MULTICOMPLEXES AND POLYNOMIALS WITH REAL ZEROS.

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Abstract. We show that each polynomial $a(z) = 1 + a_1z + \cdots + a_dz^d$ in $\mathbb{N}[z]$ having only real zeros is the f-polynomial of a multicomplex. It follows that $a(z)$ is also the h-polynomial of a Cohen-Macaulay ring and is the g-polynomial of a simplicial polytope. We conjecture that $a(z)$ is also the f-polynomial of a simplicial complex and show that the multicomplex result implies this in the special case that the zeros of $a(z)$ belong to the real interval $[-1, 0)$. We also show that for fixed d the conjecture can fail for at most finitely many polynomials having the required form.

1. Introduction

Several results in algebraic combinatorics concern simplicial complexes and polynomials

$$a(z) = 1 + a_1z + \cdots + a_dz^d \in \mathbb{N}[z] \tag{1.1}$$

having only real zeros. For example, the following results state combinatorial properties of a simplicial complex which are sufficient to prove that all zeros of a related polynomial of the form (1.1) are real.

1. (1) The f-polynomial of a matching complex has only real zeros [12].
2. (2) If a simplicial complex has a nonnegative h-vector, then the f-polynomial of its first barycentric subdivision has only real zeros [4].
3. (3) The f-polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros [9], [19], [26].
4. (4) If P is a series-parallel poset, then the f-polynomial of the distributive lattice $J(P)$ has only real zeros [28]. This is not true for an arbitrary poset P [27]. (See also [3].)

No analogous combinatorial result characterizes polynomials of the form (1.1) which have only real zeros by supplying necessary and sufficient conditions. The somewhat cumbersome nature of the noncombinatorial characterization theorems (e.g., [1, Thm. 1], [8, p. 203]) suggests studying problems converse to the above results.

Question 1.1. Let the polynomial (1.1) have only real zeros. Is $a(z)$ necessarily the f-polynomial of a simplicial complex?

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More generally, we have the following problem.

Problem 1.2. Given a sequence of d sets with cardinalities $a_1, \dots, a_d$, state conditions on the sets which are necessary to imply that the polynomial (1.1) has only real zeros.

To provide an answer to Problem 1.2, we will employ *multicomplexes*, a class of objects generalizing simplicial complexes. In Section 2 we define the f-polynomials of multicomplexes and simplicial complexes, and we summarize the well-known characterizations of these polynomials. In Section 3 we state inequalities satisfied by the coefficients of polynomials with real zeros. These inequalities lead to a proof that each polynomial (1.1) having only real zeros is the f-polynomial of a multicomplex. In Section 4 we conjecture an affirmative answer to Question 1.1 and prove some partial results.

2. Characterization of the f-vectors of multicomplexes and simplicial complexes

A *multicomplex* on a set $\{x_1, \dots, x_n\}$ of variables is a collection Σ of monomials in $x_1, \dots, x_n$ which satisfies

(1) The monomial x_i belongs to Σ, for $i = 1, \dots, n$.
(2) If the monomial u belongs to Σ and w divides u, then w also belongs to Σ.

A multicomplex Σ is called a *simplicial complex* if each monomial in Σ is square-free.

Let Σ be a multicomplex on $x_1, \dots, x_n$. We define the *f-vector* of Σ to be the sequence

$$f_\Sigma = (a_i)_{i \geq 0}, \tag{2.1}$$

where a_i is the number of monomials of degree i in Σ. Note that we necessarily have $a_0 = 1$, unless $n = 0$. Also note that the f-vector of a simplicial complex has only finitely many nonzero components.

Multicomplexes have an important interpretation in commutative algebra: if R is a graded k-algebra generated by elements $x_1, \dots, x_n$, then R has a k-basis which is a multicomplex on $x_1, \dots, x_n$. Furthermore, $a(z)$ is the f-polynomial of a finite nonempty multicomplex if and only if for any nonnegative integer c there exists a c-dimensional Cohen-Macaulay ring whose Hilbert series is

$$\frac{a(z)}{(1-z)^c}.$$

(See [24, pp. 56-57].)

Two well-known theorems characterize the f-vectors of multicomplexes and simplicial complexes in terms of functions based upon the following expression of a positive integer m as a sum of binomial coefficients. Given a positive integer i, we define the *ith Macaulay expansion* of m to be the unique expression

$$m = \binom{r_i}{i} + \binom{r_{i-1}}{i-1} + \cdots + \binom{r_j}{j}, \tag{2.2}$$

satisfying

$$r_i > r_{i-1} > \cdots > r_j \geq j \geq 1.$$

To obtain this expression we choose r_i to be the unique positive integer which satisfies

$$\binom{r_i}{i} \leq m < \binom{r_i + 1}{i}, \tag{2.3}$$

and then we compute the $(i-1)$st Macaulay expansion of $m - \binom{r_i}{i}$. Using (2.2) we then define the families $(\mu_i)_{i\geq 1}$, $(\kappa_i)_{i\geq 1}$ of functions on $\mathbb{N}$ by

