GENERATING FUNCTIONS FOR HECKE ALGEBRA CHARACTERS

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Abstract. Certain polynomials in n^2 variables which serve as generating functions for symmetric group characters are sometimes called (S_n) character immanants. We point out a close connection between the identities of Littlewood-Merris-Watkins and Goulden-Jackson, which relate S_n character immanants to the determinant, the permanent and MacMahon's Master Theorem. From these results we obtain a generalization of Muir's identity. Working with the quantum polynomial ring and the Hecke algebra $H_n(q)$, we define quantum immanants which are generating functions for Hecke algebra characters. We then prove quantum analogs of the Littlewood-Merris-Watkins identities and selected Goulden-Jackson identities which relate $H_n(q)$ character immanants to the quantum determinant, quantum permanent, and quantum Master Theorem of Garoufalidis-Lê-Zeilberger. We also obtain a generalization of Zhang's quantization of Muir's identity.

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1. Introduction

Among their abundant results expressing generating functions in terms of matrix traces and determinants, Goulden and Jackson [14] obtained several identities concerning polynomials

$$\mathrm{Imm}_\lambda(x) \underset{\text{def}}{=} \sum_{w \in S_n} \chi^\lambda(w) x_{1,w_1} \cdots x_{n,w_n}$$

in $x = (x_{1,1}, \ldots, x_{n,n})$ whose coefficients are given by irreducible characters χ^λ of S_n. We will call these polynomials *irreducible character immanants.* Using their identities, Goulden and Jackson gave new presentations of results of Littlewood, MacMahon, and Young, reiterating a little-known interpretation of MacMahon's celebrated Master Theorem which had been stated by Vere-Jones in [31]. Also giving new interpretations of Littlewood's results, Merris and Watkins [25] stated similar formulae for irreducible (and other) character immanants by summing products of permanents and determinants. As we will show in Section 3, one may use the Littlewood-Merris-Watkins identities to give a new proof of the Goulden-Jackson identities, and consequently to generalize Muir's identity.

Many of the above results have natural noncommutative extensions. Authors such as Cartier-Foata [1], Foata-Han [6], [7], [8], Garoufalidis-Lê-Zeilberger [11], Hai-Lorenz [16], Konvalinka-Pak [19], Krattenthaler-Schlosser [21], and Zhang [33] have stated quantum analogs of the Master Theorem, Muir identity, and related identities. After reviewing the relevant quantum algebras in Section 4, we will state and prove quantum analogs of identities of Littlewood-Merris-Watkins and selected identities of Goulden-Jackson in Section 5, showing that these new quantum identities are related to one another in much the same way as their classical analogs. While natural quantum analogs of the most general Goulden-Jackson identities fail to hold, we employ results of Garoufalidis-Lê-Zeilberger to quantize the Vere-Jones interpretation of the Master Theorem, and to generalize Zhang's quantization of Muir's identity.

We remark that the irreducible character immanants appeared originally in the work of Schur [26] and Littlewood [22] on representations of S_n and GL_n, and later in work connected to permanent inequalities. (See, e.g., references in [25], [30].) More recent appearances of immanants in the areas of total nonnegativity and Schur nonnegativity (e.g., [15], [17], [20], [29]) were inspired by a conjecture of Goulden and Jackson in [13], quite different in flavor from the results in [14].

2. The symmetric group, $\mathbb{C}[x]$, and immanant identities of Littlewood-Merris-Watkins

Classical generating functions for symmetric group characters are polynomials belonging to a special graded component of the ring $\mathbb{C}[x] = \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$. Recall that the symmetric group S_n is generated by the *adjacent transpositions* $s_1, \ldots, s_{n-1}$ subject to the relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j, && \text{if } |i-j| = 1, \\ s_i s_j &= s_j s_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

A standard action of S_n on rearrangements of the word $1 \cdots n$ is defined by letting s_i swap the letters in positions i and $i+1$,

$$s_i \circ a_1 \cdots a_n = a_1 \cdots a_{i-1} a_{i+1} a_i a_{i+2} \cdots a_n.$$

For each element $v = s_{i_1} \cdots s_{i_\ell} \in S_n$, we define the *one-line* notation of v to be the word $v_1 \cdots v_n = v \circ 1 \cdots n$. Thus, denoting the identity permutation of S_n by e, the one-line notation of e is $12 \cdots n$. Using this convention, the one-line notation of vw is

$$(vw)_1 \cdots (vw)_n = v \circ (w \circ e) = v \circ w_1 \cdots w_n = w_{v_1} \cdots w_{v_n}.$$

Thus, the one-line notation of $s_1 s_2 \in S_3$ is 312. We will denote the one-line notation of $v^{-1} = s_{i_\ell} \cdots s_{i_1}$ by $v_1^{-1} \cdots v_n^{-1}$.

Let $\ell(w)$ be the minimum length of an expression for w in terms of the generators. Equivalently, $\ell(w)$ is the number of inversions in the one-line notation of w. Let $\leq$ denote the Bruhat order on S_n, i.e., $v \leq w$ if every reduced expression for w contains a reduced expression for v as a subword. (See, e.g., [18].)

The ring $\mathbb{C}[x]$ is naturally graded by degree,

$$\mathbb{C}[x] = \bigoplus_{r \geq 0} \mathcal{A}_r,$$

where $\mathcal{A}_r$ is the $\mathbb{C}$-span of all monomials of total degree r, and the natural basis $\{x_{1,1}^{a_{1,1}} \cdots x_{n,n}^{a_{n,n}} \,|\, a_{1,1}, \ldots, a_{n,n} \in \mathbb{N}\}$ of $\mathbb{C}[x]$ is a disjoint union

$$\bigcup_{r \geq 0} \{x_{1,1}^{a_{1,1}} \cdots x_{n,n}^{a_{n,n}} \,|\, a_{1,1} + \cdots + a_{n,n} = r\}$$

of bases of the homogeneous components $\{\mathcal{A}_r \,|\, r \geq 0\}$.

We may further decompose each homogeneous component $\mathcal{A}_r$ by considering pairs (L, M) of multisets of integers. Thus we obtain the multigrading

$$\mathcal{A}_r = \bigoplus_{\substack{L,M \\ |L|=|M|=r}} \mathcal{A}_{L,M},$$

where $\mathcal{A}_{L,M}$ is the $\mathbb{C}$-span of monomials whose row indices and column indices (with multiplicity) are equal to the multisets L and M, respectively. Just as the $\mathbb{Z}$-graded components $\mathcal{A}_r$ and $\mathcal{A}_s$ satisfy $\mathcal{A}_r \mathcal{A}_s \subset \mathcal{A}_{r+s}$, the multigraded components $\mathcal{A}_{L,M}$ and

$\mathcal{A}_{L',M'}$ satisfy $\mathcal{A}_{L,M}\mathcal{A}_{L',M'} \subset \mathcal{A}_{L\uplus L',M\uplus M'}$, where $\uplus$ denotes the *multiset union* of two multisets,

$$1^{\ell_1}\cdots n^{\ell_n} \uplus 1^{\ell'_1}\cdots n^{\ell'_n} \underset{\text{def}}{=} 1^{\ell_1+\ell'_1}\cdots n^{\ell_n+\ell'_n}.$$

Thus $\mathcal{A}_{[n],[n]}$ is the $\mathbb{C}$-span of the monomials $\{x_{1,w_1}\cdots x_{n,w_n} \,|\, w \in S_n\}$. We will call elements of this submodule *immanants*. Defining the notation $x^{u,v} = x_{u_1,v_1}\cdots x_{u_n,v_n}$, we may rewrite the natural basis of the immanant space as $\{x^{e,w} \,|\, w \in S_n\}$.

For a function $f : S_n \to \mathbb{C}$, we follow [28] in defining the *f-immanant* to be the element

$$\mathrm{Imm}_f(x) = \sum_{w\in S_n} f(w)x_{1,w_1}\cdots x_{n,w_n}$$

of the immanant space. Two well-known examples are the determinant and permanent, whose coefficient functions are the sign and trivial characters of S_n, respectively.

Let $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ be an integer partition (with $\ell > 0$) and let $\lambda^\top$ denote the transpose (also called the *conjugate*) of the integer partition λ. (See [9].) Immanants $\mathrm{Imm}_{\chi^\lambda}(x)$ constructed from the irreducible characters $\chi^\lambda : S_n \to \mathbb{R}$ of S_n are usually abbreviated $\mathrm{Imm}_\lambda(x)$,

$$\mathrm{Imm}_\lambda(x) = \sum_{w\in S_n} \chi^\lambda(w)x^{e,w}. \tag{2.1}$$

It is well known that irreducible characters are *class functions* on S_n in the sense that if v and w have the same cycle type in S_n, then $\chi^\lambda(v)$ and $\chi^\lambda(w)$ are equal. Equivalently, we have $\chi^\lambda(v) = \chi^\lambda(w)$ if $v = uwu^{-1}$ for some $u \in S_n$.

