_k\alpha_j, \quad \beta_j\beta_k \equiv \beta_k\beta_j, \quad \mathrm{mod}\ (z_1^2, \dots, z_n^2).$$

Quantum MacMahon Master Theorem

Let $y_1, \ldots, y_n$ quasicommute, $y_j y_i = q^{\frac{1}{2}} y_i y_j$ if $i < j$.
Let $z_1, \ldots, z_n$ commute.
Define $(k)_q = 1 + q^{\frac{1}{2}} + q^{\frac{2}{2}} + \cdots + q^{\frac{k-1}{2}}$.
Define $(k)_q! = (k)_q \cdot (k-1)_q \cdots (1)_q$.

Theorem: (G-L-Z '05) Let $K = 1^{k_1} \cdots n^{k_n}$ be a multiset of $[n]$. The coefficients of $y_1^{k_1} \cdots y_n^{k_n}$ and $z_1^{k_1} \cdots z_n^{k_n}$ in

$$\Big(\sum_{j=1}^{n} x_{1,j} y_j\Big)^{k_1} \cdots \Big(\sum_{j=1}^{n} x_{n,j} y_j\Big)^{k_n} \text{ and } \frac{1}{\det(I - \operatorname{diag}(z) x; q)}$$

are both equal to $\operatorname{per}(x_{K,K}; q)/((k_1)_q! \cdots (k_n)_q!)$.