MULTICOMPLEXES AND POLYNOMIALS WITH REAL ZEROS

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Outline

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(2) Hilbert Series
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(4) Polynomials with real zeros
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Multicomplexes

A collection Γ of monomials closed under divisibility is called a *multicomplex*.

Example:

$$\begin{array}{lll} x_1^3x_2, & & \\ x_1^3, & x_1^2x_2, & \\ x_1^2, & x_1x_2, & x_2x_3, \\ x_1, & x_2, & x_3, \\ 1. & & \end{array}$$

Counting monomials by degree, we define the *f-vector* and *f-polynomial*.

Example:

$$f_\Gamma = (1, 3, 3, 2, 1),$$

$$f_\Gamma(z) = 1 + 3z + 3z^2 + 2z^3 + z^4.$$

Question 1: Which polynomials in $\mathbb{N}[z]$ are f-polynomials of multicomplexes?

Let I be a monomial ideal. Then the set of monomials in the ring $A = k[x_1, \dots, x_n]/I$ is a multicomplex.

Example: In $A = k[x_1, x_2, x_3]/\langle x_2 x_3 \rangle$, we have

$1,$

$x_1, \quad x_2, \quad x_3,$

$x_1^2, \quad x_2^2, \quad x_3^2, \quad x_1x_2, \quad x_1x_3,$

$x_1^3, \quad x_2^3, \quad x_3^3, \quad x_1^2x_2, \quad x_1x_2^2, \quad x_1^2x_3, \quad x_1x_3^2,$

$\dots$

The *Hilbert function* and *Hilbert series* of the ring count monomials by degree.

Example:

$$F_A = (1, 3, 5, 7, \dots),$$

$$F_A(z) = 1 + 3z + 5z^2 + 7z^3 + \cdots .$$

The Hilbert series of such a ring may be expressed as a rational function

$$F_A(z) = \frac{h_A(z)}{(1-z)^d}.$$

Example:

$$1 + 3z + 5z^2 + 7z^3 + \cdots = \frac{1+z}{(1-z)^2}.$$

Question 1': Which polynomials in $\mathbb{N}[z]$ can appear in the numerator of a rational expression for the Hilbert series of a Cohen-Macaulay ring?

Answer: The f-polynomials of finite multicomplexes.

Macaulay functions

Theorem: (Macaulay, 1927) The vector $(1, a_1, \ldots, a_d)$ is the f-vector of a multicomplex if and only if

$$a_{i+1} \leq \mu_i(a_i), \quad i = 1, \ldots, d-1,$$

where μ_i is the ith *Macaulay function*.

Example:

0	1	2	3	4	5	6
1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

The 3rd *Macaulay expansion* of 8 is

$$8 = 4 + 3 + 1,$$

and we have

$$\mu_3(8) = 5 + 4 + 1 = 10.$$

Example:

1	6	9	7	2
	1	11	111	1111
	2	21	211	2111
	3	22	221	
	4	31	222	
	5	32	311	
	6	33	321	
		41	322	
		42		
		43		

To construct a multicomplex from an f-vector, write down the (lexicographically) first a_i weakly decreasing i-letter words using the letters $\mathbb{N}$.

Polynomials with real zeros

Question 2: How can we tell if the polynomial $a(z) = 1 + a_1 z + \cdots + a_d z^d$ in $\mathbb{N}[z]$ has only real zeros?

Answer: Use

(1) Maple.
(2) Sturm's Algorithm.
(3) Aissen, Schoenberg, Whitney's Theorem.
(4) Gantmacher's Theorem.
(5) Theorems about $(\mathbf{3}+\mathbf{1})$-free posets.
(6) Theorems about eigenvalues.

Question 2': How can we tell if every polynomial in an infinite subset of $\mathbb{N}[z]$ has only real zeros?

Facts, problems

The f-polynomials of the following combinatorial objects have only real zeros.

(1) $(\mathbf{3}+\mathbf{1})$-free posets.
(2) Matching complexes.

Question: Do the f-polynomials of these combinatorial objects have only real zeros?

(1) Distributive Lattices.
(2) Modular Lattices.

Question: Is there some setting in which *all* polynomials in $\mathbb{N}[z]$ having real zeros arise?

