$\mathcal{H}_{\mathbf{n}}$-TRACES AND LLT POLYNOMIALS

Mark Skandera (joint work with Alejandro Morales and Jiayuan Wang)

Iwahori-Hecke algebra: $\mathcal{H}_n = \mathrm{span}_{\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]}\{T_w \,|\, w \in \mathfrak{S}_n\}$ with

$$T_w T_s = \begin{cases} qT_{ws} + (q-1)T_w & \text{if } ws < w, \\ T_{ws} & \text{if } ws > w. \end{cases}$$

Bar involution on $\mathcal{H}_n$: $\overline{\sum_w a_w T_w} = \sum_w \overline{(a_w)}\ \overline{(T_w)}$, where

$$\overline{q^{\frac{1}{2}}} = q^{\frac{-1}{2}}, \qquad \overline{T_w} = T_{w^{-1}}^{-1} = \sum_{v \leq w} q^{-\ell(v)} R_{v,w}(q^{-1}) T_v.$$

Kazhdan–Lusztig basis $\{C'_w \,|\, w \in \mathfrak{S}_n\}$ is bar-invariant with

$$w \text{ avoids pattern } 312 \quad \Rightarrow \quad \widetilde{C}_w := q^{\frac{\ell(w)}{2}} C'_w = \sum_{v \leq w} T_v.$$

$\mathcal{H}_{\mathbf{n}}$-traces

A linear functional $\theta : \mathcal{H}_n \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ is called an $\mathcal{H}_n$-*trace* if for all $D_1, D_2 \in \mathcal{H}_n$ we have $\theta(D_1 D_2) = \theta(D_2 D_1)$.

Traces form a vector space $\mathcal{T}_n$. Six bases

$$\{\chi^\lambda\}, \quad \{\epsilon^\lambda\}, \quad \{\eta^\lambda\}, \quad \{\psi^\lambda\}, \quad \{\phi^\lambda\}, \quad \{\gamma^\lambda\} \qquad (\lambda \vdash n)$$

of the trace space $\mathcal{T}_n$ correspond to bases

$$\{s_\lambda\}, \quad \{e_\lambda\}, \quad \{h_\lambda\}, \quad \{p_\lambda\}, \quad \{m_\lambda\}, \quad \{f_\lambda\} \qquad (\lambda \vdash n)$$

of homogeneous degree-n symmetric function space Λ_n.

Two more traces are

$$\delta^{1^n} : T_w \mapsto \begin{cases} 1 & \text{if } w = e, \\ 0 & \text{otherwise,} \end{cases} \qquad \zeta^n : T_w \mapsto R_{e,w}(q).$$

Evaluation of $\mathcal{H}_\mathbf{n}$-traces

$\theta \in \mathcal{T}_n$ is determined by its evaluations on $\{T_{u_\lambda} \,|\, \lambda \vdash n\}$, where u_λ is any minimum-length element of λ-conjugacy class of $\mathfrak{S}_n$:

$$\theta(D) = \sum_\lambda \delta^\lambda(D)\theta(T_{u_\lambda}),$$

where $\{\delta^\lambda \,|\, \lambda \vdash n\}$ are traces with $\delta^\lambda(T_w)$ computed recursively.

$\theta \in \mathcal{T}_n$ is determined by its evaluations on $\{\widetilde{C}_{v_\lambda} \,|\, \lambda \vdash n\}$, where v_λ is the longest element of Young subgroup $\mathfrak{S}_\lambda$:

$$\theta(D) = \sum_\lambda \frac{\phi^\lambda(D)}{[\lambda_1]_q! \cdots [\lambda_{\ell(\lambda)}]_q!}\theta(\widetilde{C}_{v_\lambda}).$$

Conj: (H '93) For all $w \in \mathfrak{S}_n$ we have $\phi^\lambda(\widetilde{C}_w) \in \mathbb{N}[q]$.

