SCHUR NONNEGATIVE POLYNOMIALS

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Outline

(1) Schur nonnegative symmetric functions
(2) Schur nonnegative polynomials
(3) Kazhdan-Lusztig immanants
(4) Applications to products of Schur functions

Minors and Schur functions

Given sets I, I', define the (I, I') *minor* of a matrix A by

$$\Delta_{I,I'}(A) = \det(a_{i,j})_{i \in I, j \in I'}.$$

Example: Let $H_{5332} = \begin{bmatrix} h_5 & h_6 & h_7 & h_8 \\ h_2 & h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 & h_3 \\ 0 & 1 & h_1 & h_2 \end{bmatrix}$. Then we have

$$\Delta_{1234,1234}(H_{5332}) = \det(H_{5332}) = s_{5332},$$

$$\Delta_{13,14}(H_{5332}) = \det \begin{bmatrix} h_5 & h_8 \\ h_1 & h_3 \end{bmatrix} = s_{73/2}.$$

Schur nonnegative symmetric functions

Call a symmetric function *Schur nonnegative* (SNN) if its Schur expansion has nonnegative coefficients.

Examples:

$$s_\kappa s_\lambda = \sum_\rho c^{\rho}_{\kappa\lambda} s_\rho,$$

$$s_{\rho/\kappa} = \sum_\lambda c^{\rho}_{\kappa\lambda} s_\lambda.$$

Questions: When are the following functions SNN?

(F-F-L-P) $\quad s_\kappa s_\lambda - s_\mu s_\nu$

(B-B-R) $\quad s_{\kappa/\alpha} s_{\lambda/\beta} - s_{\mu/\gamma} s_{\nu/\delta}$

(more generally) $\quad s_{\kappa/\alpha} - s_{\mu/\gamma}$

Littlewood-Richardson coefficients

Question: Which $\kappa, \lambda, \mu, \nu$ satisfy $c_{\kappa\lambda}^{\rho} \geq c_{\mu\nu}^{\rho}$ for all ρ?

LR coefficients appear in representations of S_n,

$$(S^{\kappa} \otimes S^{\lambda})\uparrow^{S_n} = \bigoplus_{\rho} c_{\kappa\lambda}^{\rho} S^{\rho},$$

in representations of GL_n,

$$V^{\kappa} \otimes V^{\lambda} = \bigoplus_{\rho} c_{\kappa\lambda}^{\rho} V^{\rho},$$

in $H^*(Gr(p, \mathbb{C}^n))$,

$$\sigma_{\kappa}\sigma_{\lambda} = \sum_{\rho} c_{\kappa\lambda}^{\rho} \sigma_{\rho}.$$

Examples of SNN symmetric functions

$$s_{63/1}s_{42/1} - s_{53}s_{32} = s_{5431} + 4s_{643} + 6s_{841} + \cdots$$
$$s_{63/3}s_{73/2} - s_{55/2}s_{43/1} = 9s_{743} + 3s_{9221} + s_{8222} + \cdots$$

These are

(∗) $\Delta_{13,13}(A)\Delta_{24,24}(A) - \Delta_{12,12}(A)\Delta_{34,34}(A)$,

for $A = H_{5332}$ and $A = H_{6643/311}$, respectively.

Fact: The symmetric function (∗) is SNN for *every* Jacobi-Trudi matrix $A = H_{\kappa/\alpha}$.

Monomial, Schur nonnegative polynomials

For a polynomial $p(x_{1,1}, \ldots, x_{n,n})$ and matrix $A = (a_{i,j})$, define

$$p(A) = p(a_{1,1}, \ldots, a_{n,n}).$$

Definition: Call p a *MNN polynomial* if for every JT matrix A, the symmetric function $p(A)$ is MNN.

Definition: Call p a *SNN polynomial* if for every JT matrix A, the symmetric function $p(A)$ is SNN.

Question: Which polynomials are MNN? (SNN?)

Observation: If p, q are MNN (SNN), so is $p + q$.

