MULTICOMPLEXES AND POLYNOMIALS WITH REAL ZEROS

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Outline

(1) Multicomplexes
(2) Hilbert Series
(3) Macaulay functions
(4) Polynomials with real zeros
(5) Maclaurin inequalities
(6) Results and open questions
Multicomplexes

A collection $\Gamma$ of monomials closed under divisibility is called a \textit{multicomplex}.

\textbf{Example:}

\begin{align*}
x_1^3 x_2, \\
x_1^3, \quad x_1^2 x_2, \\
x_1^2, \quad x_1 x_2, \quad x_2 x_3, \\
x_1, \quad x_2, \quad x_3, \\
1.
\end{align*}

Counting monomials by degree, we define the \textit{f-vector} and \textit{f-polynomial}.

\textbf{Example:}

\begin{align*}
f_\Gamma &= (1, 3, 3, 2, 1), \\
f_\Gamma(z) &= 1 + 3z + 3z^2 + 2z^3 + z^4.
\end{align*}
Question 1: Which polynomials in $\mathbb{N}[z]$ are $f$-polynomials of multicomplexes?
Let $I$ be a monomial ideal. Then the set of monomials in the ring $A = k[x_1, \ldots, x_n]/I$ is a multicomplex.

**Example:** In $A = k[x_1, x_2, x_3]/\langle x_2x_3 \rangle$, we have

1, 
$x_1, \ x_2, \ x_3,$
$x_1^2, \ x_2^2, \ x_3^2, \ x_1x_2, \ x_1x_3,$
$x_1^3, \ x_2^3, \ x_3^3, \ x_1^2x_2, \ x_1x_2^2, \ x_1^2x_3, \ x_1x_3^2,$
$\ldots$

The *Hilbert function* and *Hilbert series* of the ring count monomials by degree.

**Example:**

$F_A = (1, 3, 5, 7, \ldots),$  
$F_A(z) = 1 + 3z + 5z^2 + 7z^3 + \cdots.$
The Hilbert series of such a ring may be expressed as a rational function

\[ F_A(z) = \frac{h_A(z)}{(1 - z)^d}. \]

**Example:**

\[ 1 + 3z + 5z^2 + 7z^3 + \cdots = \frac{1 + z}{(1 - z)^2}. \]

**Question 1’:** Which polynomials in \( \mathbb{N}[z] \) can appear in the numerator of a rational expression for the Hilbert series of a Cohen-Macaulay ring?

**Answer:** The \( f \)-polynomials of finite multicomplexes.
Macaulay functions

**Theorem:** (Macaulay, 1927) The vector \((1, a_1, \ldots, a_d)\) is the \(f\)-vector of a multicomplex if and only if
\[ a_{i+1} \leq \mu_i(a_i), \quad i = 1, \ldots, d - 1, \]
where \(\mu_i\) is the \(i\)th Macaulay function.
Example:

The 3rd Macaulay expansion of 8 is
\[ 8 = 4 + 3 + 1, \]
and we have
\[ \mu_3(8) = 5 + 4 + 1 = 10. \]
Example:

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To construct a multicomplex from an $f$-vector, write down the (lexicographically) first $a_i$ weakly decreasing $i$-letter words using the letters $\mathbb{N}$. 
Polynomials with real zeros

**Question 2:** How can we tell if the polynomial \( a(z) = 1 + a_1 z + \cdots + a_d z^d \) in \( \mathbb{N}[z] \) has only real zeros?

**Answer:** Use

(1) Maple.
(2) Sturm’s Algorithm.
(3) Aissen, Schoenberg, Whitney’s Theorem.
(4) Gantmacher’s Theorem.
(5) Theorems about \((3 + 1)\)-free posets.
(6) Theorems about eigenvalues.

**Question 2’:** How can we tell if every polynomial in an infinite subset of \( \mathbb{N}[z] \) has only real zeros?
Facts, problems

The \( f \)-polynomials of the following combinatorial objects have only real zeros.

(1) \((3 + 1)\)-free posets.
(2) Matching complexes.

**Question:** Do the \( f \)-polynomials of these combinatorial objects have only real zeros?

(1) Distributive Lattices.
(2) Modular Lattices.

