KAZHDAN-LUSZTIG IMMANANTS AND SCHUR FUNCTIONS

Brendon Rhoades and Mark Skandera

University of Michigan, Dartmouth College

Outline

(1) The dual canonical basis and Kazhdan-Lusztig immanants
(2) Schur nonnegative symmetric functions
(3) Schur nonnegative polynomials
(4) Applications to products of Schur functions

The dual canonical basis

For a polynomial $p(x_{1,1}, \ldots, x_{n,n})$ and matrix $A = (a_{i,j})$, define

$$p(A) = p(a_{1,1}, \ldots, a_{n,n}).$$

Dual to the *canonical basis* of the Hopf algebra $U(\mathfrak{sl}_n(\mathbb{C}))$, is the *dual canonical basis* of

$$\mathcal{O}(SL_n(\mathbb{C})) \cong \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]/(\det(x) - 1).$$

Nonnegativity properties of the CB and DCB have interpretations in representation theory, symmetric functions, other areas.

Kazhdan-Lusztig immanants

Let $P_{w,w'}(q)$ be the KL polynomial indexed by $w, w' \in S_n$.
Let w_0 be the longest permutation in S_n.

For each $u \in S_n$, define $f_u : S_n \to \mathbb{Z}$ by

$$f_u(v) = (-1)^{\ell(uv)} P_{w_0 v, w_0 u}(1).$$

Define the *Kazhdan-Lusztig immanant* $\mathrm{Imm}_u(x)$ by

$$\mathrm{Imm}_u(x) = \sum_{v \geq u} f_u(v) x_{1,v(1)} \cdots x_{n,v(n)}.$$

Remark: $\det(x) = \mathrm{Imm}_1(x)$.

Minors and Schur functions

Given sets I, I', define the (I, I') *minor* of a matrix A by

$$\Delta_{I,I'}(A) = \det(a_{i,j})_{i \in I, j \in I'}.$$

Example: Let $H_{5332} = \begin{bmatrix} h_5 & h_6 & h_7 & h_8 \\ h_2 & h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 & h_3 \\ 0 & 1 & h_1 & h_2 \end{bmatrix}$. Then we have

$$\Delta_{1234,1234}(H_{5332}) = \det(H_{5332}) = s_{5332},$$

$$\Delta_{13,14}(H_{5332}) = \det \begin{bmatrix} h_5 & h_8 \\ h_1 & h_3 \end{bmatrix} = s_{73/2}.$$

Schur nonnegative symmetric functions

Call a symmetric function *Schur nonnegative* (SNN) if its Schur expansion has nonnegative coefficients.

Examples:

$$s_\kappa s_\lambda = \sum_\rho c^\rho_{\kappa\lambda} s_\rho, \qquad s_{\kappa/\lambda} = \sum_\rho c^\kappa_{\lambda\rho} s_\rho.$$

Questions: (L-L-T, O, F-F-L-P, B-B-M, W) For which $\kappa, \lambda, \mu, \nu$ are the following functions SNN?

$$s_\kappa s_\lambda - s_\mu s_\nu \qquad (c^\rho_{\kappa\lambda} \geq c^\rho_{\mu\nu} \text{ for all } \rho)$$

$$s_{\kappa/\lambda} - s_{\mu/\nu} \qquad (c^\kappa_{\lambda\rho} \geq c^\mu_{\nu\rho} \text{ for all } \rho)$$

Littlewood-Richardson coefficients

LR coefficients appear in representations of S_n,

$$(S^\kappa \otimes S^\lambda)\uparrow^{S_n} = \bigoplus_\rho c^\rho_{\kappa\lambda} S^\rho,$$

in representations of GL_n,

$$V^\kappa \otimes V^\lambda = \bigoplus_\rho c^\rho_{\kappa\lambda} V^\rho,$$

in $H^*(Gr(p, \mathbb{C}^n))$,

$$\sigma_\kappa \sigma_\lambda = \sum_\rho c^\rho_{\kappa\lambda} \sigma_\rho.$$

Examples of SNN symmetric functions

$$s_{63/1}s_{42/1} - s_{53}s_{32} = s_{5431} + 4s_{643} + 6s_{841} + \cdots$$
$$s_{63/3}s_{73/2} - s_{55/2}s_{43/1} = 9s_{743} + 3s_{9221} + s_{8222} + \cdots$$

These are

$$(*) \qquad \Delta_{13,13}(A)\Delta_{24,24}(A) - \Delta_{12,12}(A)\Delta_{34,34}(A),$$

for $A = H_{5332}$ and $A = H_{6643/311}$, respectively.

Fact: The symmetric function $(*)$ is SNN for *every* Jacobi-Trudi matrix $A = H_{\kappa/\lambda}$.

Schur nonnegative polynomials

Definition: Call $p(x_{1,1}, \ldots, x_{n,n}) \in \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ a *Schur nonnegative* (SNN) polynomial if for every JT matrix A, the symmetric function $p(A)$ is SNN.

Theorem: (Hard parts of proof by H '91, D '92) Each element of the dual canonical basis is SNN.

Observation: If p, q are SNN, so is $p + q$.

Corollary: Each element of the dual canonical cone is SNN.

Examples of SNN polynomials

(Conj. JS 90; Pf. MH 91) For any S_n character χ,

$$\mathrm{Imm}_\chi(x) = \sum_{w \in S_n} \chi(w) x_{1,w(1)} \cdots x_{n,w(n)} \quad \text{is SNN.}$$

(D-G-S 04) $\pi \leq \sigma$ in the Bruhat order iff

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \quad \text{is SNN.}$$

(R-S 04) For combinatorially specified I, I', J, J',

$$\Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x) \quad \text{is SNN.}$$

Pf: These polynomials belong to the dual canonical cone.