Defn: Call $\sum_{w} f(w) x_{1,w(1)} \cdots x_{n,w(n)}$ an *immanant.*

Fact: (Conj. G-J 89; Pf. CG 91)

$$\mathrm{Imm}_\lambda(x) = \sum_{w \in S_n} \chi^\lambda(w) x_{1,w(1)} \cdots x_{n,w(n)} \quad \text{is MNN.}$$

Fact: (Conj. JS 91; Pf. MH 92)

$$\mathrm{Imm}_\lambda(x) \quad \text{is SNN.}$$

Fact: (Pf. D-G-S 04) The following are equivalent.

(1) $\pi \leq \sigma$ in the Bruhat order.

(2) $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is SNN.

(3) $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is MNN.

Question: (F-G-J 03) What about immanants of the form

$$p(x) = \Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)?$$

Answer: Associate to the products of minors

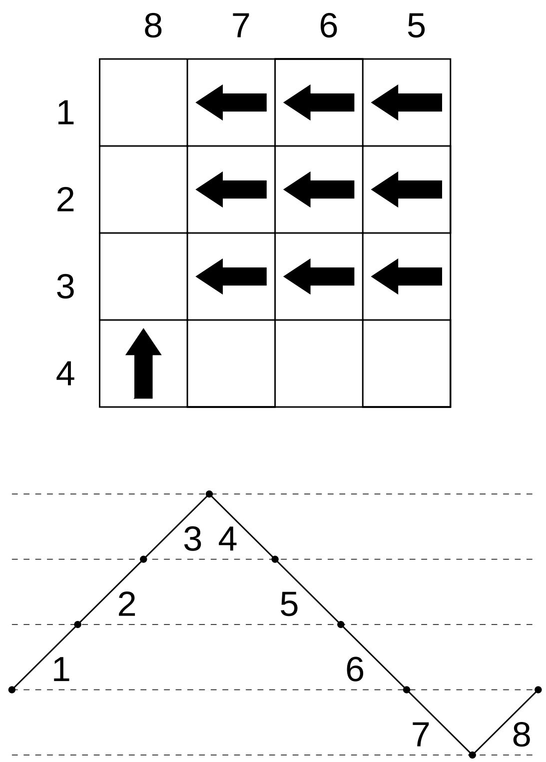
$$\Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x), \qquad \Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x),$$

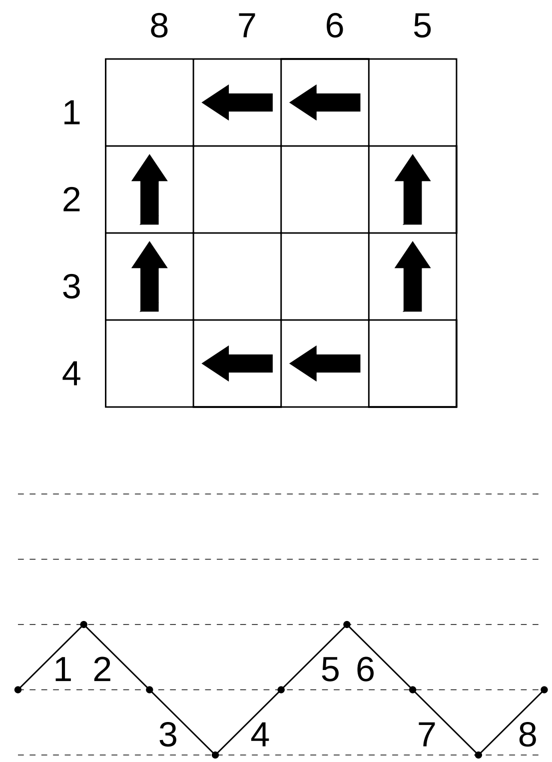
the set partitions

$$\Pi(I, I'), \qquad \Pi(J, J').$$

Theorem: (Rhoades-S 04) If $\Pi(I, I')$ refines $\Pi(J, J')$, then $p(x)$ is SNN.

