

# SCHUR NONNEGATIVE POLYNOMIALS

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## Outline

- (1) Schur nonnegative symmetric functions
- (2) Schur nonnegative polynomials
- (3) Kazhdan-Lusztig immanants
- (4) Applications to products of Schur functions

## Minors and Schur functions

Given sets  $I, I'$ , define the  $(I, I')$  *minor* of a matrix  $A$  by

$$\Delta_{I, I'}(A) = \det(a_{i, j})_{i \in I, j \in I'}.$$

**Example:** Let  $H_{5332} = \begin{bmatrix} h_5 & h_6 & h_7 & h_8 \\ h_2 & h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 & h_3 \\ 0 & 1 & h_1 & h_2 \end{bmatrix}$ . Then we have

$$\Delta_{1234, 1234}(H_{5332}) = \det(H_{5332}) = s_{5332},$$

$$\Delta_{13, 14}(H_{5332}) = \det \begin{bmatrix} h_5 & h_8 \\ h_1 & h_3 \end{bmatrix} = s_{73/2}.$$

## Schur nonnegative symmetric functions

Call a symmetric function *Schur nonnegative* (SNN) if its Schur expansion has nonnegative coefficients.

**Examples:**

$$s_{\kappa}s_{\lambda} = \sum_{\rho} c_{\kappa\lambda}^{\rho} s_{\rho},$$
$$s_{\rho/\kappa} = \sum_{\lambda} c_{\kappa\lambda}^{\rho} s_{\lambda}.$$

**Questions:** When are the following functions SNN?

(F-F-L-P)

$$s_{\kappa}s_{\lambda} - s_{\mu}s_{\nu}$$

(B-B-R)

$$s_{\kappa/\alpha}s_{\lambda/\beta} - s_{\mu/\gamma}s_{\nu/\delta}$$

(more generally)

$$s_{\kappa/\alpha} - s_{\mu/\gamma}$$

## Littlewood-Richardson coefficients

**Question:** Which  $\kappa, \lambda, \mu, \nu$  satisfy  $c_{\kappa\lambda}^\rho \geq c_{\mu\nu}^\rho$  for all  $\rho$ ?

LR coefficients appear in representations of  $S_n$ ,

$$(S^\kappa \otimes S^\lambda) \uparrow^{S_n} = \bigoplus_{\rho} c_{\kappa\lambda}^\rho S^\rho,$$

in representations of  $GL_n$ ,

$$V^\kappa \otimes V^\lambda = \bigoplus_{\rho} c_{\kappa\lambda}^\rho V^\rho,$$

in  $H^*(Gr(p, \mathbb{C}^n))$ ,

$$\sigma_\kappa \sigma_\lambda = \sum_{\rho} c_{\kappa\lambda}^\rho \sigma_\rho.$$

## Examples of SNN symmetric functions

$$\begin{aligned}
 s_{63/1} s_{42/1} - s_{53} s_{32} &= s_{5431} + 4s_{643} + 6s_{841} + \cdots \\
 s_{63/3} s_{73/2} - s_{55/2} s_{43/1} &= 9s_{743} + 3s_{9221} + s_{8222} + \cdots
 \end{aligned}$$

These are

$$(*) \quad \Delta_{13,13}(A) \Delta_{24,24}(A) - \Delta_{12,12}(A) \Delta_{34,34}(A),$$

for  $A = H_{5332}$  and  $A = H_{6643/311}$ , respectively.

**Fact:** The symmetric function  $(*)$  is SNN for *every* Jacobi-Trudi matrix  $A = H_{\kappa/\alpha}$ .

## Monomial, Schur nonnegative polynomials

For a polynomial  $p(x_{1,1}, \dots, x_{n,n})$  and matrix  $A = (a_{i,j})$ , define

$$p(A) = p(a_{1,1}, \dots, a_{n,n}).$$

**Definition:** Call  $p$  a *MNN polynomial* if for every JT matrix  $A$ , the symmetric function  $p(A)$  is MNN.

**Definition:** Call  $p$  a *SNN polynomial* if for every JT matrix  $A$ , the symmetric function  $p(A)$  is SNN.

**Question:** Which polynomials are MNN? (SNN?)

**Observation:** If  $p, q$  are MNN (SNN), so is  $p + q$ .

**Defn:** Call  $\sum_w f(w)x_{1,w(1)} \cdots x_{n,w(n)}$  an *immanant*.

**Fact:** (Conj. G-J 89; Pf. CG 91)

$$\text{Imm}_\lambda(x) = \sum_{w \in S_n} \chi^\lambda(w)x_{1,w(1)} \cdots x_{n,w(n)} \quad \text{is MNN.}$$

**Fact:** (Conj. JS 91; Pf. MH 92)

$$\text{Imm}_\lambda(x) \quad \text{is SNN.}$$

**Fact:** (Pf. D-G-S 04) The following are equivalent.

- (1)  $\pi \leq \sigma$  in the Bruhat order.
- (2)  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$  is SNN.
- (3)  $x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$  is MNN.

