SCRIPT FOR PETCHA KUTCHA TALK

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1. The Hecke algebra

(0:00) The Hecke algebra, as a module over Laurent polynomials in root q, has two famous bases.
(0:06) It is usually defined in terms of a *natural basis* consisting of symbols T_w where w is a permutation in $\mathfrak{S}_n$.
(0:13) Multiplication boils down to considering T_w times T_s where s is a generator of $\mathfrak{S}_n$. The formula for T_wT_s has two cases, depending on whether ws is less than or greater than w in the Bruhat order.
(0:25) The second basis, due to Kazhdan and Lusztig, is defined in terms of a *bar involution* on the Hecke algebra, which takes root q to its reciprocal, and T_w to the inverse of $T_{w^{-1}}$.
(0:35) Expanding that in terms of the natural basis we obtain coefficients known as the *R-polynomials.*
(0:40) The Kazhdan–Lusztig basis, with elements written C_w, is bar-invariant, and if we modify each element by multiplying by root q to the length of w, then resulting elements indexed by 312-avoiding permutations w are just sums of natural basis elements T_v with $v \leq w$ in the Bruhat order.

2. $H_n(q)$ traces

(1:00) A linear functional θ from the Hecke algebra to the Laurent polynomials in root q is called a *trace* if its evaluation at each product D_1D_2 is the same as its evaluation at D_2D_1.
(1:12) Traces form a vector space, and six bases of this trace space correspond to six well-known bases of symmetric functions.
(1:20) Three of these are character bases: irreducible characters, induced sign characters, and induced trivial characters, which correspond to Schur, elementary, and homogeneous symmetric functions.
(1:31) Three more non-character bases we could call power sum, monomial, and forgotten traces, because they correspond to power sum, monomial, and forgotten symmetric functions.
(1:42) Another trace, delta 1^n, maps T_w to 1 if the cycle type of w is 1^n and to 0 otherwise.
(1:52) Another trace, zeta n, maps T_w to the R-polynomial $R_{e,w}$, which we saw a minute ago.

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3. Evaluation of $H_n(q)$-traces

(2:00) A trace θ is determined by its evaluations at elements T_{u_λ}, where u_λ is any minimum-length element of the λ conjugacy class of $\mathfrak{S}_n$:

(2:10) to evaluate θ at some element D of the Hecke algebra, we first evaluate θ at all the elements T_{u_λ}, and then take a linear combination of these evaluations. The λ coefficient involves a trace function delta λ and can be computed recursively. We saw delta 1^n thirty seconds ago.

(2:26) Alternatively, a trace θ is determined by its evaluations at Kazhdan–Lusztig basis elements C_{v_λ}, where v_λ is the longest element of the λ Young subgroup of $\mathfrak{S}_n$:

(2:39) to compute $\theta(D)$, we first evaluate θ at all the elements C_{v_λ}, and then take a linear combination of these evaluations. The λ coefficient involves the monomial trace ϕ^λ, which we saw a minute ago.

(2:50) If D is *any* Kazhdan–Lusztig basis element, the evaluation $\phi^\lambda(D)$ is a polynomial in q. Haiman conjectured it to have nonnegative coefficients.

4. Chromatic (quasi-) symmetric functions

(3:00) It is known that chromatic symmetric functions are related to Hecke algebra trace evaluations.

(3:05) Specifically, given a graph G, we can define a quasisymmetric function in terms of proper colorings of G, each weighted by q to a statistic on the coloring.

(3:15) If G is the natural indifference graph corresponding to a 312-avoiding permutation w, then the quasisymmetric function is in fact symmetric, and it is a generating function for evaluations of Hecke algebra traces at the Kazhdan–Lusztig basis element C_w.

(3:32) Expansions in the Schur basis, monomial basis, and forgotten basis yield coefficients which are evaluations of irreducible characters, induced sign characters, and induced trivial characters, respectively.

(3:45) In the elementary expansion, we see coefficients that are monomial trace evaluations. So Haiman's conjectured monomial trace positivity implies e-positivity of the chromatic symmetric functions.

5. Unicellular LLT polynomials

(4:00) Related to chromatic symmetric functions by plethysm are the unicellular LLT polynomials,

(4:05) These can be parametrized by 312-avoiding permutations so that the one corresponding to w is combinatorially defined in terms of arbitrary colorings (not necessarily proper) of the indifference graph corresponding to w.

(4:25) Like the chromatic symmetric function X_G, the symmetric function LLT_w also is a generating function for evaluations of certain traces at the Kazhdan–Lusztig basis element C_w.

(4:36) Let's call these traces *LLT analogs* of the standard bases of the trace space.

(4:42) Expansions of unicelular LLT polynomials in the Schur basis, monomial basis, and forgotten basis yield coefficients which are evaluations of *LLT analogs of* irreducible characters, induced sign characters, and induced trivial characters, respectively.

6. Formulas for LLT-analog of induced sign and trivial characters

(5:00) We have concise descriptions of LLT-analogs of induced sign and trivial characters in terms of induction from Young subalgebras H_λ.

(5:06) In particular,

- (5:07) The LLT-analog of the induced sign character is a tensor product of the delta traces indexed by all ones, induced from H_λ to H_n;
- (5:18) The LLT-analog of the induced trivial character is a tensor product of the zeta traces (defined in terms of R-polynomials), induced from H_λ to H_n.

(5:29) In the special case that $\lambda = n$, we can also express these characters in terms of irreducible characters.

(5:35) Formulas involving hooklengths and binomial coefficients come from principal specializations of Schur functions.

(5:42) So now we can understand not just chromatic symmetric functions, but also LLT polynomials from the point of view of Hecke algebra characters.

(5:55) Thank you.