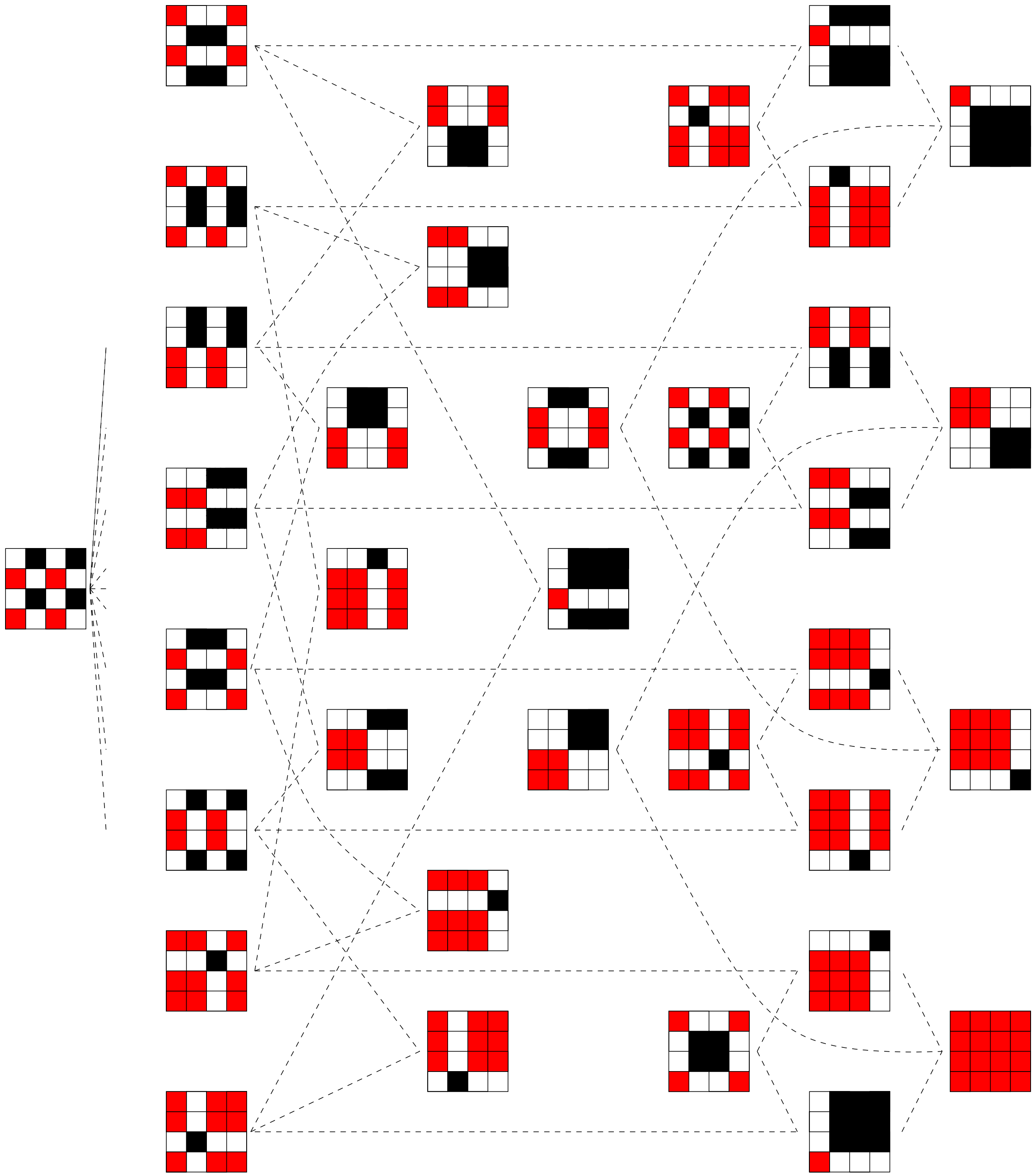
$$\Delta_{123,234}(x)\Delta_{4,1}(x) \qquad \Delta_{14,23}(x)\Delta_{23,14}(x)$$