**Question:** (F-G-J 03) What about immanants of the form

$$p(x) = \Delta_{J,J'}(x)\Delta_{\bar{J},\bar{J}'}(x) - \Delta_{I,I'}(x)\Delta_{\bar{I},\bar{I}'}(x)?$$

**Answer:** Associate to the products of minors

$$\Delta_{I,I'}(x)\Delta_{\bar{I},\bar{I}'}(x), \quad \Delta_{J,J'}(x)\Delta_{\bar{J},\bar{J}'}(x),$$

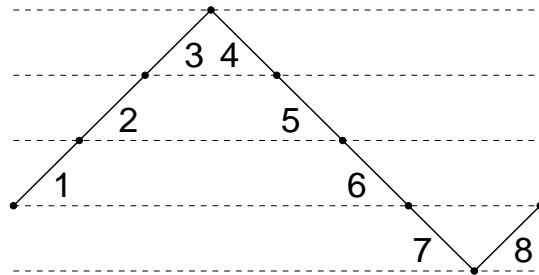
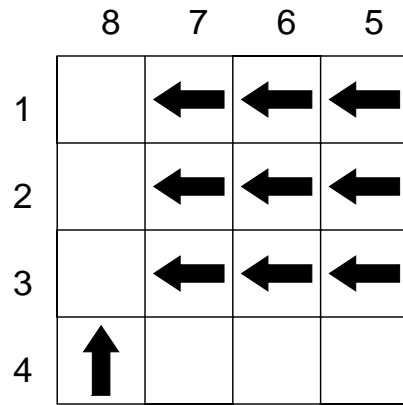
the set partitions

$$\Pi(I, I'), \quad \Pi(J, J').$$

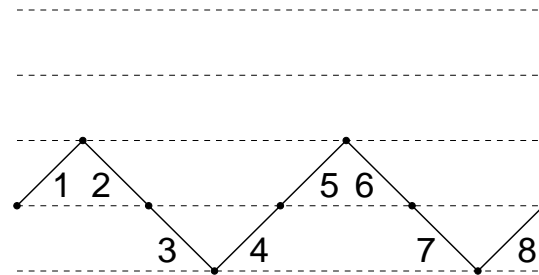
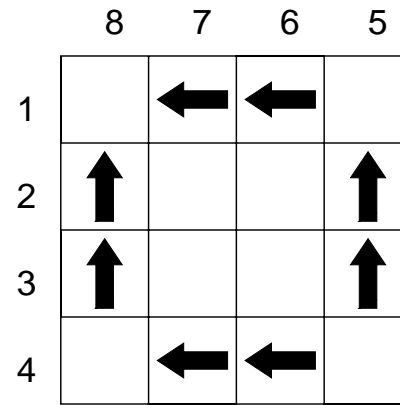
**Theorem:** (Rhoades-S 04) If  $\Pi(I, I')$  refines  $\Pi(J, J')$ , then  $p(x)$  is SNN.