$$\mu_i(m) = \begin{cases} \binom{r_i+1}{i+1} + \binom{r_{i-1}+1}{i} + \cdots + \binom{r_j+1}{j+1} & \text{if } m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\kappa_i(m) = \begin{cases} \binom{r_i}{i+1} + \binom{r_{i-1}}{i} + \cdots + \binom{r_j}{j+1} & \text{if } m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Recursive formulas for these functions follow immediately from their definitions:

$$\mu_i(m) = \binom{r_i+1}{i+1} + \mu_{i-1}(m - \binom{r_i}{i}), \tag{2.4}$$

$$\kappa_i(m) = \binom{r_i}{i+1} + \kappa_{i-1}(m - \binom{r_i}{i}). \tag{2.5}$$

The characterization of f-vectors of multicomplexes is due to Macaulay [15].

Theorem 2.1. *An integer sequence* $(a_0, a_1, \dots)$ *is the* f*-vector of a nonempty multicomplex on* n *variables if and only if we have* $a_0 = 1$, $a_1 = n$ *and*

$$0 \leq a_{i+1} \leq \mu_i(a_i)$$

for $i \geq 1$.

The characterization of f-vectors of simplicial complexes is due (independently) to Kruskal [14], Katona [13], and Schützenberger [17].

Theorem 2.2. *An integer sequence* $(a_0, \dots, a_d)$ *is the* f*-vector of a nonempty simplicial complex on* n *variables if and only if we have* $a_0 = 1$, $a_1 = n$ *and*

$$0 < a_{i+1} \leq \kappa_i(a_i)$$

for $i = 1, \dots, d-1$.

(See [5], [10] for proofs of these theorems, and [24, pp. 55-56] for constructions.)

It is customary to define the f-vector of a finite multicomplex Σ to be only the nonzero subsequence of the sequence (2.1),

$$f_\Sigma = (a_0, \dots, a_d).$$

We then define the *f-polynomial* of Σ to be

$$f_\Sigma(z) = a_0 + a_1 z + \cdots + a_d z^d.$$

We may also associate f-vectors and f-polynomials to posets. In particular, the set of chains of a poset P forms a simplicial complex $\mathcal{O}(P)$ called the *order complex* of P. (See [25, Ch. 3].) We then define the f-vector f_P and f-polynomial $f_P(z)$ of P to be $f_{\mathcal{O}(P)}$ and $f_{\mathcal{O}(P)}(z)$, respectively.

The functions μ_i and κ_i may be expressed in terms of one another very easily.

Proposition 2.3. *For any positive integers m, i, we have*

$$\kappa_i(m) + m = \mu_i(m).$$

Proof. We use induction on i. Observe that for $i = 1$ we have

$$\kappa_1(m) + m = \binom{m}{2} + m = \binom{m+1}{2} = \mu_1(m).$$

Now fix $i > 1$ and suppose that the claim is true for $1, \dots, i-1$. Let r_i be the positive integer which satisfies

$$\binom{r_i}{i} \le m < \binom{r_i + 1}{i}$$

and define $q = m - \binom{r_i}{i}$. By the recursive formulas (2.4) and (2.5) we have

$$\begin{aligned} \mu_i(m) - \kappa_i(m) &= \binom{r_i+1}{i+1} - \binom{r_i}{i+1} + \mu_{i-1}(q) - \kappa_{i-1}(q) \\ &= \binom{r_i}{i} + q \\ &= m, \end{aligned}$$

as desired. $\square$

3. MAIN RESULTS

Let the polynomial $a(z) = 1 + a_1 z + \cdots + a_d z^d$ in $\mathbb{R}[z]$ have positive coefficients. Conditions on the sequence $(1, a_1, \dots, a_d)$ which are both necessary and sufficient for $a(z)$ to have only real zeros are known but somewhat cumbersome. (See e.g., [1, Thm. 1], [8, p. 203].) On the other hand, several well-known conditions which are merely necessary are quite simple. In the event that $a(z)$ has only real zeros, the sequence $(1 = a_0, \dots, a_d)$ is *unimodal*,

$$a_0 \leq \cdots \leq a_j \geq \cdots \geq a_d \quad \text{for some } j,$$

and *log-concave*,

$$a_i^2 \geq a_{i-1}a_{i+1} \quad \text{for } i = 1, \dots, d-1.$$

It also has Newton's log-concavity property,

$$\left(\frac{a_i}{\binom{d}{i}}\right)^2 \geq \frac{a_{i-1}}{\binom{d}{i-1}} \frac{a_{i+1}}{\binom{d}{i+1}} \quad \text{for } i = 1, \dots, d-1, \tag{3.1}$$

from which one can derive Maclaurin's inequalities,

$$\frac{a_1}{d} \geq \sqrt{\frac{a_2}{\binom{d}{2}}} \geq \sqrt[3]{\frac{a_3}{\binom{d}{3}}} \geq \cdots \geq \sqrt[d]{a_d}. \tag{3.2}$$

(See e.g. [11, p. 52].)