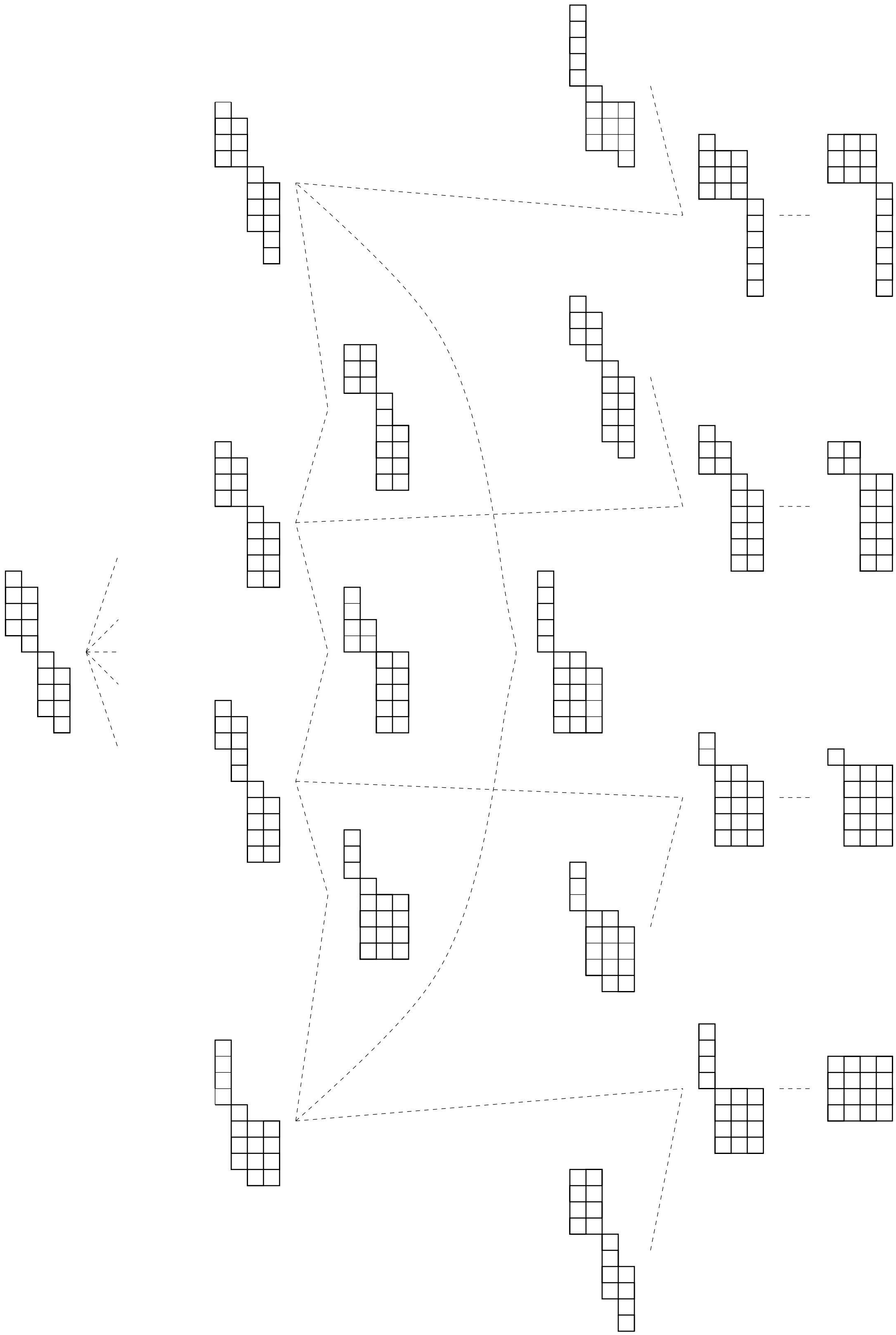




$$\Pi(123,234) = 16|25|34|78, \qquad \Pi(14,23) = 1256|3478.$$

Thus, $\Delta_{14,23}(x)\Delta_{23,14}(x) - \Delta_{123,234}(x)\Delta_{4,1}(x)$ is SNN.





Kazhdan-Lusztig immanants

Let $P_{w,w'}(q)$ be the KL polynomial indexed by $w, w' \in S_n$.
Let w_0 be the longest permutation in S_n.

For each $u \in S_n$, define $f_u : S_n \to \mathbb{Z}$ by

$$f_u(v) = (-1)^{\ell(uv)} P_{w_0 v, w_0 u}(1).$$

Define the *Kazhdan-Lusztig immanant* $\mathrm{Imm}_u(x)$ by

$$\mathrm{Imm}_u(x) = \sum_{v \geq u} f_u(v) x_{1,v(1)} \cdots x_{n,v(n)}.$$

Remark: $\det(x) = \mathrm{Imm}_1(x)$.