Immanants which are somewhat better understood than irreducible character immanants correspond to characters $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$ induced from the sign character of Young subgroups of S_n and to characters $\{\eta^\lambda \,|\, \lambda \vdash n\}$ induced from the trivial character of Young subgroups of S_n. Simple formulas for these immanants employ determinants and permanents of submatrices

$$x_{I,J} \underset{\text{def}}{=} (x_{i,j})_{i\in I, j\in J}$$

of x. In particular, Littlewood [22] and Merris and Watkins [25] showed the following.

Theorem 2.1. *Fix a partition $\lambda = (\lambda_1,\ldots,\lambda_\ell) \vdash n$. Then we have*

$$\begin{aligned}\mathrm{Imm}_{\epsilon^\lambda}(x) &= \sum_{(I_1,\ldots,I_\ell)} \det(x_{I_1,I_1})\cdots\det(x_{I_\ell,I_\ell}),\\ \mathrm{Imm}_{\eta^\lambda}(x) &= \sum_{(I_1,\ldots,I_\ell)} \mathrm{per}(x_{I_1,I_1})\cdots\mathrm{per}(x_{I_\ell,I_\ell}),\end{aligned} \tag{2.2}$$

where the sums are over all sequences $(I_1,\ldots,I_\ell)$ of pairwise disjoint subsets of $[n]$ satisfying $|I_j| = \lambda_j$.

We will give our own proof of this fact in Section 5.

Each of the sets $\{\chi^\lambda \,|\, \lambda \vdash n\}$, $\{\eta^\lambda \,|\, \lambda \vdash n\}$, $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$ forms a basis for the space of class functions on S_n. We may express the first basis in terms of the others by

$$\chi^\lambda = \sum_\mu K^{-1}_{\mu,\lambda} \eta^\mu = \sum_\mu K^{-1}_{\mu,\lambda^\top} \epsilon^\mu, \tag{2.3}$$

and we therefore may express irreducible character immanants in terms of the induced character immanants by

$$\mathrm{Imm}_\lambda(x) = \sum_\mu K^{-1}_{\mu,\lambda} \mathrm{Imm}_{\eta^\mu}(x) = \sum_\mu K^{-1}_{\mu,\lambda^\top} \mathrm{Imm}_{\epsilon^\mu}(x). \tag{2.4}$$

The coefficients appearing in these identities are called the *inverse Kostka numbers* and may be defined by

$$\det(\xi_{\lambda_i+j-i})_{i,j=1}^{\ell} = \sum_{\mu \vdash n} K^{-1}_{\mu,\lambda} \xi_{\mu_1} \cdots \xi_{\mu_\ell}, \tag{2.5}$$

where $\{\xi_i \,|\, i > 0\}$ are commuting indeterminates, and where we define $\xi_0 = 1$, and $\xi_i = 0$ for $i < 0$. This definition implies that we have $K^{-1}_{\mu,\lambda} = 0$ unless $\lambda \leq \mu$ in majorization. (See [9].) Thus the sums in (2.3) and (2.4) may be taken over $\mu \geq \lambda$ and $\mu \geq \lambda^\top$, respectively.

While the immanant space is just one graded component of $\mathbb{C}[x]$, one can understand all other components in terms of immanants as well. In particular, given multisets $L = 1^{\ell_1} 2^{\ell_2} \cdots n^{\ell_n}$, $M = 1^{m_1} 2^{m_2} \cdots n^{m_n}$ with $\ell_1 + \cdots + \ell_n = m_1 + \cdots + m_n = r$, we define the L, M *generalized submatrix* of x to be the $r \times r$ matrix $x_{L,M}$ obtained from x by repeating the ith row ℓ_i times and ith column m_i times for $i = 1, \ldots, n$. For example when $L = 1112$, $M = 1334$ we have

$$(\ell_1, \ell_2, \ell_3, \ell_4) = (3, 1, 0, 0), \quad (m_1, m_2, m_3, m_4) = (1, 0, 2, 1),$$

$$x_{L,M} = \begin{bmatrix} x_{1,1} & x_{1,3} & x_{1,3} & x_{1,4} \\ x_{1,1} & x_{1,3} & x_{1,3} & x_{1,4} \\ x_{1,1} & x_{1,3} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,3} & x_{2,3} & x_{2,4} \end{bmatrix}.$$

In general, it is easy to see that elements of the component $\mathcal{A}_{L,M}$ of $\mathbb{C}[x]$ are precisely the specializations of the $r \times r$ immanants

$$\{\mathrm{Imm}_f(y_{1,1}, \ldots, y_{r,r}) \,|\, f : S_r \to \mathbb{C}\}$$

at $y = x_{L,M}$.

Using generalized submatrices, we may generalize the Littlewood-Merris-Watkins identities (2.2) as follows.

Proposition 2.2. *Fix an r-element multiset $M = 1_1^m \cdots n^{m_n}$ of $[n]$ and a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of r. Then we have*

$$\mathrm{Imm}_{\epsilon^\lambda}(x_{M,M}) = m_1! \cdots m_n! \sum_{(J_1,\dots,J_\ell)} \det(x_{J_1,J_1}) \cdots \det(x_{J_\ell,J_\ell})$$

where the sum is over all sequences $(J_1, \dots, J_\ell)$ of subsets of $[n]$ satisfying $|J_i| = \lambda_i$ and $J_1 \uplus \cdots \uplus J_\ell = M$.

Proof. Let $y = (y_{1,1}, \dots, y_{r,r})$. The map $y \mapsto x_{M,M}$ defines a ring homomorphism $\mathbb{C}[y] \to \mathbb{C}[x]$ and thus preserves the identities (2.2). We therefore have

$$\mathrm{Imm}_{\epsilon^\lambda}(x_{M,M}) = \sum_{(I_1,\dots,I_\ell)} \det((x_{M,M})_{I_1,I_1}) \cdots \det((x_{M,M})_{I_\ell,I_\ell}).$$

Each term in the above sum has the form $\det(x_{J_1,J_1}) \cdots \det(x_{J_\ell,J_\ell})$, where $(J_1, \dots, J_\ell)$ is a sequence of submultisets of M satisfying $|J_i| = \lambda_i$ and $J_1 \uplus \cdots \uplus J_\ell = M$. Since repeated rows and columns cause the determinant to vanish, we may sum over sequences of subsets. Now if $J_1, \dots, J_\ell$ are all sets, then the number of times $\det(x_{J_1,J_1}) \cdots \det(x_{J_\ell,J_\ell})$ appears in this sum is $m_1! \cdots m_n!$. □

Similarly, we have the following generalization of the permanent identity in (2.2).

Proposition 2.3. *Fix an r-element multiset $M = 1^{m_1} \cdots n^{m_n}$ of $[n]$ and a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of r. Then we have*

$$\mathrm{Imm}_{\eta^\lambda}(x_{M,M}) = \sum_{(J_1,\dots,J_\ell)} \theta(J_1, \dots, J_\ell) \mathrm{per}(x_{J_1,J_1}) \cdots \mathrm{per}(x_{J_\ell,J_\ell}),$$

where the sum is over all sequences $(J_1, \dots, J_\ell)$ of submultisets $J_i = 1^{j_{i,1}} \cdots n^{j_{i,n}}$ of $[n]$ satisfying $|J_i| = \lambda_i$, $J_1 \uplus \cdots \uplus J_\ell = M$, and where

$$\theta(J_1, \dots, J_\ell) = \prod_{i=1}^{\ell} \binom{m_i}{j_{i,1}, \dots, j_{i,n}}.$$

Proposition 2.2 leads to another generalization of Theorem 2.1.

Corollary 2.4. *For $k = 1, \dots, n$ define*

$$\alpha_k = \sum_{\substack{I \subset [n] \\ |I| = k}} \det(x_{I,I}).$$

Fix a partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash r$. Then we have

$$\alpha_{\lambda_1} \cdots \alpha_{\lambda_\ell} = \sum_M \frac{\mathrm{Imm}_{\epsilon^\lambda}(x_{M,M})}{m_1! \cdots m_n!},$$

where the sum is over all r-element multisets $M = 1^{m_1} \cdots n^{m_n}$ of $[n]$.

3. The Goulden-Jackson identities, MacMahon Master Theorem and Muir identity

Goulden and Jackson [14] stated several identities relating the irreducible character immanants $\{\mathrm{Imm}_\lambda(x) \,|\, \lambda \vdash n\}$ to multivariate generating functions and to MacMahon's Master Theorem. We will summarize their results and give new proofs which expose connections to the Littlewood-Merris-Watkins identities in Theorem 2.1 and Propositions 2.2-2.3.

To begin, we state the following fact [14, Eqn. (6)] concerning a power series and its inverse. (For a proof, see [10, Ch. 1 §4], [23, pp. 22-23].)

Proposition 3.1. *Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition and let $(\alpha_k)_{k\geq 0}$ and $(\beta_k)_{k\geq 0}$ be sequences of elements of a commutative ring R. If the sequences are related by the identity*

$$\frac{1}{\sum_{k\geq 0}\alpha_k(-t)^k} = \sum_{k\geq 0}\beta_k t^k$$

in $R[[t]]$, then the $\lambda_1 \times \lambda_1$ matrix $A = (\alpha_{\lambda^\top_i - i+j})$ and the $\ell\times\ell$ matrix $B = (\beta_{\lambda_i-i+j})$ satisfy $\det(A) = \det(B)$.