Note that we may interpret (3.2) as a generalization of the Artithmetic Mean - Geometric Mean Inequality by factoring $a(z)$ as $(1+\beta_1 z)\cdots(1+\beta_d z)$. From Maclaurin's inequalities we obtain the following upper bound for each coefficient a_i in terms of a_1.

Observation 3.1. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{R}[z]$ have positive coefficients and only real zeros. Then for $i = 2, \dots, d$ we have*

$$a_i \leq \binom{d}{i}\left(\frac{a_1}{d}\right)^i.$$

Setting $i = d$ in Observation 3.1 and assuming that all coefficients are integers, we obtain an upper bound on the degree in terms of a_1.

Observation 3.2. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ have only real zeros. Then d is no greater than a_1.*

The combination of these two facts yields a third.

Observation 3.3. *For any fixed n there are only finitely many polynomials of the form $1 + nz + a_2 z^2 + \cdots + a_d z^d$ in $\mathbb{N}[z]$ which have only real zeros.*

Maclaurin's inequalities also give us a lower bound for each coefficient a_i in terms of a_d. In particular we have the following.

Observation 3.4. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ have only real zeros. Then for $i = 1, \dots, d-1$ we have*

$$a_i \geq a_d^{i/d}\binom{d}{i} \geq \binom{d}{i}.$$

Thus it is easy to see that a polynomial such as $1 + 4z + 9z^2 + 10z^3 + 5z^4 + z^5$ has nonreal zeros.

To relate Maclaurin's inequalities to the Macaulay functions, it will be convenient to define numbers $\tau_0, \dots, \tau_d$ by $\tau_0 = 1$ and

$$a(z) = 1 + a_1 z + \cdots + a_d z^d = \sum_{i=0}^{d} \tbinom{d}{i} \tau_i^i z^i,$$

and to write Maclaurin's inequalities as

$$\tau_1 \geq \cdots \geq \tau_d. \tag{3.3}$$

Proposition 3.5. *Let* $a(z) = 1 + a_1 z + \cdots + a_d z^d = \sum_{i=0}^{d} \binom{d}{i} \tau_i^i z^i \in \mathbb{N}[z]$ *satisfy Maclaurin's inequalities and define the integer sequence* $(n_1, \dots, n_d)$ *by*

$$\binom{n_i}{i} \leq a_i < \binom{n_i + 1}{i}.$$

Then for $i = 1, \dots, d$ *we have*

$$\tau_i < \frac{n_i - i + 1}{d - i}. \tag{3.4}$$

Proof. Suppose (3.4) is false for some i. Then we have $n_i - i + 1 < \tau_i(d - i)$, and since $\tau_i > 1$, we also have

$$n_i - i + 1 + j < \tau_i(d - i + j)$$

for all $j \geq 0$. From this inequality we obtain

$$\binom{n_i + 1}{i} = \frac{1}{i!} \prod_{j=1}^{i} (n_i - i + 1 + j) < \frac{1}{i!} \prod_{j=1}^{i} [\tau_i(d - i + j)] = \binom{d}{i} \tau_i^i = a_i,$$

a contradiction. □

Theorem 3.6. *Let* $a(z) = 1 + a_1 z + \cdots + a_d z^d = \sum_{i=0}^{d} \binom{d}{i} \tau_i^i z_i \in \mathbb{N}[z]$ *satisfy Maclaurin's inequalities (3.2). Then* $a(z)$ *is the* f*-polynomial of a multicomplex.*

Proof. Let i be any integer between 1 and $d - 1$. By Maclaurin's inequalities we have

$$a_{i+1} = \binom{d}{i+1} \tau_{i+1}^{i+1} \leq \binom{d}{i+1} \tau_i^{i+1} = \frac{(d - i)\tau_i}{i + 1} \binom{d}{i} \tau_i^i. \tag{3.5}$$

Now define the integer n_i by

$$\binom{n_i}{i} \leq a_i < \binom{n_i + 1}{i}.$$

Applying Proposition 3.5 to rightmost expression in (3.5) we have

$$a_{i+1} \leq \frac{n_i - i + 1}{i + 1} \binom{d}{i} \tau_i^i = \frac{n_i - i + 1}{i + 1} a_i \leq \frac{n_i - i + 1}{i + 1} \binom{n_i + 1}{i} = \binom{n_i + 1}{i + 1}.$$

Since $\binom{n_i+1}{i+1}$ is no greater than $\mu_i(a_i)$, we have the desired result. □

Corollary 3.7. *Let $a(z) = 1+a_1z+\cdots+a_dz^d \in \mathbb{N}[z]$ satisfy Maclaurin's inequalities. Then for any $c \in \mathbb{N}$ there exists a Cohen-Macaulay ring whose Hilbert series has the form*

$$\frac{a(z)}{(1-z)^c}.$$

Equivalently, $a(z)$ is the h-polynomial of a Cohen-Macaulay complex.