Define a bilinear form $\langle\ ,\ \rangle$ on

$$\mathrm{span}_{\mathbb{C}}\{x_{1,w(1)} \cdots x_{n,w(n)} \,|\, w \in S_n\} \times \mathbb{C}[S_n]$$

by

$$\langle x_{1,u(1)} \cdots x_{n,u(n)}, v\rangle = \delta_{u,v}.$$

Then we have

$$\langle \mathrm{Imm}_u(x), C'_v(1)\rangle = \delta_{u,v},$$

where

$$C'_v(1) = \sum_{u \leq v} P_{u,v}(1) u$$

is the Kazhdan-Lusztig basis element indexed by v.

Theorem: (MH 93) KL immanants are SNN.

Theorem: (MH 93) $\mathrm{Imm}_\lambda(x)$ is SNN.

Proof idea: $\mathrm{Imm}_\lambda(x)$ is a nonnegative linear combination of KL immanants.

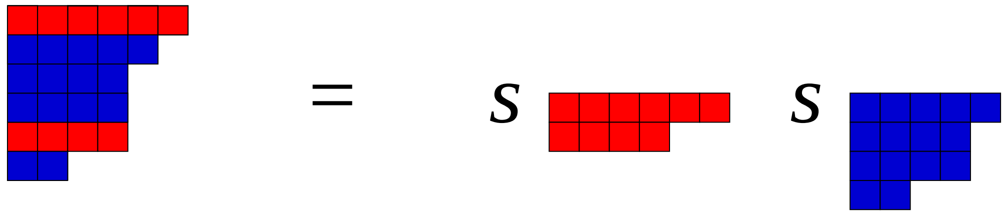
Theorem: (Rhoades-S 04) The polynomial

$$p(x) = \Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$$

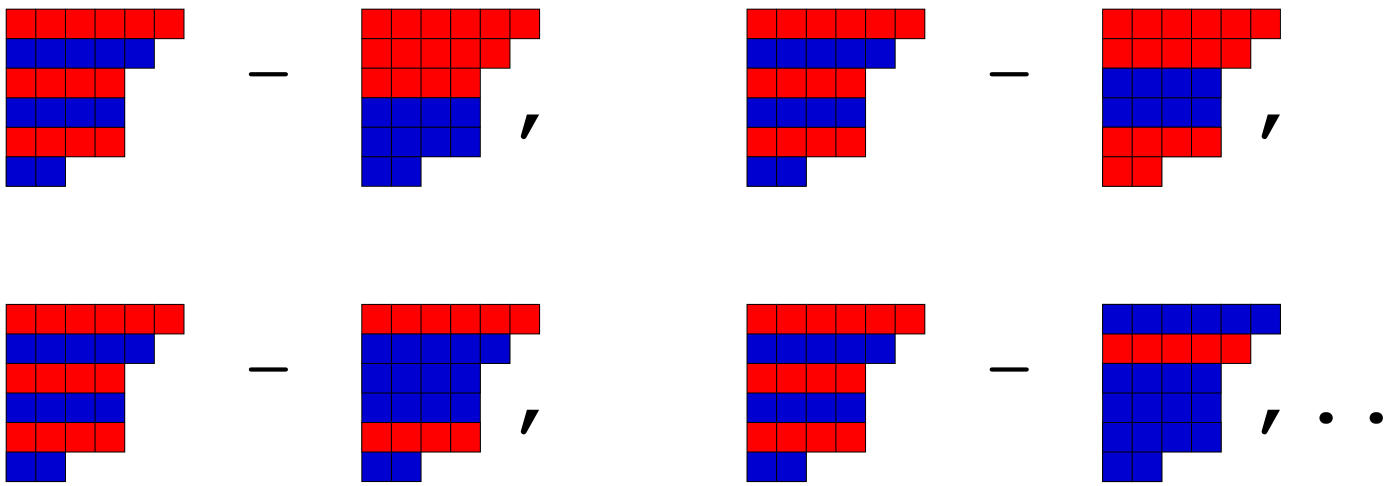
is SNN if $\Pi(I, I')$ refines $\Pi(J, J')$.