Chromatic (quasi-)symmetric functions

Given $G = (V, E)$, define the chromatic quasisymmetric function

$$X_{G,q} = \sum_{\kappa} q^{\mathrm{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots ,$$

where the sum is over all proper colorings $\kappa : G \to \mathbb{P}$, and

$$\mathrm{asc}(\kappa) = \#\{(i,j) \in E \mid i < j,\ \kappa(i) < \kappa(j)\}.$$

Thm: (CHSS ’16) For $w \in \mathfrak{S}_n$ avoiding 312, there is a natural indifference graph $G(w)$ satisfying

$$X_{G(w),q} = \sum_{\lambda} \chi^{\lambda^{\top}}(\widetilde{C}_w) s_{\lambda} = \sum_{\lambda} \epsilon^{\lambda}(\widetilde{C}_w) m_{\lambda} = \sum_{\lambda} \eta^{\lambda}(\widetilde{C}_w) f_{\lambda}$$

$$= \sum_{\lambda} \frac{(-1)^{n-\ell(\lambda)}}{z_{\lambda}} \psi^{\lambda}(\widetilde{C}_w) p_{\lambda} = \sum_{\lambda} \phi^{\lambda}(\widetilde{C}_w) e_{\lambda} = \sum_{\lambda} \gamma^{\lambda}(\widetilde{C}_w) h_{\lambda}.$$

Unicellular LLT polynomials

Given 312-avoiding permutation w, define the unicellular LLT polynomial

$$\mathrm{LLT}_{w,q} = \sum_{\kappa} q^{\mathrm{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots ,$$

where the sum is over all arbitrary colorings $\kappa : G(w) \rightarrow \mathbb{P}$.

Thm: (MSW '24) For $w \in \mathfrak{S}_n$ avoiding 312, there are bases

$$\{\chi^{\lambda}_{\mathrm{LLT}} \,|\, \lambda \vdash n\}, \ldots, \{\gamma^{\lambda}_{\mathrm{LLT}} \,|\, \lambda \vdash n\}$$

of the trace space $\mathcal{T}_n$ such that

$$\mathrm{LLT}_{w,q} = \sum_{\lambda} \chi^{\lambda^{\top}}_{\mathrm{LLT}}(\widetilde{C}_w) s_{\lambda} = \sum_{\lambda} \epsilon^{\lambda}_{\mathrm{LLT}}(\widetilde{C}_w) m_{\lambda}$$

$$= \sum_{\lambda} \eta^{\lambda}_{\mathrm{LLT}}(\widetilde{C}_w) f_{\lambda} = \cdots = \sum_{\lambda} \gamma^{\lambda}_{\mathrm{LLT}}(\widetilde{C}_w) h_{\lambda}.$$

LLT analogs of induced sign and trivial characters

Thm (MSW '24): For $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ we have

$$\epsilon^{\lambda}_{\mathrm{LLT}} = (\delta^{1^{\lambda_1}} \otimes \cdots \otimes \delta^{1^{\lambda_r}})\uparrow^{\mathcal{H}_n}_{\mathcal{H}_\lambda},$$

$$\eta^{\lambda}_{\mathrm{LLT}} = (\zeta^{\lambda_1} \otimes \cdots \otimes \zeta^{\lambda_r})\uparrow^{\mathcal{H}_n}_{\mathcal{H}_\lambda},$$

where $\mathcal{H}_\lambda$ is the Young subalgebra of $\mathcal{H}_n$ indexed by λ.

Thm (MSW '24): In the special case $\lambda = n$, we have

$$\epsilon^{n}_{\mathrm{LLT}} = (1-q)^n \sum_{\lambda} \frac{q^{b(\lambda)}}{\prod_{c\in\lambda}(1-q^{h(c)})}\chi^{\lambda},$$

$$\eta^{n}_{\mathrm{LLT}} = (1-q)^n \sum_{\lambda} \frac{q^{b(\lambda^\top)}}{\prod_{c\in\lambda}(1-q^{h(c)})}\chi^{\lambda},$$

where $h(c)$ is the hooklength of cell c and $b(\lambda) = \sum_i \binom{\lambda_i^\top}{2}$.