A second consequence of Theorem 3.6 concerns simplicial polytopes. (See [2] for definitions.)

Corollary 3.8. *Let $a(z) = 1+a_1z+\cdots+a_dz^d \in \mathbb{N}[z]$ satisfy Maclaurin's inequalities. Then for any $c \in \mathbb{N}$ greater than or equal to $2d$, there exists a simplicial c-polytope whose g-polynomial is $a(z)$.*

A third consequence of Maclaurin's inequalties relates polynomials with real zeros to the Upper Bound Conjecture for f-vectors of simplicial convex polytopes. (See [24, p. 59] for definitions.)

Corollary 3.9. *Let $a(z) = 1+a_1z+\cdots+a_dz^d \in \mathbb{N}[z]$ satisfy Maclaurin's inequalities and let $f = (1, f_0, \ldots, f_{d-1})$ be the f-vector of the cyclic polytope $C(a_1, d)$. Then for $i = 1, \ldots, d$ we have*

$$a_i \leq f_{i-1}.$$

Proof. Define the polynomial

$$b(z) = (1 + \tfrac{a_1}{d}z)^d = 1 + b_1z + \cdots + b_dz^d.$$

By (3.2) a_i is no greater than b_i for $i = 1, \ldots, d$. Therefore it suffices to show that b_i is no greater than f_{i-1} for $i = 1, \ldots, d$.

By a result of McMullen (see [24, p. 59]), the coefficients of $b(z)$ satisfy the conditions of the Upper Bound Conjecture if the coefficients of the polynomial

$$h(z) = (1 + \tfrac{a_1-d}{d}z)^d = 1 + h_1z + \cdots + h_dz^d$$

satisfy

$$h_i \leq \binom{a_1 - d + i - 1}{i}.$$

Computing an upper bound for h_i we have

$$h_i = \binom{d}{i}\left(\frac{a_1-d}{d}\right)^i = \frac{d(d-1)\cdots(d-i+1)(a_1-d)^i}{i!\,d^i} \leq \frac{(a_1-d)^i}{i!},$$

which is clearly less than or equal to

$$\frac{1}{i!}\prod_{j=0}^{i-1}(a_1 - d + j) = \binom{a_1 - d + i - 1}{i}.$$

□

Yet another consequence of Maclaurin's inequalities is a family of inequalities satisfied by the minors of totally nonnegative matrices. Denote by $\binom{[n]}{k}$ the collection of k-element subsets of $[n] = \{1, \ldots, n\}$. For any matrix A of size at least $n \times n$ and any elements S, T of $\binom{[n]}{k}$ define $\Delta_{S,T}$ to be the (S,T) *minor* of A, the determinant of the submatrix of A corresponding to rows S and columns T. A matrix is called *totally nonnegative* if all of its minors are nonnegative.

Corollary 3.10. *Let A be an $n \times n$ totally nonnegative matrix and let $k < \ell$ be two integers in $[n]$. Then we have*

$$\binom{n}{\ell}^k \Big(\sum_{S \in \binom{[n]}{k}} \Delta_{S,S} \Big)^\ell - \binom{n}{k}^\ell \Big(\sum_{S \in \binom{[n]}{\ell}} \Delta_{S,S} \Big)^k \geq 0. \tag{3.6}$$

Proof. Suppose A is totally nonnegative. A well-known result states that A has only nonnegative real eigenvalues and therefore that the polynomial

$$\det(Az + I) = 1 + a_1 z + \cdots + a_n z^n$$

has only negative real zeros. Since these coefficients are given by

$$a_i = \sum_{S \in \binom{[n]}{i}} \Delta_{S,S},$$

we may apply (3.2) to obtain the desired result. □

Corollary 3.10 gives an example of a multivariate polynomial in matrix entries which is nonnegative for all totally nonnegative matrices. Such a polynomial is itself called *totally nonnegative*. (See e.g. [20, Cor. 3.3], [29, Sec. 3.1].)

4. Results concerning simplicial complexes

Theorem 3.6 implies that each polynomial $1+a_1z+\cdots+a_dz^d \in \mathbb{N}[z]$ having only real zeros is the f-polynomial of a multicomplex. We conjecture that this multicomplex may be chosen to be a simplicial complex.

Conjecture 4.1. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ have only real zeros. Then $a(z)$ is the f-polynomial of a simplicial complex.*

One special case of the conjecture, which depends upon the locations of the real zeros of the polynomial, follows from Theorem 3.6. The hypothesis in the following proposition is equivalent to the condition that the zeros of a monic polynomial in $\mathbb{N}[z]$ be real and less than or equal to -1.