$$\Delta_{123,234}(x)\Delta_{4,1}(x)$$



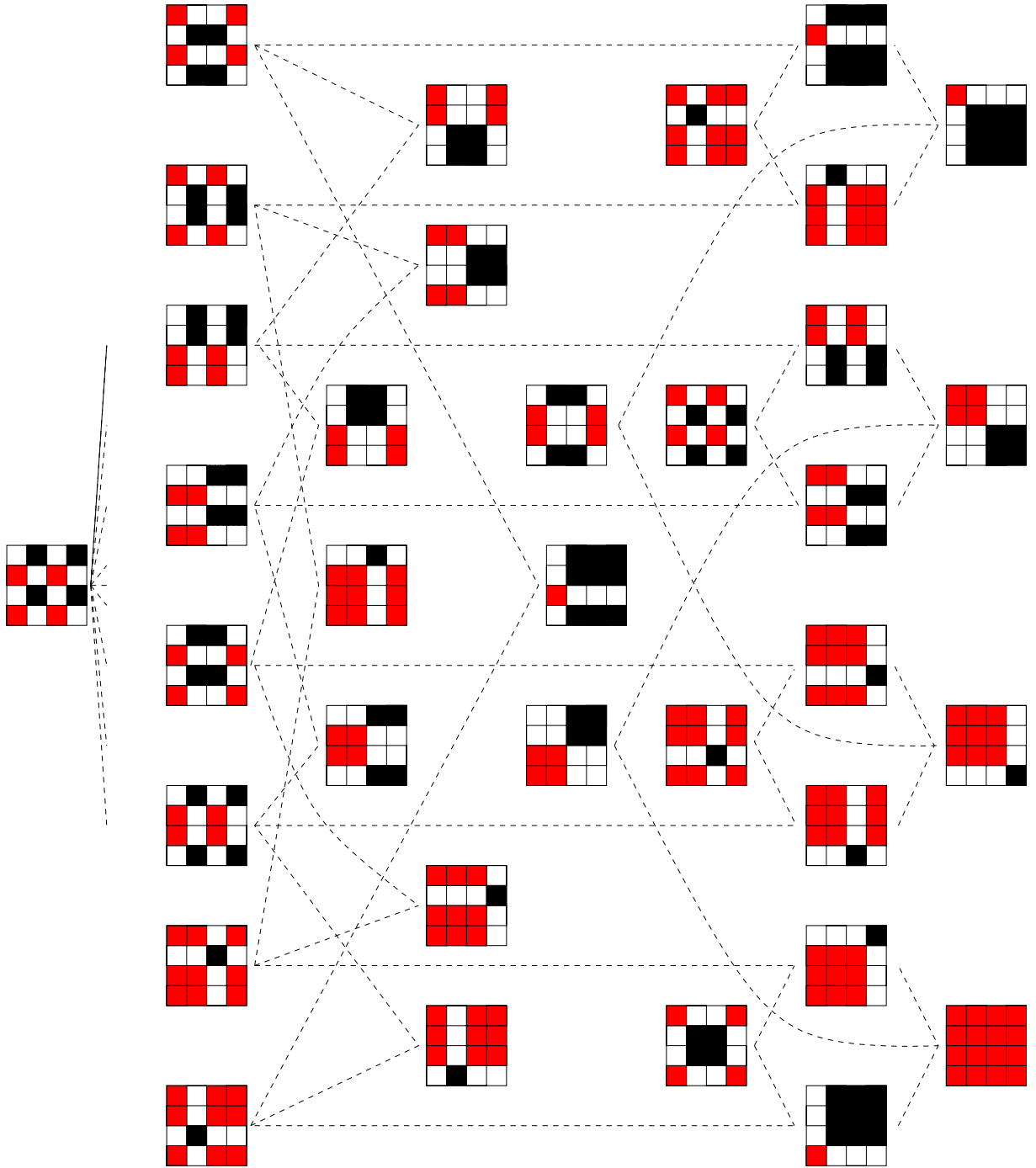
$$\Delta_{14,23}(x)\Delta_{23,14}(x)$$

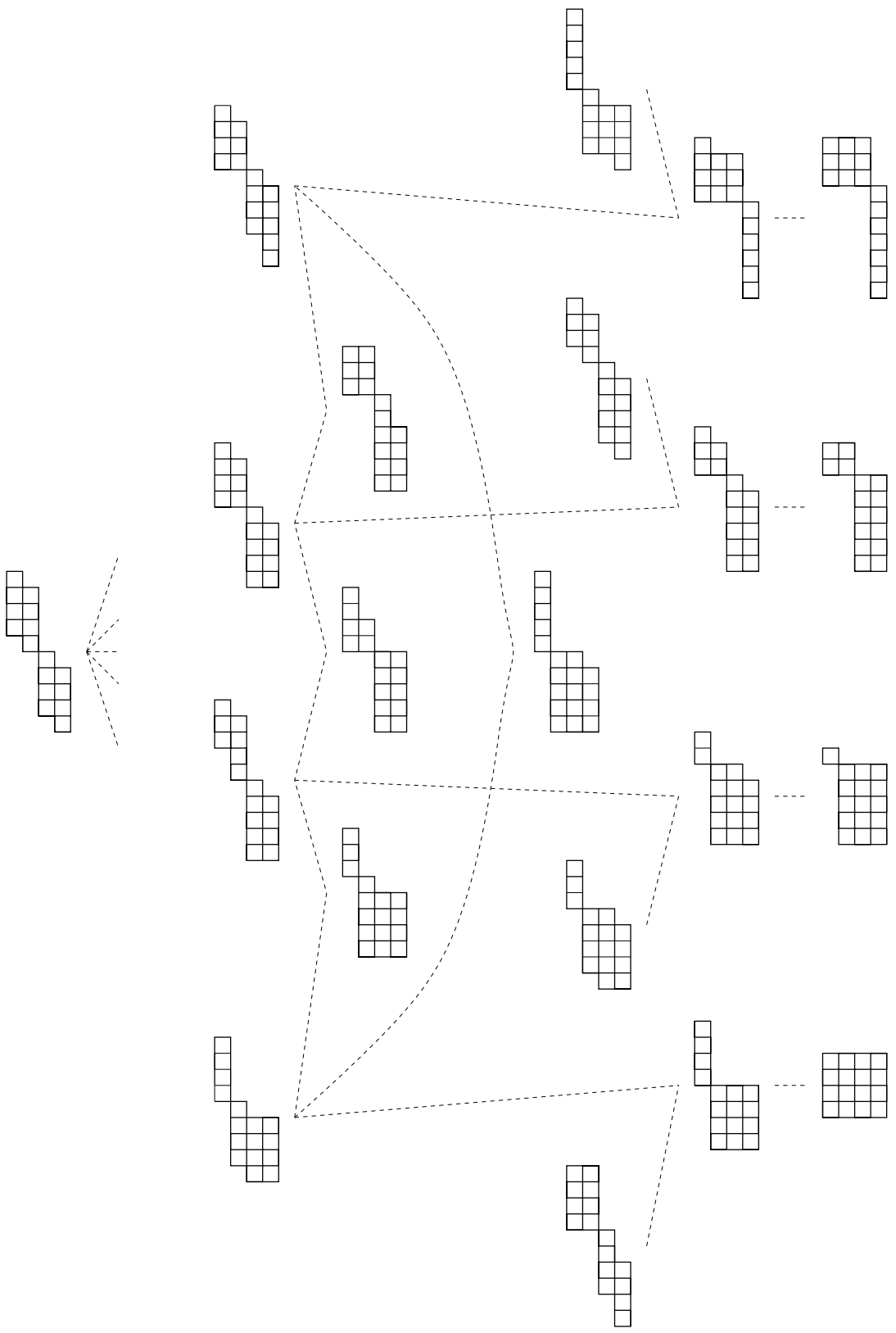


$$\Pi(123, 234) = 16|25|34|78,$$

$$\Pi(14, 23) = 1256|3478.$$

Thus,  $\Delta_{14,23}(x)\Delta_{23,14}(x) - \Delta_{123,234}(x)\Delta_{4,1}(x)$  is SNN.





## Kazhdan-Lusztig immanants

Let  $P_{w,w'}(q)$  be the KL polynomial indexed by  $w, w' \in S_n$ .  
Let  $w_0$  be the longest permutation in  $S_n$ .

For each  $u \in S_n$ , define  $f_u : S_n \rightarrow \mathbb{Z}$  by

$$f_u(v) = (-1)^{\ell(uv)} P_{w_0v, w_0u}(1).$$

Define the *Kazhdan-Lusztig immanant*  $\text{Imm}_u(x)$  by

$$\text{Imm}_u(x) = \sum_{v \geq u} f_u(v) x_{1,v(1)} \cdots x_{n,v(n)}.$$

Remark:  $\det(x) = \text{Imm}_1(x)$ .

Define a bilinear form  $\langle \cdot, \cdot \rangle$  on

$$\text{span}_{\mathbb{C}}\{x_{1,w(1)} \cdots x_{n,w(n)} \mid w \in S_n\} \times \mathbb{C}[S_n]$$

by

$$\langle x_{1,u(1)} \cdots x_{n,u(n)}, v \rangle = \delta_{u,v}.$$

Then we have

$$\langle \text{Imm}_u(x), C'_v(1) \rangle = \delta_{u,v},$$

where

$$C'_v(1) = \sum_{u \leq v} P_{u,v}(1)u$$

is the Kazhdan-Lusztig basis element indexed by  $v$ .

**Theorem:** (MH 93) KL immanants are SNN.

**Theorem:** (MH 93)  $\text{Imm}_\lambda(x)$  is SNN.

Proof idea:  $\text{Imm}_\lambda(x)$  is a nonnegative linear combination of KL immanants.