To state the main results [14, Thm. 2.1, Cor. 2.3] on irreducible character immanants, we define the sequences $(\alpha_k)_{k\in\mathbb{Z}}$, $(\beta_k)_{k\in\mathbb{Z}}$, $(\gamma_k)_{k\in\mathbb{Z}}$, $(\delta_k)_{k\in\mathbb{Z}}$ of polynomials in $\mathbb{Z}[x]$ by the generating functions

$$\text{(3.1)}\qquad \begin{aligned} \det(I+tx) &= \sum_{k=0}^{n}\alpha_k t^k, & \frac{1}{\det(I-tx)} &= \sum_{k\geq 0}\beta_k t^k,\\ \mathrm{per}(I+tx) &= \sum_{k=0}^{n}\gamma_k t^k, & \frac{1}{\mathrm{per}(I-tx)} &= \sum_{k\geq 0}\delta_k t^k,\end{aligned}$$

and by the requirement that polynomials with indices not appearing here be zero. Now fix a partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash r$ and define the $\lambda_1 \times \lambda_1$ matrices A, D and the $\ell \times \ell$ matrices B, C by

$$\text{(3.2)}\qquad A = (\alpha_{\lambda^\top_i-i+j}), \qquad B = (\beta_{\lambda_i-i+j}), \qquad C = (\gamma_{\lambda_i-i+j}), \qquad D = (\delta_{\lambda^\top_i-i+j}).$$

It is easy to see that the determinant of each of these matrices belongs to $\mathcal{A}_r$. By Proposition 3.1 we also have $\det(A) = \det(B)$ and $\det(C) = \det(D)$. Moreover, we have the following more specific descriptions of these polynomials.

Theorem 3.2. *Fix a partition $\lambda \vdash r$ and define matrices A, B as in (3.2). Then we have*

$$\text{(3.3)}\qquad \det(A) = \det(B) = \sum_M \frac{\mathrm{Imm}_\lambda(x_{M,M})}{m_1!\cdots m_n!},$$

where the sum is over all r-element multisets $M = 1^{m_1} \cdots n^{m_n}$ of $[n]$.

Proof. Observe that the polynomials $\{\alpha_k \mid k \in \mathbb{Z}\}$ in the commutative ring $\mathbb{Z}[x]$ satisfy $\alpha_0 = 1$, $\alpha_k = 0$ if $k < 0$, and

$$\alpha_k = \sum_{\substack{I \subset [n] \\ |I| = k}} \det(x_{I,I}) \tag{3.4}$$

if $k > 0$. Thus by Propositions 2.2-3.1 and the definition of the inverse Kostka numbers, we have

$$\begin{aligned}
\det(B) = \det(A) &= \sum_{\mu \geq \lambda^\top} K^{-1}_{\mu,\lambda^\top} \alpha_{\mu_1} \cdots \alpha_{\mu_{\lambda_1}} \\
&= \sum_{\mu \geq \lambda^\top} K^{-1}_{\mu,\lambda^\top} \sum_M \frac{\mathrm{Imm}_{\epsilon^\mu}(x_{M,M})}{m_1! \cdots m_n!} \\
&= \sum_M \frac{\mathrm{Imm}_\lambda(x_{M,M})}{m_1! \cdots m_n!}.
\end{aligned}$$

□

In the special case $r = n$, we have that the projection of $\det(A) = \det(B)$ onto the immanant space is equal to $\mathrm{Imm}_\lambda(x)$. A weaker result applies to $\det(C)$ and $\det(D)$.

Theorem 3.3. *Fix a partition $\lambda \vdash r$ and define matrices C, D as in (3.2). Then we have $\det(C) = \det(D)$ and for any r-element subset $I \subset [n]$, the projection of this polynomial onto $\mathcal{A}_{I,I}$ is equal to $\mathrm{Imm}_\lambda(x_{I,I})$.*

Proof. The polynomials $\{\gamma_k \mid k \in \mathbb{Z}\}$ satisfy $\gamma_0 = 1$, $\gamma_k = 0$ if $k < 0$, and

$$\gamma_k = \sum_{\substack{I \subset [n] \\ |I| = k}} \mathrm{per}(x_{I,I}) \tag{3.5}$$

if $k > 0$. By (2.2), Proposition 3.1 and the definition of the inverse Kostka numbers, we have

$$\begin{aligned}
\det(D) = \det(C) &= \sum_{\mu \geq \lambda} K^{-1}_{\mu,\lambda} \gamma_{\mu_1} \cdots \gamma_{\mu_{\lambda_1}} \\
&= \sum_{\mu \geq \lambda} K^{-1}_{\mu,\lambda} \Big(\sum_{\substack{M \subset [n] \\ |M| = r}} \mathrm{Imm}_{\eta^\mu}(x_{M,M}) + \sum_N \mathrm{Imm}_{f_{N,\mu}}(x_{N,N}) \Big) \\
&= \sum_{\substack{M \subset [n] \\ |M| = r}} \mathrm{Imm}_\lambda(x_{M,M}) + \sum_N \mathrm{Imm}_{g_N}(x_{N,N}),
\end{aligned}$$

where the last sums on the above two lines are over r-element multisets N of $[n]$ in which some element appears with multiplicity at least two, and $f_{N,\mu}$, g_N are functions from S_r to $\mathbb{Z}$. □

Again, in the special case $r = n$, we have that the projection of $\det(C) = \det(D)$ onto the immanant space is equal to $\mathrm{Imm}_\lambda(x)$.

The special case of Theorem 3.2 corresponding to the partition $\lambda = (r)$,

$$\beta_r = \sum_M \frac{\mathrm{per}(x_{M,M})}{m_1! \cdots m_n!}, \tag{3.6}$$

is equivalent to MacMahon's Master Theorem [24]. This easy equivalence seems to have been first stated explicitly by Vere-Jones [31]. (See also [32], and the earlier paper [27], which does not explicitly mention the Master Theorem.) Goulden and Jackson [14, Thm. 3.3] state and prove this more explicit version of the Master Theorem as follows.

Theorem 3.4. *Fix a multiset $M = 1^{m_1} \cdots n^{m_n}$ of $[n]$. The coefficient of $z_1^{m_1} \cdots z_n^{m_n}$ in*

$$\Big(\sum_{j=1}^n x_{1,j} z_j\Big)^{m_1} \cdots \Big(\sum_{j=1}^n x_{n,j} z_j\Big)^{m_n} \tag{3.7}$$

and the projection of

$$\frac{1}{\det(I - x)} \tag{3.8}$$

onto $\mathcal{A}_{M,M}$ are both equal to

$$\frac{\mathrm{per}(x_{M,M})}{m_1! \cdots m_n!}.$$

Proof. The coefficient of $z_1^{m_1} \cdots z_n^{m_n}$ in (3.7) may be interpreted as follows. Circle n entries of the matrix $x_{M,[n]}$ so that each row contains exactly one circled entry and column i contains exactly m_i circled entries for $i = 1, \ldots, n$, and take the sum of products of all such collections of circled entries. But this sum is the same as that obtained by circling one entry per row and one entry per column in the matrix $x_{M,M}$ in all possible ways, divided by $m_1! \cdots m_n!$. Thus the coefficient in question is equal to $\mathrm{per}(x_{M,M})/(m_1! \cdots m_n!)$.

On the other hand, the projection of (3.8) onto $\mathcal{A}_{M,M}$ is equal to the projection of β_r onto $\mathcal{A}_{M,M}$, which by (3.6) is $\mathrm{per}(x_{M,M})/(m_1! \cdots m_n!)$. □

This same special case (3.6) of Theorem 3.2 provides a proof of (a generalization of) Muir's indentity.

Corollary 3.5. *Let $M = 1^{m_1} \cdots n^{m_n}$ be an r-element multiset of $[n]$, $r > 0$. Then we have*

$$\sum_{k=0}^{\min\{r,n\}} (-1)^k \sum_{\substack{I \subset M \\ |I|=k}} \frac{\det(x_{I,I})\mathrm{per}(x_{M \smallsetminus I, M \smallsetminus I})}{(m_1 - i_1)! \cdots (m_n - i_n)!} = 0,$$

where the second sum is over sets $I = 1^{i_1} \cdots n^{i_n}$ contained in M.

Proof. By the definitions in Equation (3.1), the sequences $(\alpha_k)_{k \geq 0}$ and $(\beta_k)_{k \geq 0}$ satisfy

$$\sum_{k=0}^{\min\{r,n\}} (-1)^k \alpha_k \beta_{r-k} = 0 \tag{3.9}$$

for $r > 0$. Using (3.4) and (3.6) and projecting both sides of (3.9) onto $\mathcal{A}_{M,M}$, we have the desired result. □

In the special case $M = [n]$, we obtain Muir's classical identity

$$\sum_{k=0}^{n} (-1)^k \sum_{\substack{I \subset [n] \\ |I|=k}} \det(x_{I,I})\mathrm{per}(x_{[n] \smallsetminus I, [n] \smallsetminus I}) = 0.$$

4. The Hecke algebra and quantum polynomial ring

In order to state and prove quantum analogs of the identities concerning generating functions for symmetric group characters, we present the *Hecke algebra*, a quantum analog of $\mathbb{C}[S_n]$, and the *quantum polynomial ring*, a quantum analog of the ordinary polynomial ring in n^2 variables.