Proposition 4.2. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ have only real zeros, all of which lie between -1 and 0. Then $a(z)$ is the f-polynomial of a simplicial complex.*

Proof. Factoring $a(z)$ as

$$a(z) = \prod_{i=1}^{d} (1 + \beta_i z),$$

we have that $\beta_1, \ldots, \beta_d$ are real and greater than or equal to one. Now define the polynomial $h(z)$ by

$$h(z) = \prod_{i=1}^{d} [1 + (\beta_i - 1)z].$$

Clearly $h(z)$ belongs to $\mathbb{N}[z]$ and has only real zeros. By Theorem 3.6 $h(z)$ is the f-polynomial of a multicomplex, and by [24, Thm. 3] it is therefore the h-polynomial of a $(d-1)$-dimensional Cohen-Macaulay simplicial complex Δ. (See [24, pp. 33-35, 53-58] for infomation about h-polynomials and Cohen-Macaulay complexes.) It follows that $a(z)$ is the f-polynomial of Δ. $\square$

For the remainder of this section, we shall use the notation

$$m^{(i)} = m(m-1)\cdots(m-i+1).$$

On the other hand, we will also use parentheses as necessary in fractional exponents and the expression $m^{(i+1)/i}$ should be interpreted as m raised to the power $\frac{i+1}{i}$. In order to prove the main result (Theorem 4.8) of this section, we state and prove several technical lemmas. In these lemmas, e is the natural constant $2.71828\cdots$.

Lemma 4.3. *Fix an integer $d \geq 3$. Then for $i = 1, \ldots, d-1$ and for all $t \geq d^2$, we have*

$$\frac{t^{(i+1)}}{((t+1)^{(i)})^{(i+1)/i}} \geq \frac{d^{(i+1)}}{((d)^{(i)})^{(i+1)/i}}. \tag{4.1}$$

Proof. Verifying (4.1) for $d = 3, 4$ reduces to straightforward computations. Assume therefore that d is at least 5, and let $G(t)$ be the function on the left-hand side of

(4.1). Note that we have

$$\begin{aligned}
G'(t) &= G(t)\left(\sum_{j=0}^{i}\frac{1}{t-j} - \frac{i+1}{i}\sum_{j=0}^{i-1}\frac{1}{t+1-j}\right)\\
&= G(t)\left(\frac{1}{t-i} - \frac{1}{t+1} + \frac{1}{t-i+1} - \frac{1}{i}\sum_{j=0}^{i-1}\frac{1}{t+1-j}\right),\\
&\geq G(t)\left(\frac{1}{t-i} - \frac{1}{t+1} + \frac{1}{t-i+1} - \frac{1}{t-i+2}\right),
\end{aligned}$$

which is nonnegative. Thus it is sufficient to prove the claim for $t = d^2$. We have

$$\begin{aligned}
G(d^2) &= \frac{(d^2-i+1)(d^2-i)}{(d^2+1)\sqrt[i]{(d^2+1)d^2\cdots(d^2-i+2)}}\\
&\geq \frac{(d^2-i)^2}{(d^2+1)(d^2-\frac{i-3}{2})}\\
&= 1 - \frac{(3i+5)d^2-(2i^2+i-3)}{(d^2+1)(2d^2-i+3)}\\
&\geq 1 - \frac{(3i+5)d^2}{(d^2+1)(2d^2-i)}\\
&\geq 1 - \frac{(3i+5)}{2d^2-d}.
\end{aligned}$$

Now we claim that

$$\frac{3i+5}{2d^2-d} \leq \frac{1}{d-i+1}.$$

To see this, consider the parabola

$$H(t) = (3t+5)(d-t+1) - (2d^2-d)$$

which opens downward and has a maximum at $t = \frac{d}{2} - \frac{1}{3}$. Since

$$H(\tfrac{d}{2}-\tfrac{1}{3}) = -\tfrac{5}{4}d^2 + 5d + \tfrac{16}{3}$$

is negative when $d \geq 5$, we have

$$\begin{aligned}
G(d^2) &\geq 1 - \frac{1}{d-i+1}\\
&= \frac{d-i}{d-i+1}\\
&\geq \frac{d^{(i+1)}}{(d^{(i)})^{(i+1)/i}},
\end{aligned}$$

as desired. □

Lemma 4.4. *The sequence* $(\sqrt[n-1]{n!}/n)_{n\geq 2}$ *monotonically decreases with* n.

Proof. It is straightforward to verify that the sequence decreases for $n = 2, \ldots, 13$. Assume therefore that $n \geq 13$, and let $F(n)$ be the ratio of the nth and $(n+1)$st terms. We will show that $F(n)$ is at least 1, or equivalently that

$$\log F(n) = \log(1 + \tfrac{1}{n}) + \frac{\log n!}{n(n-1)} - \frac{\log(n+1)}{n}$$

is positive.