**Theorem:** (Rhoades-S 04) The polynomial

$$p(x) = \Delta_{J,J'}(x)\Delta_{\overline{J},\overline{J'}}(x) - \Delta_{I,I'}(x)\Delta_{\overline{I},\overline{I'}}(x)$$

is SNN if  $\Pi(I, I')$  refines  $\Pi(J, J')$ .

Proof idea:  $p(x)$  is a nonnegative linear combination of KL immanants  $\text{Imm}_w(x)$ , with  $w$  321-avoiding.

(Uses Temperley-Lieb algebra.)

Notation:

The diagram shows a Young diagram on the left, which is the product of two Young diagrams on the right. The left diagram has 5 rows: the first row is red (5 boxes), the second and third rows are blue (4 and 3 boxes respectively), the fourth row is red (3 boxes), and the fifth row is blue (2 boxes). This is equal to the Schur functor  $S$  applied to a red Young diagram (1 row of 5 boxes) multiplied by the Schur functor  $S$  applied to a blue Young diagram (3 rows: 3, 2, 1 boxes).

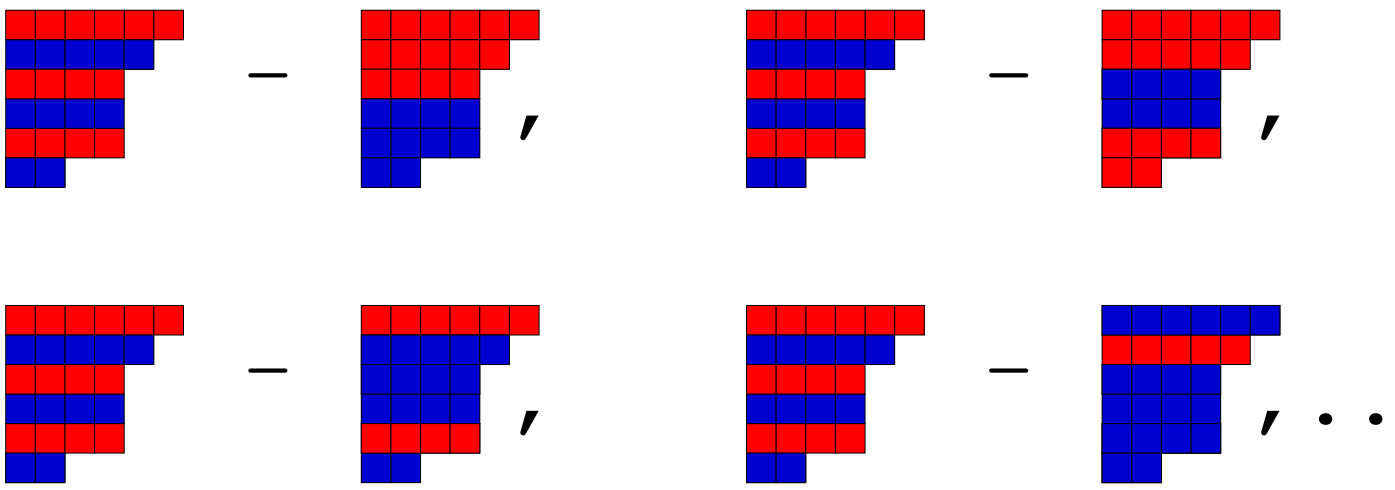
**Conjecture:** (F-F-L-P) These symmetric functions are SNN.

The diagram shows two rows of Young diagrams. Each diagram is a product of two Young diagrams, with a minus sign between them and a slash at the end. The first row shows two examples of such products. The second row shows two examples, with the second one followed by three dots, indicating a sequence of operations.

Notation:

The diagram shows an equality between two 5x5 grids. The left grid has a red top row, a blue second row, a red third row, a blue fourth row, and a red bottom row. This is equal to the symmetric function  $S$  applied to a grid with a red top row and a blue second row, followed by  $S$  applied to a grid with a blue top row and a red second row.

**Fact:** (Lewicke-S) These symmetric functions are SNN.





## Open problems

**Problem:** Use results to describe families of SNN symmetric functions of the form  $s_{\kappa/\alpha} s_{\lambda/\beta} - s_{\mu/\gamma} s_{\nu/\delta}$  without explicitly mentioning minors.

**Problem:** Use results to prove special cases of the conjectures of F-F-L-P and B-B-R.

**Problem:** Extend results to products of 3 or more minors.

**Problem:** Describe the extremal rays of the cone of SNN immanants. Which of these rays are KL immanants?

**Question:** Is this cone different from the cone of MNN (TNN) immanants?