The Hecke algebra $H_n(q)$ is a noncommutative $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-algebra generated either by the set $\{T_{s_i} \,|\, 1 \leq i \leq n-1\}$ of *natural generators*, or equivalently by the set $\{\widetilde{T}_{s_i} \,|\, 1 \leq i \leq n-1\}$ of *modified natural generators*, subject to the relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + q, & \widetilde{T}_{s_i}^2 &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}_{s_i} + 1, & &\text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, & \widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j}, & &\text{if } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, & \widetilde{T}_{s_i}\widetilde{T}_{s_j} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i}, & &\text{if } |i-j| \geq 2. \end{aligned}$$

The natural and modified natural generators are related by $\widetilde{T}_{s_i} = q^{-\frac{1}{2}}T_{s_i}$. If $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for $w \in S_n$ we define

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}, \qquad \widetilde{T}_w = q^{-\frac{\ell}{2}}T_w = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_\ell}}, \qquad T_e = \widetilde{T}_e = 1.$$

We shall call the elements $\{T_w \,|\, w \in S_n\}$ and $\{\widetilde{T}_w \,|\, w \in S_n\}$ the *standard basis* and *modified basis*, respectively, of $H_n(q)$ as a $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module. Specializing $H_n(q)$ at $q^{\frac{1}{2}} = 1$, we obtain the classical group algebra $\mathbb{C}[S_n]$ of the symmetric group.

One multiplies modified basis elements by recursively using either of the formulae

$$\widetilde{T}_{s_i}\widetilde{T}_w = \begin{cases} \widetilde{T}_{s_iw} & \text{if } s_iw > w, \\ \widetilde{T}_{s_iw} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}_w & \text{if } s_iw < w, \end{cases}$$
$$\widetilde{T}_w\widetilde{T}_{s_i} = \begin{cases} \widetilde{T}_{ws_i} & \text{if } ws_i > w, \\ \widetilde{T}_{ws_i} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}_w & \text{if } ws_i < w. \end{cases} \tag{4.1}$$

This procedure yields elements $c^w_{u,v} \in \mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ occurring as coefficients in the expression

$$\widetilde{T}_u\widetilde{T}_v = \sum_w c^w_{u,v}\widetilde{T}_w. \tag{4.2}$$

By the symmetry of the formulae (4.1), we see immediately that these coefficients satisfy $c^w_{u,v} = c^{w^{-1}}_{v^{-1},u^{-1}}$. We also have the following equalities.

Lemma 4.1. *The coefficients* $\{c^w_{u,v} \,|\, u, v, w \in S_n\}$ *satisfy* $c^w_{u,v} = c^{u^{-1}}_{v,w^{-1}}$.

Proof. First consider the case $w = e$. It is clear from (4.1) that we have $c^e_{u,u^{-1}} = 1$. On the other hand, it is also clear that if $\ell(u) \neq \ell(v)$, or if $\ell(u) = \ell(v)$ and $v \neq u^{-1}$, then $\widetilde{T}_e$ does not appear in the expansion of $\widetilde{T}_u\widetilde{T}_v$. Thus we have

$$c^e_{u,v} = \begin{cases} 1 & \text{if } v = u^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$

which is clearly equal to $c^{u^{-1}}_{v,e}$.

Now consider the identity

$$\widetilde{T}_u\widetilde{T}_v\widetilde{T}_{w^{-1}} = \Big(\sum_{y \in S_n} c^y_{u,v}\widetilde{T}_y\Big)\widetilde{T}_{w^{-1}} = \widetilde{T}_u\Big(\sum_{y \in S_n} c^y_{v,w^{-1}}\widetilde{T}_y\Big).$$

Equating the coefficients of $\widetilde{T}_e$ in these expressions, we have $c^w_{u,v} = c^{u^{-1}}_{v,w^{-1}}$. □

A second noncommutative $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-algebra is the quantum polynomial ring $\mathcal{A}(q)$. It is generated by n^2 variables $x = (x_{1,1}, \ldots, x_{n,n})$ representing matrix entries, subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k}, \\ x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{4.3}$$

for all indices $i < j$, $k < \ell$. The quantum polynomial ring often arises in conjunction with the quantum coordinate ring of $SL(n, \mathbb{C})$, which may be expressed as a quotient $\mathcal{O}_q SL(n,\mathbb{C}) \cong \mathcal{A}(q)/(\det_q(x) - 1)$, where

$$\det_q(x) \underset{\text{def}}{=} \sum_{w \in S_n} (-q^{-\frac{1}{2}})^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n} = \sum_{w \in S_n} (-q^{-\frac{1}{2}})^{\ell(w)} x_{w_1,1} \cdots x_{w_n,n} \tag{4.4}$$

is the *quantum determinant.* (We caution the reader that the second equality above is implied by the third relation in (4.3), and does not hold in an arbitrary noncommutative ring in n^2 variables.) Specializing $\mathcal{A}(q)$ at $q^{\frac{1}{2}} = 1$, we obtain the commutative polynomial ring $\mathbb{C}[x]$.

As a $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module, $\mathcal{A}(q)$ is spanned by monomials in lexicographic order, and we can use the relations above to convert any other monomial to this standard form. It is easy to see that the monomials $\{x^{u,v} \mid u, v \in S_n\}$ satisfy

$$x^{s_iu,v} = \begin{cases} x^{u,s_iv} & \text{if } s_iu > u \text{ and } s_iv > v, \text{ or if } s_iu < u \text{ and } s_iv < v, \\ x^{u,s_iv} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x^{u,v} & \text{if } s_iu > u \text{ and } s_iv < v, \\ x^{u,s_iv} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x^{u,v} & \text{if } s_iu < u \text{ and } s_iv > v. \end{cases} \tag{4.5}$$

Thus for all $w \in S_n$ we have the identity

$$x^{w,e} = x^{e,w^{-1}}. \tag{4.6}$$

On the other hand, we do not in general have the equality of $x^{v,w}$ and $x^{w^{-1},v^{-1}}$. Applying (4.5) recursively to express $x^{v,w}$ in terms of the natural basis, we obtain an expression of the form

$$x^{v,w} = x^{e,v^{-1}w} + \sum_{u > v^{-1}w} d^u_{v,w} x^{e,u}, \tag{4.7}$$

where the coefficients $d^u_{v,w}$ belong to $\mathbb{N}[q^{\frac{1}{2}} - q^{-\frac{1}{2}}]$.

Like $\mathbb{C}[x]$, the noncommutative ring $\mathcal{A}(q)$ has a natural grading by degree,

$$\mathcal{A}(q) = \bigoplus_{r \geq 0} \mathcal{A}_r(q),$$

where $\mathcal{A}_r(q)$ is the $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-span of all monomials of total degree r. As before, the natural basis $\{x_{1,1}^{a_{1,1}} \cdots x_{n,n}^{a_{n,n}} \mid a_{1,1}, \ldots, a_{n,n} \in \mathbb{N}\}$ of $\mathcal{A}(q)$ is a disjoint union of bases of the homogeneous components $\{\mathcal{A}_r(q) \,|\, r \geq 0\}$, and we may further decompose each homogeneous component $\mathcal{A}_r(q)$ as

$$\mathcal{A}_r(q) = \bigoplus_{\substack{L,M \\ |L|=|M|=r}} \mathcal{A}_{L,M}(q),$$

where $\mathcal{A}_{L,M}(q)$ is the $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-span of monomials whose row indices and column indices (with multiplicity) are equal to the multisets L and M, respectively. Again we have $\mathcal{A}_{L,M}(q)\mathcal{A}_{L',M'}(q) \subset \mathcal{A}_{L \uplus L', M \uplus M'}(q)$.