**Question:** Is there some setting in which \textit{all} polynomials in \( \mathbb{N}[z] \) having real zeros arise?
Maclaurin’s inequalities

**Proposition:** Let $1 + a_1 z + \cdots + a_d z^d$ in $\mathbb{N}[z]$ have only real zeros. Then we have

$$\frac{a_1}{d} \geq \sqrt[2]{\frac{a_2}{d(2)}} \geq \sqrt[3]{\frac{a_3}{d(3)}} \geq \cdots \geq \sqrt[d]{a_d} \geq 1.$$
Corollary: Factoring the polynomial as 
\[ a(z) = (1 + \beta_1 z) \cdots (1 + \beta_d z), \]
we obtain the Arithmetic Mean - Geometric Mean Inequality,
\[ \frac{\beta_1 + \cdots + \beta_d}{d} \geq \sqrt[d]{\beta_1 \cdots \beta_d}. \]
Corollary: For all $i$ we have

$$a_i \geq \binom{d}{i}.$$

Example: $1 + 4z + 5z^2 + 4z^3 + z^4$ has (at least) a pair of imaginary zeros.
Corollary: For all $i$ we have

$$a_{i+1} \leq \binom{d}{i+1} \left( \frac{a_i}{\binom{d}{i}} \right)^{(i+1)/i}.$$


Using Maclaurin’s inequalities and a technical lemma, we have the following.

**Proposition:** (Bell-S 2002) Let the polynomial $a(z) = 1 + a_1z + \cdots + a_dz^d$ in $\mathbb{N}[z]$ have only real zeros. Then we have

$$a_{i+1} \leq \mu_i(a_i),$$

for $i = 1, \ldots, d - 1$.

Equivalently, $a(z)$ is the $f$-polynomial of a multicomplex.

Equivalently, for every nonnegative integer $c$, there exists a Cohen-Macaulay ring with Hilbert series

$$\frac{a(z)}{(1 - z)^c}.$$
Question: Which Cohen-Macaulay rings correspond to polynomials with real zeros?

Question: Which multicomplexes correspond to polynomials with real zeros? Can these be chosen to be simplicial complexes?
Kruskal-Katona functions

**Theorem:** (Schutzenberger, Kruskal, Katona) The vector $(1, a_1, \ldots, a_d)$ is the $f$-vector of a simplicial complex if and only if

$$a_{i+1} \leq \kappa_i(a_i), \quad i = 1, \ldots, d - 1,$$

where $\kappa_i$ is the $i$th Kruskal-Katona function.
Example:

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

The 3rd Macaulay expansion of 8 is

\[8 = 4 + 3 + 1,\]

and we have

\[\kappa_3(8) = 0 + 1 + 1 = 2.\]
Observation: As $a_i$ gets large, the function $\kappa_i(a_i)$ approaches
\[
\frac{(i!a_i)^{(i+1)/i}}{(i + 1)!},
\]
which is greater than the upper bound
\[
\binom{d}{i + 1} \left( \frac{a_i}{\binom{d}{i}} \right)^{(i+1)/i}
\]
implied by the Maclaurin inequalities for polynomials with real zeros.
Corollary: (of Maclaurin’s inequalities)
If $a(z) = 1 + a_1 z + \cdots + a_d z^d$ in $\mathbb{N}[z]$ has only real zeros, then
$$a_i \leq \binom{d}{i} \left( \frac{a_1}{d} \right)^i.$$  

Setting $i = d$, we see that the degree $d$ is no greater than $a_1$.

Thus for fixed $a_1$, there are only finitely many polynomials in $\mathbb{N}[z]$ of the above form which have only real zeros.
Example: For $a_1 = 5$ we have

\[
\begin{align*}
1 + 5z \\
1 + 5z + z^2 \\
1 + 5z + 2z^2 \\
1 + 5z + 3z^2 \\
1 + 5z + 4z^2 \\
1 + 5z + 5z^2 \\
1 + 5z + 6z^2 \\
1 + 5z + 5z^2 + z^3 \\
1 + 5z + 6z^2 + z^3 \\
1 + 5z + 6z^2 + 2z^3 \\
1 + 5z + 7z^2 + 2z^3 \\
1 + 5z + 7z^2 + 3z^3 \\
1 + 5z + 8z^2 + 4z^3 \\
1 + 5z + 8z^2 + 5z^3 + z^4 \\
1 + 5z + 9z^2 + 7z^3 + 2z^4 \\
1 + 5z + 10z^2 + 10z^3 + 5z^4 + z^5.
\end{align*}
\]
Partial results

If \( a(z) = 1 + a_1 z + \cdots + a_d z^d \) in \( \mathbb{N}[z] \) has only real zeros, then it is the \( f \)-vector of a simplicial complex if

1. the coefficients are large.
2. the coefficients are small \( (a_1 \leq 10) \).
3. the degree is small \( (d \leq 4) \).
4. \( a(z) = (1 + \beta_1 z) \cdots (1 + \beta_d z) \), and \( \beta_i \geq 1 \) for all \( i \).

Furthermore, \( a_{i+1} \leq \kappa_i(a_i) \) for \( i = 1, \ldots, \frac{2d}{3} \).