Noting that the Taylor expansion of $\log(1+x)$ is an alternating series and evaluating at $x = \frac{1}{n}$, we have

$$\frac{1}{n} - \frac{1}{2n^2} \leq \log(1 + \tfrac{1}{n}) \leq \frac{1}{n}. \tag{4.2}$$

Evaluating the Taylor expansion of e^x at $x = n$, we have $e^n > n^n/n!$ or equivalently

$$\log(n!) > n \log n - n.$$

Thus $\log F(n)$ is at least

$$\tfrac{1}{n} - \tfrac{1}{2n^2} + \tfrac{1}{n(n-1)}(n \log n - n) - \tfrac{1}{n}\log(n+1) = \tfrac{1}{n} - \tfrac{1}{2n^2} - \tfrac{1}{n}\log(1 + \tfrac{1}{n}) + \tfrac{1}{n(n-1)}\log n - \tfrac{1}{n-1}.$$

Using the right inequality in (4.2), we then have

$$F(n) \geq -\tfrac{1}{n(n-1)} - \tfrac{1}{2n^2} - \tfrac{1}{n^2} + \tfrac{1}{n(n-1)}\log n,$$

which is at least

$$-\tfrac{1}{n(n-1)} - \tfrac{1}{2n(n-1)} - \tfrac{1}{n(n-1)} + \tfrac{1}{n(n-1)}\log n = \frac{\log n - 2.5}{n(n-1)},$$

which is nonnegative since $n \geq 13$ and $e^{2.5} \approx 12.18$. □

Lemma 4.5. *Fix positive integers* $i < d$. *Then we have*

$$\sqrt[i]{d^{(i)}} \geq \frac{d}{e},$$

Proof. Note that

$$\sqrt[i]{d^{(i)}} = \sqrt[i-1]{d^{(i-1)}} \frac{\sqrt[i]{d-i+1}}{\sqrt[i(i-1)]{d^{(i-1)}}} \leq \sqrt[i-1]{d^{(i-1)}}.$$

Thus $\sqrt[i]{d^{(i)}}$ decreases as i increases from 1 to $d-1$. In particular we have

$$\sqrt[i]{d^{(i)}} \geq \sqrt[d-1]{d^{(d-1)}} = \sqrt[d-1]{d!}.$$

Dividing by d and applying Stirling's formula, we have

$$\lim_{d\to\infty} \frac{\sqrt[d-1]{d!}}{d} = \frac{1}{e}.$$

Combining this fact with Lemma 4.4, we have

$$\sqrt[d-1]{d!} \geq \frac{d}{e},$$

which gives us the desired inequality. □

Using Lemmas 4.3-4.5 we may now prove that if the coefficients of $a(z)$ are large enough, then it is the f-polynomial of a simplicial complex.

Proposition 4.6. *Let* $a(z) = 1 + a_1 z + \cdots + a_d z^d = \sum_{i=0}^{d} \binom{d}{i} \tau_i^i z^i \in \mathbb{N}[z]$ *have only real zeros. For* $i = 1, \ldots, d-1$*, the inequality*

$$a_i \geq \binom{d}{i} (ed)^i \tag{4.3}$$

implies that $a_{i+1} \leq \kappa_i(a_i)$*.*

Proof. If $d = 2$, then without appealing to (4.3) we may use the quadratic formula and Observation 3.4 to see that $a_2 \leq \kappa_1(a_1)$ and therefore that $a(z)$ is the f-polynomial of a simplicial complex.

Suppose therefore that $d \geq 3$ and choose an index $1 \leq i \leq d-1$ which satisfies (4.3). Let r be the unique integer which satisfies

$$\binom{r}{i} \leq a_i < \binom{r+1}{i} \tag{4.4}$$

so that we have

$$\kappa_i(a_i) \geq \binom{r}{i+1}.$$

By Lemma 4.5 and our choice of i we have

$$\binom{d^2}{i} \leq \frac{d^{2i}}{i!} = \binom{d}{i} \left(\frac{d^2}{\sqrt[i]{d^{(i)}}} \right)^i \leq \binom{d}{i} (ed)^i \leq a_i,$$

which implies that $r \geq d^2$. Thus by Lemma 4.3, we have

$$\kappa_i(a_i) \geq \binom{r}{i+1} \frac{d^{(i+1)}}{(d^{(i)})^{(i+1)/i}} \frac{((r+1)^{(i)})^{(i+1)/i}}{r^{(i+1)}} = \binom{d}{i+1} \left(\frac{(r+1)^{(i)}}{\binom{d}{i} i!} \right)^{(i+1)/i}.$$

By Maclaurin's inequalities (3.3), we may multiply the expression on the right by $(\tau_{i+1}/\tau_i)^{i+1}$ to obtain

$$\kappa_i(a_i) \geq a_{i+1} \left(\frac{\binom{r+1}{i}}{a_i} \right)^{(i+1)/i},$$

which by (4.4) is at least a_{i+1}. □

While the inequality $a_{d+1} \leq \kappa_d(a_d)$ is meaningless for a degree-d polynomial, the following corollary of Proposition 4.6 shows that it is enough to check the inequality (4.3) for $i = d$ in order to assert that $a(z)$ is the f-polynomial of a simplicial complex.