Thus $\mathcal{A}_{[n],[n]}(q)$ is the $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-submodule of $\mathcal{A}(q)$ spanned by the monomials $\{x^{e,w} = x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in S_n\}$. We will call elements of this submodule *quantum immanants*, and for any $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-linear function $f : H_n(q) \to \mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, we define the *quantum f-immanant* to be

$$\mathrm{Imm}_f(x) = \sum_{w \in S_n} f(\widetilde{T}_w) x^{e,w}. \tag{4.8}$$

We will see in Theorems 5.4, 5.7 and 5.9 that the modified basis $\{\widetilde{T}_w \,|\, w \in S_n\}$ is a more natural choice for the definition (4.8) than is the standard basis $\{T_w \,|\, w \in S_n\}$. Corresponding to the Hecke algebra sign character $\chi_q^{1^n} : \widetilde{T}_w \mapsto (-q^{-\frac{1}{2}})^{\ell(w)}$ is the quantum determinant

$$\det_q(x) = \sum_{w \in S_n} (-q^{-\frac{1}{2}})^{\ell(w)} x^{e,w},$$

and corresponding to the Hecke algebra trivial character $\chi_q^{n} : \widetilde{T}_w \mapsto (q^{\frac{1}{2}})^{\ell(w)}$ is the quantum permanent

$$\mathrm{per}_q(x) = \sum_{w \in S_n} (q^{\frac{1}{2}})^{\ell(w)} x^{e,w}.$$

When $n = 3$ we have

$$\begin{aligned}
\det_q(x) &= x_{1,1}x_{2,2}x_{3,3} - q^{-\frac{1}{2}} x_{1,1}x_{2,3}x_{3,2} - q^{-\frac{1}{2}} x_{1,2}x_{2,1}x_{3,3} \\
&\qquad + q^{-1} x_{1,2}x_{2,3}x_{3,1} + q^{-1} x_{1,3}x_{2,1}x_{3,2} - q^{-\frac{3}{2}} x_{1,3}x_{2,2}x_{3,1}, \\
\mathrm{per}_q(x) &= x_{1,1}x_{2,2}x_{3,3} + q^{\frac{1}{2}} x_{1,1}x_{2,3}x_{3,2} + q^{\frac{1}{2}} x_{1,2}x_{2,1}x_{3,3} \\
&\qquad + q x_{1,2}x_{2,3}x_{3,1} + q x_{1,3}x_{2,1}x_{3,2} + q^{\frac{3}{2}} x_{1,3}x_{2,2}x_{3,1}.
\end{aligned}$$

5. Formulae for quantum character immanants

In analogy to the irreducible S_n character immanants, we shall construct quantum immanants $\mathrm{Imm}_{\chi_q^\lambda}(x)$ from the irreducible characters $\chi_q^\lambda : H_n(q) \to \mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ of

$H_n(q)$,

$$\mathrm{Imm}_{\chi_q^\lambda}(x;q) = \sum_{w \in S_n} \chi_q^\lambda(\widetilde{T}_w) x^{e,w} = \sum_{w \in S_n} \chi_q^\lambda(T_w)(-q^{-\frac{1}{2}})^{\ell(w)} x^{e,w}. \tag{5.1}$$

Two examples, as we have mentioned, are the quantum determinant and quantum permanent. The following table shows the values of irreducible characters on modified basis elements of $H_4(q)$.

w	$\chi_q^{1111}(\widetilde{T}_w)$	$\chi_q^{211}(\widetilde{T}_w)$	$\chi_q^{22}(\widetilde{T}_w)$	$\chi_q^{31}(\widetilde{T}_w)$	$\chi_q^{4}(\widetilde{T}_w)$
1234	1	3	2	3	1
1243, 1324, 2134	$-q^{-\frac{1}{2}}$	$q^{\frac{1}{2}} - 2q^{-\frac{1}{2}}$	$q^{\frac{1}{2}} - q^{-\frac{1}{2}}$	$2q^{\frac{1}{2}} - q^{-\frac{1}{2}}$	$q^{\frac{1}{2}}$
1423, 1432, 3124, 2314	q^{-1}	$q^{-1} - 1$	-1	$q - 1$	q
2143	q^{-1}	$q^{-1} - 2$	$q^{-1} + q$	$q - 2$	q
4123, 2413, 3142, 2341	$-q^{-\frac{3}{2}}$	$q^{-\frac{1}{2}}$	0	$-q^{\frac{1}{2}}$	$q^{\frac{3}{2}}$
1342, 3214	$-q^{-\frac{3}{2}}$	$-q^{-\frac{3}{2}}$	0	$q^{\frac{3}{2}}$	$q^{\frac{3}{2}}$
4132, 4213, 2431, 3241	q^{-2}	0	-1	0	q^2
3412	q^{-2}	-1	$q^{-1} + q$	-1	q^2
4312, 3421	$-q^{-\frac{5}{2}}$	$q^{-\frac{1}{2}}$	$q^{-\frac{1}{2}} - q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	$q^{\frac{5}{2}}$
4231	$-q^{-\frac{5}{2}}$	$-q^{-\frac{3}{2}}$	$q^{-\frac{1}{2}} - q^{\frac{1}{2}}$	$q^{\frac{3}{2}}$	$q^{\frac{5}{2}}$
4321	q^{-3}	$-q^{-1}$	2	$-q$	q^3

Note that the trace χ_q of a matrix representation of $H_n(q)$ is not an S_n-class function, in the sense that $\chi_q(\widetilde{T}_v)$ and $\chi_q(\widetilde{T}_{u^{-1}vu})$ are not in general equal (equivalently, $\chi_q(T_v)$ and $\chi_q(T_{u^{-1}vu})$ are not in general equal). On the other hand, χ_q does have the conjugation property one would expect: $\chi_q(\widetilde{T}_v) = \chi_q(\widetilde{T}_u^{-1}\widetilde{T}_v\widetilde{T}_u)$. We will use the term $H_n(q)$ *character* to refer to any function $\chi_q : H_n(q) \to \mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ having this conjugation property, whether or not the function is the trace of a matrix representation. Quantum analogs of the immanants $\{\mathrm{Imm}_{\epsilon^\lambda}(x) \mid \lambda \vdash n\}$ and $\{\mathrm{Imm}_{\eta^\lambda}(x) \mid \lambda \vdash n\}$ correspond to characters $\{\epsilon_q^\lambda \mid \lambda \vdash n\}$ and $\{\eta_q^\lambda \mid \lambda \vdash n\}$ induced from the sign character and trivial character of Hecke algebras of Young subgroups of S_n.

We will use the following standard notation and facts about Young subgroups. Let J be a subset of the standard generators $\{s_1, \dots, s_{n-1}\}$ of $W = S_n$ and let W_J be the corresponding Young subgroup of W. Let W/W_J be the set of cosets of the form uW_J. Each such coset is an interval in the Bruhat order and thus has a unique minimal element and a unique maximal element. Let W_-^J be the set of minimal representatives of cosets in W/W_J. It is well known that we have

$$
\begin{aligned}
W_-^J &= \{w \in S_n \mid ws_i > w \text{ for all } s_i \in J\} \\
&= \{w \in S_n \mid s_i w^{-1} > w^{-1} \text{ for all } s_i \in J\} \\
&= \{w \mid i \text{ appears before } i+1 \text{ in } w_1 \cdots w_n \text{ for all } s_i \in J\} \\
&= \{w \mid w_i^{-1} < w_{i+1}^{-1} \text{ for all } s_i \in J\}.
\end{aligned} \tag{5.2}
$$

To prove quantum analogs of the formulae (2.2), we consider elements of $H_n(q)$ which are often used in conjunction with Young subalgebras. (See, e.g., [2], [3], [5].) For each permutation $u \in W_-^J$, define the Hecke algebra elements

$$T_{uW_J} = \widetilde{T}_u \sum_{y \in W_J} (-q^{-\frac{1}{2}})^{\ell(y)} \widetilde{T}_y, \qquad T'_{uW_J} = \widetilde{T}_u \sum_{y \in W_J} (q^{\frac{1}{2}})^{\ell(y)} \widetilde{T}_y. \tag{5.3}$$

Note that if $J = \emptyset$ then each coset W/W_J is a single element $u \in S_n$ and we have $T'_{uW_J} = T_{uW_J} = \widetilde{T}_u$.

The elements (5.3) are used to construct induced representations as follows. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n, choose any rearrangement $\nu = (\nu_1, \ldots, \nu_\ell)$ of λ and define the subset $J = J(\nu)$ of generators of S_n by

$$J = \{s_1, \ldots, s_{n-1}\} \smallsetminus \{s_{\nu_1}, s_{\nu_1+\nu_2}, \ldots, s_{\nu_1+\cdots+\nu_{\ell-1}}\}. \tag{5.4}$$

Letting $H_n(q)$ act by left multiplication on the $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-spans of coset sums

$$\mathrm{span}_{\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]} \{T_{uW_J} \mid u \in W_-^J\}, \quad \mathrm{span}_{\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]} \{T'_{uW_J} \mid u \in W_-^J\}, \tag{5.5}$$

we obtain the two $H_n(q)$ modules corresponding to induction of the sign and trivial representations (respectively) of Young subalgebras of type λ. For each $w \in S_n$, the matrices representing the two actions of $\widetilde{T}_w$ have entries indexed by permutations $u, v \in W_-^J$, which we describe as follows.