Maclaurin's inequalities

Proposition: Let $1 + a_1 z + \cdots + a_d z^d$ in $\mathbb{N}[z]$ have only real zeros. Then we have

$$\frac{a_1}{d} \geq \sqrt{\frac{a_2}{\binom{d}{2}}} \geq \sqrt[3]{\frac{a_3}{\binom{d}{3}}} \geq \cdots \geq \sqrt[d]{a_d} \geq 1.$$

Corollary: Factoring the polynomial as $a(z) = (1 + \beta_1 z) \cdots (1 + \beta_d z)$, we obtain the Arithmetic Mean - Geometric Mean Inequality,

$$\frac{\beta_1 + \cdots + \beta_d}{d} \geq \sqrt[d]{\beta_1 \cdots \beta_d}.$$

Corollary: For all i we have
$$a_i \geq \binom{d}{i}.$$

Example: $1 + 4z + 5z^2 + 4z^3 + z^4$ has (at least) a pair of imaginary zeros.

Corollary: For all i we have

$$a_{i+1} \leq \binom{d}{i+1} \left(\frac{a_i}{\binom{d}{i}} \right)^{(i+1)/i} .$$

Using Maclaurin's inequalities and a technical lemma, we have the following.

Proposition: (Bell-S 2002) Let the polynomial $a(z) = 1 + a_1 z + \cdots + a_d z^d$ in $\mathbb{N}[z]$ have only real zeros. Then we have

$$a_{i+1} \leq \mu_i(a_i),$$

for $i = 1, \ldots, d-1$.

Equivalently, $a(z)$ is the f-polynomial of a multicomplex.

Equivalently, for every nonnegative integer c, there exists a Cohen-Macaulay ring with Hilbert series

$$\frac{a(z)}{(1-z)^c}.$$

Question: Which Cohen-Macaulay rings correspond to polynomials with real zeros?

Question: Which multicomplexes correspond to polynomials with real zeros? Can these be chosen to be simplicial complexes?

Kruskal-Katona functions

Theorem: (Schutzenberger, Kruskal, Katona) The vector $(1, a_1, \ldots, a_d)$ is the f-vector of a simplicial complex if and only if

$$a_{i+1} \leq \kappa_i(a_i), \quad i = 1, \ldots, d-1,$$

where κ_i is the ith *Kruskal-Katona function.*

Example:

0	1	2	3	4	5	6
1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

The 3rd *Macaulay expansion* of 8 is

$$8 = 4 + 3 + 1,$$

and we have

$$\kappa_3(8) = 0 + 1 + 1 = 2.$$

Observation: As a_i gets large, the function $\kappa_i(a_i)$ approaches

$$\frac{(i!a_i)^{(i+1)/i}}{(i+1)!},$$

which is greater than the upper bound

$$\binom{d}{i+1}\left(\frac{a_i}{\binom{d}{i}}\right)^{(i+1)/i}$$

implied by the Maclaurin inequalities for polynomials with real zeros.

Corollary: (of Maclaurin's inequalities) If $a(z) = 1 + a_1 z + \cdots + a_d z^d$ in $\mathbb{N}[z]$ has only real zeros, then

$$a_i \leq \binom{d}{i} \left(\frac{a_1}{d}\right)^i .$$

Setting $i = d$, we see that the degree d is no greater than a_1.

Thus for fixed a_1, there are only finitely many polynomials in $\mathbb{N}[z]$ of the above form which have only real zeros.

Example: For $a_1 = 5$ we have

$$
\begin{aligned}
&1 + 5z \\
&1 + 5z + z^2 \\
&1 + 5z + 2z^2 \\
&1 + 5z + 3z^2 \\
&1 + 5z + 4z^2 \\
&1 + 5z + 5z^2 \\
&1 + 5z + 6z^2 \\
&1 + 5z + 5z^2 + z^3 \\
&1 + 5z + 6z^2 + z^3 \\
&1 + 5z + 6z^2 + 2z^3 \\
&1 + 5z + 7z^2 + 2z^3 \\
&1 + 5z + 7z^2 + 3z^3 \\
&1 + 5z + 8z^2 + 4z^3 \\
&1 + 5z + 8z^2 + 5z^3 + z^4 \\
&1 + 5z + 9z^2 + 7z^3 + 2z^4 \\
&1 + 5z + 10z^2 + 10z^3 + 5z^4 + z^5.
\end{aligned}
$$

Partial results

If $a(z) = 1 + a_1 z + \cdots + a_d z^d$ in $\mathbb{N}[z]$ has only real zeros, then it is the f-vector of a simplicial complex if

(1) the coefficients are large.
(2) the coefficients are small ($a_1 \leq 10$).
(3) the degree is small ($d \leq 4$).
(4) $a(z) = (1+\beta_1 z)\cdots(1+\beta_d z)$, and $\beta_i \geq 1$ for all i.

Furthermore, $a_{i+1} \leq \kappa_i(a_i)$ for $i = 1, \ldots, \frac{2d}{3}$.