Proof idea: $p(x)$ is a nonnegative linear combination of KL immanants $\mathrm{Imm}_w(x)$, with w 321-avoiding.
(Uses Temperley-Lieb algebra.)

Notation:



Conjecture: (F-F-L-P) These symmetric functions are SNN.



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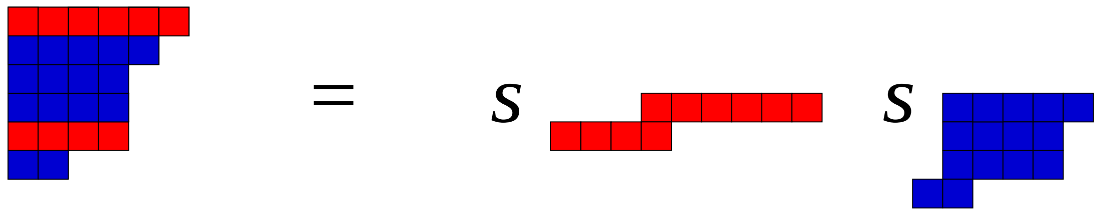
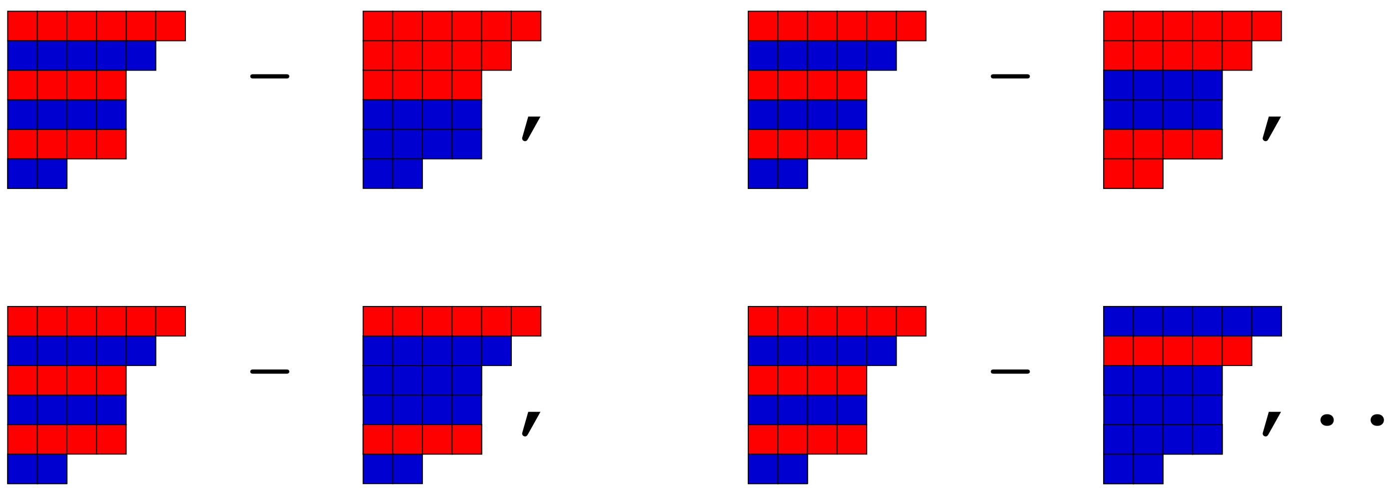


Fact: (Lewicke-S) These symmetric functions are SNN.



Open problems

Problem: Use results to describe families of SNN symmetric functions of the form $s_{\kappa/\alpha}s_{\lambda/\beta} - s_{\mu/\gamma}s_{\nu/\delta}$ without explicitly mentioning minors.

Problem: Use results to prove special cases of the conjectures of F-F-L-P and B-B-R.

Problem: Extend results to products of 3 or more minors.

Problem: Describe the extremal rays of the cone of SNN immanants. Which of these rays are KL immanants?

Question: Is this cone different from the cone of MNN (TNN) immanants?