Lemma 5.1. *Fix w in S_n and u, v in W_-^J. For the above constructions of the induced sign and trivial $H_n(q)$ modules, the u, v entries of the matrices representing $\widetilde{T}_w$ are equal to the coefficients of $\widetilde{T}_w$ in $\widetilde{T}_u T_{W_J} \widetilde{T}_{v^{-1}}$ and $\widetilde{T}_u T'_{W_J} \widetilde{T}_{v^{-1}}$ respectively.*

Proof. The two u, v entries are equal to the coefficients of T_{uW_J} in $\widetilde{T}_w T_{vW_J}$ and of T'_{uW_J} in $\widetilde{T}_w T'_{vW_J}$. Equivalently, they are equal to the coefficients of $\widetilde{T}_u$ in $\widetilde{T}_w T_{vW_J}$ and $\widetilde{T}_w T'_{vW_J}$. By the definitions (4.2) and (5.3), these coefficients are

$$\sum_{y \in W_J} (-q^{-\frac{1}{2}})^{\ell(y)} c^u_{w,vy}, \qquad \sum_{y \in W_J} (q^{\frac{1}{2}})^{\ell(y)} c^u_{w,vy}.$$

On the other hand, the coefficients of $\widetilde{T}_w$ in

$$\widetilde{T}_u T_{W_J} \widetilde{T}_{v^{-1}} = \sum_{y \in W_J} (-q^{-\frac{1}{2}})^{\ell(y)} \widetilde{T}_u \widetilde{T}_{y^{-1}v^{-1}}, \qquad \widetilde{T}_u T'_{W_J} \widetilde{T}_{v^{-1}} = \sum_{y \in W_J} (q^{\frac{1}{2}})^{\ell(y)} \widetilde{T}_u \widetilde{T}_{y^{-1}v^{-1}}$$

are equal to

$$\sum_{y \in W_J} (-q^{-\frac{1}{2}})^{\ell(y)} c^w_{u,y^{-1}v^{-1}}, \qquad \sum_{y \in W_J} (q^{\frac{1}{2}})^{\ell(y)} c^w_{u,y^{-1}v^{-1}}.$$

Since $c^w_{u,y^{-1}v^{-1}} = c^{u^{-1}}_{y^{-1}v^{-1},w^{-1}} = c^u_{w,vy}$ by Lemma 4.1, we have the desired result. □

From this fact, we obtain the following Hecke algebra "generating functions" for induced characters.

Lemma 5.2. *Let λ, ν and J be as above. Then we have*

$$\sum_{v \in W^J_-} \widetilde{T}_v T_{W_J} \widetilde{T}_{v^{-1}} = \sum_{w \in S_n} \epsilon^\lambda_q(\widetilde{T}_w) \widetilde{T}_w, \qquad \sum_{v \in W^J_-} \widetilde{T}_v T'_{W_J} \widetilde{T}_{v^{-1}} = \sum_{w \in S_n} \eta^\lambda_q(\widetilde{T}_w) \widetilde{T}_w.$$

Proof. Using the formulae in Lemma 5.1 and summing over all diagonal matrix entries, we obtain the desired equalities. □

These Hecke algebra generating functions in turn are related to quantum immanants by the actions of $H_n(q)$ on $\mathcal{A}_{[n],[n]}(q)$ defined by

$$\widetilde{T}_{s_i} \circ x^{e,v} = x^{s_i,v} = \begin{cases} x^{e,s_iv} & \text{if } s_iv > v, \\ x^{e,s_iv} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) x^{e,v} & \text{if } s_iv < v, \end{cases}$$

$$x^{e,v} \circ \widetilde{T}_{s_i} = x^{v^{-1},e} \circ \widetilde{T}_{s_i} = x^{v^{-1},s_i} = \begin{cases} x^{e,vs_i} & \text{if } vs_i > v, \\ x^{e,vs_i} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) x^{e,v} & \text{if } vs_i < v. \end{cases} \tag{5.6}$$

A straightforward but tedious computation shows that the left and right actions commute. By the definitions, it is easy to see that we have

$$\widetilde{T}_v \circ x^{e,e} = x^{e,e} \circ \widetilde{T}_v = x^{e,v} \tag{5.7}$$

for all $v \in S_n$. On the other hand, we do not in general have the equality of $\widetilde{T}_v \circ x^{e,w}$ and $x^{e,w} \circ \widetilde{T}_v$. One consequence of the definitions (5.6) is the following formula.

Lemma 5.3. *For all $v \in W^J_-$, $y \in W_J$, we have $\widetilde{T}_v \circ x^{e,y} \circ \widetilde{T}_{v^{-1}} = x^{v^{-1},yv^{-1}}$.*

Proof. By (5.2) and (5.6) we have

$$\widetilde{T}_v \circ x^{e,y} \circ \widetilde{T}_{v^{-1}} = x^{e,vy} \circ \widetilde{T}_{v^{-1}} = x^{y^{-1}v^{-1},e} \circ \widetilde{T}_{v^{-1}} = x^{y^{-1}v^{-1},v^{-1}}.$$

By (4.5), this is equal to $x^{v^{-1},yv^{-1}}$. □

Now we quantize the Littlewood-Merris-Watkins identities in Theorem 2.1 as follows.

Theorem 5.4. *Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n and let $\nu = (\nu_1, \ldots, \nu_\ell)$ be any rearrangement of λ. Then we have*

$$\begin{aligned} \mathrm{Imm}_{\epsilon_q^\lambda}(x) &= \sum_{(I_1,\ldots,I_\ell)} \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_\ell,I_\ell}), \\ \mathrm{Imm}_{\eta_q^\lambda}(x) &= \sum_{(I_1,\ldots,I_\ell)} \mathrm{per}_q(x_{I_1,I_1}) \cdots \mathrm{per}_q(x_{I_\ell,I_\ell}), \end{aligned} \tag{5.8}$$

where the sums are over all sequences $(I_1, \ldots, I_\ell)$ of pairwise disjoint subsets of $[n]$ satisfying $|I_j| = \nu_j$.

Proof. Define the set $J = J(\nu)$ of generators as in (5.4). By definition of the induced character immanants and by Lemma 5.2 we have

$$\mathrm{Imm}_{\epsilon_q^\lambda}(x) = \sum_{v \in S_n} \epsilon_q^\lambda(\widetilde{T}_v) \circ x^{e,v} = \sum_{v \in S_n} \epsilon_q^\lambda(\widetilde{T}_v) \widetilde{T}_v \circ x^{e,e} = \sum_{v \in W_-^J} \widetilde{T}_v T_{W_J} \widetilde{T}_{v^{-1}} \circ x^{e,e}.$$

Similarly, $\mathrm{Imm}_{\eta_q^\lambda}(x) = \sum_{v \in W_-^J} \widetilde{T}_v T'_{W_J} \widetilde{T}_{v^{-1}} \circ x^{e,e}$.

On the other hand, we may describe terms in the sums on the right hand side of (5.8) by using the last expression in (5.2) to deduce that all monomials which appear have the form $x^{v^{-1}, yv^{-1}}$ for some $v \in W_-^J$ and $y \in W_J$. In particular, choose $v \in W_-^J$ and define sets $L_1, \ldots, L_\ell$ of indices by $L_1 = \{v_1^{-1}, \ldots, v_{\nu_1}^{-1}\}$ and

$$L_i = \{v^{-1}_{\nu_1 + \cdots + \nu_{i-1}+1}, \ldots, v^{-1}_{\nu_1 + \cdots + \nu_i}\}, \quad i = 2, \ldots, \ell.$$

Then we have

$$\det_q(x_{L_1,L_1}) \cdots \det_q(x_{L_\ell,L_\ell}) = \sum_{y \in W_J} (-q^{-\frac{1}{2}})^{\ell(y)} x^{v^{-1}, yv^{-1}}. \tag{5.9}$$

By (5.7) and Lemma 5.3, this sum is equal to

$$\sum_{y \in W_J} (-q^{-\frac{1}{2}})^{\ell(y)} \widetilde{T}_v \circ x^{e,y} \circ \widetilde{T}_{v^{-1}} = \widetilde{T}_v T_{W_J} \circ x^{e,e} \circ \widetilde{T}_{v^{-1}} = \widetilde{T}_v T_{W_J} \widetilde{T}_{v^{-1}} \circ x^{e,e}.$$

Furthermore, by letting v vary over W_-^J in (5.9), we obtain all possible products of complementary principal quantum minors of sizes $\nu_1 \times \nu_1, \ldots, \nu_\ell \times \nu_\ell$ (in order). Thus we have

$$\sum_{(I_1,\ldots,I_\ell)} \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_\ell,I_\ell}) = \sum_{v \in W_-^J} \widetilde{T}_v T_{W_J} \widetilde{T}_{v^{-1}} \circ x^{e,e} = \mathrm{Imm}_{\epsilon_q^\lambda}(x).$$

Similarly, the identity $\mathrm{per}_q(x_{L_1,L_1}) \cdots \mathrm{per}_q(x_{L_\ell,L_\ell}) = \widetilde{T}_v T'_{W_J} \widetilde{T}_{v^{-1}} \circ x^{e,e}$ leads to the claimed formula for $\mathrm{Imm}_{\eta_q^\lambda}(x)$. □

We caution the reader that the quantum identities in Theorem 5.4 do not specialize as simply via generalized submatrices as do their classical analogs. For instance, we may use the following table of characters (in which the (w, χ_q) entry gives $\chi_q(\widetilde{T}_w)$)