Corollary 4.7. *Let* $a(z) = 1 + a_1 z + \cdots + a_d z^d = \sum_{i=0}^{d} \binom{d}{i} \tau_i^i z^i \in \mathbb{N}[z]$ *have only real zeros. If*

$$a_d \geq (ed)^d \tag{4.5}$$

then $a(z)$ *is the* f*-vector of a simplicial complex.*

Proof. Rewriting Equation (4.5) as $\tau_d \geq ed$ and applying Maclaurin's inequalities we have

$$\tau_i \geq ed, \qquad i = 1, \dots, d-1,$$

or equivalently,

$$a_i \geq \binom{d}{i} (ed)^i, \qquad i = 1, \dots, d-1.$$

By Proposition 4.6, this gives the desired result. □

Using Proposition 4.6 we can now show that for fixed d, Conjecture 4.1 fails for at most finitely many polynomials.

Theorem 4.8. *For any fixed* d*, there are at most finitely many polynomials of the form* $1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ *which have only real zeros and are not* f*-polynomials of simplicial complexes.*

Proof. Suppose first that we have $a_{d-1} \geq d(ed)^{d-1}$, or equivalently, $\tau_{d-1} \geq ed$. Then by Maclaurin's inequalities we have $\tau_i \geq ed$ for $i = 1, \dots, d-2$, and by Proposition 4.6 $a(z)$ is the f-polynomial of a simplicial complex.

Now suppose that we have $a_{d-1} < d(ed)^{d-1}$. Maclaurin's inequalities then bound a_d by

$$a_d \leq (ed)^d.$$

Log-concavity and Observation 3.4 then bound the coefficients $a_1, \dots, a_{d-2}$ by

$$a_i \leq \frac{a_{i+1}^2}{a_{i+2}} \leq \frac{a_{i+1}^2}{\binom{d}{i+2}}.$$

□

As we have already mentioned, Conjecture 4.1 holds if we fix $d = 2$. It would be interesting therefore to obtain tighter bounds in Proposition 4.6 and Corollary 4.7 to perhaps prove special cases of the conjecture corresponding to a few more values of d. On the other hand, the following fact [6] shows that analytic results like Maclaurin's inequalities will not suffice to prove Conjecture 4.1.

Proposition 4.9. *A polynomial $a(z) = 1 + a_1 z + \cdots + a_d z^d$ having positive real coefficients and only real zeros need not satisfy $a_{i+1} \leq \kappa_i(a_i)$, even for the coefficients a_i which are integers.*

Proof. Choose an integer $i > 2$, define the positive real number β by

$$\beta = \sqrt[i]{\frac{\binom{d}{i} + i - 1}{\binom{d}{i}}},$$

and consider the polynomial

$$b(z) = (1 + \beta z)^d = 1 + b_1 z + \cdots + b_d z^d.$$

Note that we have

$$b_i = \binom{d}{i} + i - 1,$$
$$b_{i+1} = \binom{d}{i+1}\beta^{i+1}.$$

Since the ith Macaulay expansion of b_i is

$$\binom{d}{i} + \binom{i-1}{i-1} + \cdots + \binom{1}{1},$$

we have

$$\kappa_i(b_i) = \binom{d}{i+1},$$

which is less than b_{i+1}, since β is greater than 1. □

Thus if Conjecture 4.1 is true, it cannot be true for purely analytic reasons. It is equally clear, however, that the conjecture can not be true for purely algebraic reasons, because it involves establishing an inequality.

Facts concerning conjugate algebraic integers (that is, *irreducible* polynomials of the required form) may provide the tools necessary to make progress on Connjecture 4.1. It is easy to see that the conjecture is equivalent to the assertion that each *irreducible* polynomial $a(z) = 1+a_1z+\cdots+a_dz^d$ in $\mathbb{N}[z]$ having only real zeros is the f-polynomial of a simplicial complex, because for any pair of simplicial complexes (Δ, Σ) there exists a third simplicial complex Γ whose f-polynomial is equal to $f_\Delta(z)f_\Sigma(z)$. By restricting attention to irreducible polynomials, one can obtain coefficient bounds which strengthen Observation 3.2. For example, Schur [16, Thm. XI] showed that for any number $0 \leq \gamma < \sqrt{e} \approx 1.6487$, all but finitely many irreducible polynomials $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ having only real zeros satisfy $a_1 > \gamma d$. Improving upon Schur's result, Siegel [18, p. 303] replaced $\sqrt{e}$ with a slightly greater number and observed that the assertion becomes false if we replace $\sqrt{e}$ by any number greater than 2. He essentially posed the following question which is of interest in light of Conjecture 4.1.

Question 4.1. What is the greatest number γ such that we have $a_1 > \gamma d$ for all but finitely many of the irreducible polynomials $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ which have only real zeros?

Smyth [22, p. 2] made the following progress on Question 4.1.