$w\backslash\chi_q$	$\chi_q^{111} = \epsilon_q^3$	χ_q^{21}	$\chi_q^3 = \eta_q^3$	$\eta_q^{111} = \epsilon_q^{111}$	η_q^{21}	ϵ_q^{21}
123	1	2	1	6	3	3
132	$-q^{-\frac{1}{2}}$	$q^{\frac{1}{2}} - q^{-\frac{1}{2}}$	$q^{\frac{1}{2}}$	$3(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$	$2q^{\frac{1}{2}} - q^{-\frac{1}{2}}$	$q^{\frac{1}{2}} - 2q^{-\frac{1}{2}}$
213	$-q^{-\frac{1}{2}}$	$q^{\frac{1}{2}} - q^{-\frac{1}{2}}$	$q^{\frac{1}{2}}$	$3(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$	$2q^{\frac{1}{2}} - q^{-\frac{1}{2}}$	$q^{\frac{1}{2}} - 2q^{-\frac{1}{2}}$
231	q^{-1}	-1	q	$q - 2 + q^{-1}$	$q - 1$	$q^{-1} - 1$
312	q^{-1}	-1	q	$q - 2 + q^{-1}$	$q - 1$	$q^{-1} - 1$
321	$-q^{-\frac{3}{2}}$	0	$q^{\frac{3}{2}}$	$q^{\frac{3}{2}} - q^{\frac{-3}{2}}$	$q^{\frac{3}{2}}$	$-q^{-\frac{3}{2}}$

to compute

$$\begin{aligned}\mathrm{Imm}_{\epsilon_q^{21}}(x_{112,112}) &= (3 + q^{\frac{1}{2}} - 2q^{-\frac{1}{2}})x_{1,1}^2x_{2,2} + (q^{\frac{1}{2}} - 2q^{-\frac{1}{2}} + q^{-1} - 1)x_{1,1}x_{1,2}x_{2,1} \\ &\quad + (q^{-1} - 1 - q^{-\frac{3}{2}})x_{1,2}x_{1,1}x_{2,1} \\ &= (3 + q^{\frac{1}{2}} - 2q^{-\frac{1}{2}})x_{1,1}^2x_{2,2} + (-q^{-\frac{1}{2}} - 1)x_{1,1}x_{1,2}x_{2,1}.\end{aligned}$$

This is not equal to the specialization of the first sum in (5.8) at $x = x_{112,112}$,

$$\det_q(x_{11,11})x_{2,2} + 2\det_q(x_{12,12})x_{1,1} = (3 - q^{-\frac{1}{2}})x_{1,1}^2x_{2,2} - 2q^{-\frac{1}{2}}x_{1,1}x_{1,2}x_{2,1}.$$

The authors do not know quantum analogs of Propositions 2.2-2.3 or Corollary 2.4. Nevertheless, we will succeed in quantizing a special case of the Goulden-Jackson identities in Theorem 3.2 by applying Theorem 5.4.

Just as inverse Kostka numbers describe the expansions of induced sign and trivial characters of S_n in terms of irreducible S_n characters, these numbers also describe the expansions of induced sign and trivial characters of $H_n(q)$ in terms of irreducible $H_n(q)$ characters. (No "quantum analog" of inverse Kostka numbers is needed for this purpose. See [12, §9.1.9].) Specifically we have

$$\chi_q^\lambda = \sum_\mu K_{\mu,\lambda}^{-1}\eta_q^\mu = \sum_\mu K_{\mu,\lambda^\top}^{-1}\epsilon_q^\mu. \tag{5.10}$$

Now let us quantize the Goulden-Jackson generating functions (3.1). We define the sequences $(\alpha_k)_{k\in\mathbb{Z}}$, $(\beta_k)_{k\in\mathbb{Z}}$, $(\gamma_k)_{k\in\mathbb{Z}}$, $(\delta_k)_{k\in\mathbb{Z}}$ of polynomials in $\mathcal{A}(q)$ by the generating functions

$$\begin{aligned}\det_q(I + tx) &= \sum_{k=0}^n \alpha_k t^k, \qquad \frac{1}{\det_q(I - tx)} = \sum_{k\geq 0}\beta_k t^k, \\ \mathrm{per}_q(I + tx) &= \sum_{k=0}^n \gamma_k t^k, \qquad \frac{1}{\mathrm{per}_q(I - tx)} = \sum_{k\geq 0}\delta_k t^k,\end{aligned} \tag{5.11}$$

in $\mathcal{A}(q)[[t]]$, and again by the requirement that polynomials with indices not appearing here be zero. In terms of these sequences, define the $\lambda_1 \times \lambda_1$ matrices A, D and the $\ell \times \ell$ matrices B, C as before,

$$A = (\alpha_{\lambda^{\top}_i - i+j}), \qquad B = (\beta_{\lambda_i - i+j}), \qquad C = (\gamma_{\lambda_i - i+j}), \qquad D = (\delta_{\lambda^{\top}_i - i+j}). \tag{5.12}$$

To provide a quantum analog of Theorems 3.2-3.3, we must evaluate some form of the determinant at the matrices defined in (5.12). Perhaps surprisingly, we will use the classical (commutative) determinant and will justify this by stating the following commutative properties of the sequences defined in (5.11).

Theorem 5.5. *The polynomials $\{\alpha_k \,|\, k \in \mathbb{Z}\}$ in $\mathcal{A}(q)$ pairwise commute, as do the polynomials $\{\beta_k \,|\, k \in \mathbb{Z}\}$.*

Proof. The pairwise commutation of the polynomials $\{\alpha_k \,|\, k \in \mathbb{Z}\}$ is due to Domokos and Lenagan [4, Thm. 6.1]. Using (5.11) to write $\beta_k = \beta_{k-1}\alpha_1 - \cdots \pm \beta_1\alpha_{k-1} \mp \alpha_k$, we see that the polynomials $\{\beta_k \,|\, k \in \mathbb{Z}\}$ pairwise commute as well. □

While it is not in general true that the polynomials $\{\gamma_k \,|\, k \in \mathbb{Z}\}$ or $\{\delta_k \,|\, k \in \mathbb{Z}\}$ in $\mathcal{A}(q)$ pairwise commute, we do have the following weaker result. Let $\mathcal{R}(q)$ be the quotient of $\mathcal{A}(q)$ modulo the ideal generated by all monomials of the forms $x_{i,k}x_{j,k}$ and $x_{i,k}x_{i,\ell}$.

Theorem 5.6. *In $\mathcal{R}(q)$, the canonical images of the polynomials $\{\gamma_k \,|\, k \in \mathbb{Z}\}$ pairwise commute, as do the canonical images of the polynomials $\{\delta_k \,|\, k \in \mathbb{Z}\}$.*

Proof. It is easy to see that we have

$$\gamma_i = \sum_{\substack{I \subset [n] \\ |I| = i}} \mathrm{per}_q(x_{I,I}). \tag{5.13}$$

Choose indices $i < j$, define the partition $\lambda = (j, i)$, and consider the product $\gamma_j\gamma_i$ in $\mathcal{R}(q)$. Each term of the form $\mathrm{per}_q(x_{J,J})\mathrm{per}_q(x_{I,I})$ vanishes in $\mathcal{R}(q)$ unless I and J are disjoint. Collecting terms of the form $\mathrm{per}_q(x_{J,J})\mathrm{per}_q(x_{I,I})$ for each $(i+j)$-element subset K of $[n]$, we have

$$\gamma_j\gamma_i = \sum_K \sum_{(J,I)} \mathrm{per}_q(x_{J,J})\mathrm{per}_q(x_{I,I}),$$

where K varies over all $(i+j)$-element subsets of $[n]$ and (J, I) varies over all pairs of disjoint subsets of $[n]$ satisfying $|J| = j$, $|I| = i$, $I \cup J = K$. On the other hand, for each subset K of $[n]$, we have by Theorem 5.4 that the induced character immanant $\mathrm{Imm}_{\epsilon_q^\lambda}(x_{K,K})$ is equal to this second sum, and also to the similar sum over pairs (I, J) satisfying $|I| = i$, $|J| = j$. It follows that $\gamma_j\gamma_i = \gamma_i\gamma_j$ in $\mathcal{R}(q)$.

As in the previous proof, we may write each polynomial δ_j as a sum of products of the polynomials $\gamma_1, \dots, \gamma_j$ to see that we have $\delta_i \delta_j = \delta_j \delta_i$ in $\mathcal{R}(q)$. □

Now we compute the determinants of A, B, C, D in $\mathcal{R}(q)$. For the matrices A, B, whose classical determinants are well-defined even in $\mathcal{A}(q)$, this is equivalent to computing the determinants in $\mathcal{A}(q)$ and projecting them onto the immanant space.

Theorem 5.7. *In* $\mathcal{R}(q)$*, the nonquantum determinants* $\det(A)$*,* $\det(B)$*,* $\det(C)$*, and* $\det(D)$ *are all equal to* $\mathrm{Imm}_{\chi_q^\lambda}(x)$*.*

Proof. By Proposition 3.1, we have $\det(A) = \det(B)$. Computing in $\mathcal{R}(q)$, we have

$$\det(A) = \det(B) = \sum_{\substack{\mu \geq \lambda^\top \\ \mu=(\mu_1,\dots,\mu_m)}} K^{-1}_{\mu,\lambda^\top} \sum_{(I_1,\dots,I_m)} \det{}_q(x_{I_1,I_1}) \cdots \det{}_q(x_{I_m,I_m}), \tag{5.14}$$

where the second sum is over all sequences $(I_1, \dots, I_m)$ of pairwise disjoint subsets of $[n]$ satisfying $|I_j| = \mu_j$. By Theorem 5.4, this is equal to

$$\sum_{\mu \geq \lambda^\top} K^{-1}_{\mu,\lambda^\top} \mathrm{Imm}_{\epsilon_q^\mu}(x) = \mathrm{Imm}_{\chi_q^\lambda}(x).$$

Similarly, by Proposition 3.1 we have $\det(C) = \det(D)$ in $\mathcal{R}(q)$ and thus

$$\det(C) = \det(D) = \sum_{\substack{\mu \geq \lambda \\ \mu=(\mu_1,\dots,\mu_m)}} K^{-1}_{\mu,\lambda} \sum_{(I_1,\dots,I_m)} \mathrm{per}_q(x_{I_1,I_1}) \cdots \mathrm{per}_q(x_{I_m,I_m}),$$

where the second sum is over all sequences of pairwise disjoint subsets of $[n]$ satisfying $|I_j| = \mu_j$. By Theorem 5.4, this is

$$\sum_{\mu \geq \lambda} K^{-1}_{\mu,\lambda} \mathrm{Imm}_{\eta_q^\mu}(x) = \mathrm{Imm}_{\chi_q^\lambda}(x).$$

□

When quantizing an identity, one often replaces nonnegative integers and the ordinary factorial function with appropriate quantum analogs. We will do this as follows. For each nonnegative integer k, define the *quantum analog of* k to be the expression

$$(k)_q = \begin{cases} 0 & \text{if } k = 0, \\ 1 + q^{\frac{1}{2}} + q^{\frac{2}{2}} + \cdots + q^{\frac{k-1}{2}} & \text{otherwise.} \end{cases}$$

Define the *quantum factorial function* ! by

$$(k)_q! = \begin{cases} 1 & \text{if } k = 0, \\ 1 \cdot (2)_q \cdots (k)_q & \text{otherwise.} \end{cases}$$

Note that the evaluation of the above expressions at $q^{\frac{1}{2}} = 1$ gives the classical non-negative integers and factorial function.

The most natural quantization of Theorem 3.2 would assert the equality of $\det(A)$, $\det(B)$ and

$$\sum_M \frac{\mathrm{Imm}_{\chi_q^\lambda}(x_{M,M})}{(m_1)_q! \cdots (m_n)_q!}. \tag{5.15}$$

Unfortunately, this equality does not hold in general. For instance, when $\lambda = 21$, we have

$$\det \begin{bmatrix} \alpha_2 & \alpha_3 \\ 1 & \alpha_1 \end{bmatrix} = (\det_q(x_{12,12}) + \det_q(x_{13,13}) + \det_q(x_{23,23}))(x_{1,1} + x_{2,2} + x_{3,3}) - \det_q(x)$$
$$= \mathrm{Imm}_{\chi_q^{21}}(x) + (x_{1,1}^2 x_{2,2} - q^{-\frac{1}{2}} x_{1,1}x_{1,2}x_{2,1}) + \cdots + (x_{2,2}x_{3,3}^2 - q^{-\frac{1}{2}} x_{2,3}x_{3,2}x_{3,3}).$$

Computing $\mathrm{Imm}_{\chi_q^{21}}(x_{112,112})$, $\mathrm{Imm}_{\chi_q^{111}}(x_{112,112}) = \det_q(x_{112,112})$, etc., we see that this is equal to

$$\mathrm{Imm}_{\chi_q^{21}}(x) + \frac{\mathrm{Imm}_{\chi_q^{21}}(x_{112,112}) - \det_q(x_{112,112})}{2_q!1_q!} + \cdots + \frac{\mathrm{Imm}_{\chi_q^{21}}(x_{233,233}) - \det_q(x_{233,233})}{2_q!1_q!},$$

which differs from the formula (5.15).

Problem 5.8. *Fix a partition λ and define a matrix A as in (5.12). Express* $\det(A)$ *as a linear combination of quantum (irreducible) character immanants of generalized submatrices of A.*

On the other hand, the special case of Problem 5.8 corresponding to the partition $\lambda = 1^r$ does lead to the identity one would expect, and is equivalent to a quantization of the MacMahon Master Theorem. This quantized Master Theorem was first stated by Garoufalidis-Lê-Zeilberger [11, Thm. 1] in a slightly more general form, using "right quantum" variables which satisfy only three of the four relations in (4.3). In the spirit of Vere-Jones, we state the quantized Master Theorem as follows.

Theorem 5.9. *Let $z = (z_1, \dots, z_n)$ be a vector of quasicommuting variables satisfying $z_j z_i = q^{\frac{1}{2}} z_i z_j$ if $i < j$, and commuting with x. Let $M = 1^{m_1} \cdots n^{m_n}$ be an r-element multiset of $[n]$. Then the coefficient of $z_1^{m_1} \cdots z_n^{m_n}$ in*

$$\Big(\sum_{j=1}^n x_{1,j} z_j\Big)^{m_1} \cdots \Big(\sum_{j=1}^n x_{n,j} z_j\Big)^{m_n} \tag{5.16}$$

and the projection of

$$\frac{1}{\det_q(I - x)} \tag{5.17}$$

onto $\mathcal{A}_{M,M}(q)$ are both equal to

$$\frac{\mathrm{per}_q(x_{M,M})}{(m_1)_q! \cdots (m_n)_q!}.$$

Proof. The equality of the coefficient of $z_1^{m_1} \cdots z_n^{m_n}$ in (5.16) and the projection of (5.17) onto $\mathcal{A}_{M,M}(q)$ is due to Garoufalidis-Lê-Zeilberger [11, Thm. 1].

Let $S(M)$ be the set of rearrangements of M, and for each word $w = w_1 \cdots w_r$ in $S(M)$ define $\ell(w)$ to be the number of pairs (i, j) of indices satisfying $i < j$ and $w_i > w_j$. Now define the $r \times n$ matrix

$$F = \mathrm{diag}(\underbrace{z_1, \ldots, z_1}_{m_1}, \underbrace{z_2, \ldots, z_2}_{m_2}, \ldots, \underbrace{z_n, \ldots, z_n}_{m_n}) x_{M,[n]},$$

and expand the sum

$$\sum_{w \in S(M)} (q^{\frac{1}{2}})^{\ell(w)} F^{e,w}.$$

Using the relations $z_j z_i = q^{\frac{1}{2}} z_i z_j$ to express terms of this expansion as lexicographically ordered monomials, we obtain the coefficient of $z_1^{m_1} \cdots z_n^{m_n}$ in (5.16). But this is equal to

$$\frac{1}{(m_1)_q! \cdots (m_n)_q!} \sum_{w \in S_r} (q^{\frac{1}{2}})^{\ell(w)} (x_{M,M})^{e,w} = \frac{\mathrm{per}_q(x_{M,M})}{(m_1)_q! \cdots (m_n)_q!}.$$

□

As a consequence, we obtain the following natural quantum analog of (3.6).

Corollary 5.10. *The functions $(\beta_r)_{r \in \mathbb{Z}}$ defined in (5.11) satisfy*

$$\beta_r = \sum_M \frac{\mathrm{per}_q(x_{M,M})}{(m_1)_q! \cdots (m_n)_q!}, \tag{5.18}$$

where the sum is over all r-element multisets $M = 1^{m_1} \cdots n^{m_n}$ of $[n]$.

As another consequence, we obtain the following quantization of the generalized Muir identity in Corollary 3.5.

Corollary 5.11. *Let $M = 1^{m_1} \cdots n^{m_n}$ be an r-element multiset of $[n]$, $r > 0$. Then we have*

$$\sum_{k=0}^{\min\{r,n\}} (-1)^k \sum_{\substack{I \subset M \\ |I| = k}} \frac{\det_q(x_{I,I}) \mathrm{per}_q(x_{M \smallsetminus I, M \smallsetminus I})}{(m_1 - i_1)_q! \cdots (m_n - i_n)_q!} = 0,$$

where the second sum is over sets $I = 1^{i_1} \cdots n^{i_n}$ contained in M.

Proof. As before, we deduce from (5.11) that

$$\sum_{k=0}^{\min\{r,n\}} (-1)^k \alpha_k \beta_{r-k} = 0 \tag{5.19}$$

for $r \geq 1$, and it is easy to see that we have

$$\alpha_k = \sum_{\substack{I \subset [n] \\ |I|=k}} \det_q(x_{I,I})$$

for $k = 0, \ldots, n$. Now using (5.18) and projecting both sides of (5.19) onto $\mathcal{A}_{M,M}(q)$, we have the desired result. □

In the special case $M = [n]$, we obtain Zhang's quantization [33, Thm. 3.2] of Muir's identity,

$$\sum_{k=0}^{n} (-1)^k \sum_{\substack{I \subset [n] \\ |I|=k}} \det_q(x_{I,I}) \mathrm{per}_q(x_{[n] \smallsetminus I, [n] \smallsetminus I}) = 0.$$

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