Theorem 4.10. *Let $a(z) = 1 + a_1 z + \cdots + a_d z^d$ be an irreducible polynomial in $\mathbb{N}[z]$ which has only real zeros. Then we have*

$$a_1 > 1.7719\, d$$

unless $a(z)$ is equal to one of the five exceptions

$$\begin{aligned} &1 + z, \\ &1 + 3z + z^2, \\ &1 + 6z + 5z^2 + z^3, \\ &1 + 7z + 13z^2 + 7z^3 + z^4, \\ &1 + 8z + 14z^2 + 7z^3 + z^4. \end{aligned}$$

Further results [23] suggest that the answer to Question 4.1 is strictly less than 2. By Proposition 4.6, we have $a_{i+1} \leq \kappa_i(a_i)$ whenever a_i is large enough. It is possible to use Theorem 4.10 to tighten the bound given by Proposition 4.6, but the details are somewhat tedious and will not be given here. It would be interesting to see if Proposition 4.6 could be improved in a less tedious fashion by employing an appropriate generalization of Theorem 4.10. For instance, we have the following question.

Question 4.2. What is the greatest number γ_i such that we have $a_i > \gamma_i d$ for all but finitely many of the irreducible polynomials $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ which have only real zeros?

5. Containments of classes of f-polynomials

To finish, we will consider the cardinalities of various sets of f-polynomials and the containment relations satisfied by these sets.

We define a simplicial complex with f-polynomial $a(z) = 1 + a_1 z + \cdots + a_d z^d$ to be *balanced* if there exists a coloring of the vertex set such that no face contains two vertices of the same color. A characterization of the f-polynomials of balanced complexes due to Frankl, Furedi and Kalai [7] is analogous to Theorems 2.2 and 2.1.

Question 5.1. Let $a(z) = 1 + a_1 z + \cdots + a_d z^d \in \mathbb{N}[z]$ have only real zeros. Is $a(z)$ the f-polynomial of a balanced complex?

We define the f-polynomial of a poset P by letting a_i count the number of i-element chains in P. This is a special case of our definition of the f-vector of a simplicial complex, since the chains of a poset form a simplicial complex. We define a poset to be a *unit interval order* if it has no induced subposet which is isomorphic to the four-element posets $\mathbf{2+2}$ or $\mathbf{3+1}$. (See [19] for definitions.) More generally, we define a poset to be $(\mathbf{3+1})$*-free* if it contains no subposet isomorphic to $\mathbf{3+1}$. It is known that the f-polynomial of a $(\mathbf{3+1})$-free poset, and therefore of a unit interval order, has only real zeros.

For fixed n, let us define five sets of polynomials as follows,

$$\begin{aligned}
U_n &= \{f_P(z) \,|\, P \text{ a unit interval order on } n \text{ elements}\},\\
R_n &= \{a(z) = 1 + nz + a_2 z^2 + \cdots + a_d z^d \in \mathbb{N}[z] \,|\, a(z) \text{ has only real zeros}\},\\
B_n &= \{f_\Delta(z) \,|\, \Delta \text{ a balanced simplicial complex on } n \text{ variables}\},\\
C_n &= \{f_\Delta(z) \,|\, \Delta \text{ a simplicial complex on } n \text{ variables}\},\\
M_n &= \{f_\Delta(z) \,|\, \Delta \text{ a multicomplex on } n \text{ variables with no monomial of degree } n+1\}.
\end{aligned}$$

It is unnecessary for us to consider the set of f-polynomials of $(\mathbf{3+1})$-free posets on n elements, because by [21] this is equal to U_n.

The containments $B_n \subset C_n \subset M_n$ are immediate and one can show (using Maple and C programs) that we have the containments

$$U_n \subset R_n \subset B_n \subset C_n \subset M_n,$$

for $n \leq 10$. The following table shows the cardinalities of some of these sets and the ratios of these to $|R_n|$.

n	$\lvert U_n\rvert$	$\lvert R_n\rvert$	$\lvert B_n\rvert$	$\lvert C_n\rvert$	$\lvert M_n\rvert$	$\frac{\lvert U_n\rvert}{\lvert R_n\rvert}$	$\frac{\lvert B_n\rvert}{\lvert R_n\rvert}$	$\frac{\lvert C_n\rvert}{\lvert R_n\rvert}$	$\frac{\lvert M_n\rvert}{\lvert R_n\rvert}$
1	1	1	1	1	1	1.00	1.00	1.00	1.00
2	2	2	2	2	4	1.00	1.00	1.00	2.00
3	4	4	4	5	36	1.00	1.00	1.25	9.00
4	8	8	10	16	941	1.00	1.25	2.00	117.62
5	16	16	30	70	91308	1.00	1.88	4.38	5706.75
6	34	36	124	457	37780101	.94	3.65	13.44	1049447.30
7	75	78	712	4908	?	.96	9.49	65.44	?
8	170	185	6600	95248	?	.92	35.68	514.85	?
9	407	452	105336	3617645	?	.90	233.04	8003.64	?

Given the values in the table, it is clear that even an affirmative answer to Question 5.1 would be far from a characterization of R_n. It also seems that U_n approximates R_n less closely as n increases. This leaves us with the following problem.

Problem 5.2. Combinatorially describe a class of simplicial complexes whose f-polynomials approximate R_n as closely as possible.

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