COMBINATORIAL INTERPRETATIONS OF KAZHDAN-LUSZTIG BASIS ELEMENTS INDEXED BY 45312-AVOIDING PERMUTATIONS IN S_6

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Outline

(1) The symmetric group and pattern avoidance
(2) The Hecke algebra and Kazhdan-Lusztig bases
(3) Planar networks and path families
(4) Combinatorial interpretations of Kazhdan-Lusztig basis elements

The symmetric group S_n and group algebra $\mathbb{Z}[S_n]$

Generators: $s_1, \ldots, s_{n-1}$.

Relations:

$$\begin{aligned} s_i^2 &= e && \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j| = 1, \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2. \end{aligned}$$

Call $s_{i_1} \cdots s_{i_\ell}$ *reduced* if it is equal to no shorter product; call ℓ the *length* of this element of S_n.

$\mathbb{Z}[S_n] = \mathbb{Z}$-linear combinations of S_n-elements, e.g.,

$$1 - s_1 + s_2 + 2s_1 s_2 - s_2 s_1 + 4 s_1 s_2 s_1.$$

One-line notation and pattern avoidance

Let s_i act on words by swapping letters in positions $i, i+1$,

$$s_1 s_2(1234) = s_1(1324) = 3124.$$

Say that $w \in S_n$ *avoids the pattern* $a_1 \cdots a_k$ if no subword consists of letters in the same relative order.

45321 avoids the patterns 3412 and 4231;	45312 does not.
45123 avoids the pattern 321;	45321 does not.
45321 avoids the pattern 56781234;	so do all $w \in S_5$.

Call $w \in S_n$ *321-hexagon-avoiding* if it avoids the patterns 321, 56781234, 46781235, 56718234, 46718235.

The Hecke algebra $H_n(q)$

Generators over $\mathbb{Z}[q]$: $T_{s_1}, \ldots, T_{s_{n-1}}$.

Relations:

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e && \text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} && \text{for } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} && \text{for } |i-j| \geq 2. \end{aligned}$$

Natural basis: $\{T_w \,|\, w \in S_n\}$,

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}, \qquad (w = s_{i_1} \cdots s_{i_\ell} \text{ reduced}); \qquad T_e = 1.$$

$H_n(1) \cong \mathbb{Z}[S_n]$.

Typical elements look like

$$1 - T_{s_1} + qT_{s_2} + (1+q)T_{s_1 s_2} - q^2 T_{s_2 s_1} + (1+q)^2 T_{s_1 s_2 s_1}.$$

The Kazhdan-Lusztig basis of $H_n(q)$

Quantum groups lead to another basis $\{\widetilde{C}_w(q) \,|\, w \in S_n\}$,

$$\widetilde{C}_w(q) = \sum_{v \leq w} P_{v,w}(q) T_v; \qquad \widetilde{C}_e(q) = 1,$$

where $\leq$ is the Bruhat order, and coefficients $P_{v,w}(q) \in \mathbb{N}[q]$ are the recursively defined *Kazhdan-Lusztig* polynomials.

For example we have

$$\widetilde{C}_{321}(q) = 1 + T_{s_1} + T_{s_2} + T_{s_1 s_2} + T_{s_2 s_1} + T_{s_1 s_2 s_1}.$$

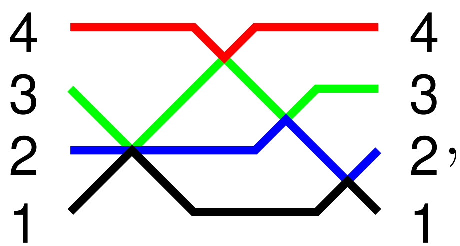
Question: Can we graphically encode $\{\widetilde{C}_w(q) \,|\, w \in S_n\}$ to avoid recursive definitions?

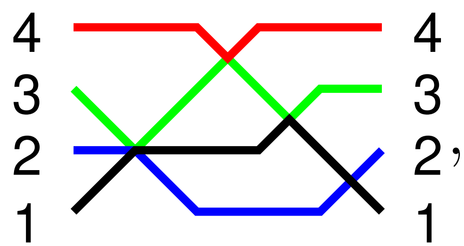
Path families in a planar network G

Let G be a planar acyclic network with n sources, n sinks, and edges oriented from left to right. Say that a family $\pi = (\pi_1, \ldots, \pi_n)$ of paths *covers* G and has *type* $w = w_1 \cdots w_n$ if

- π uses all edges of G,
- π_i is a path from source i on left to sink w_i on right.

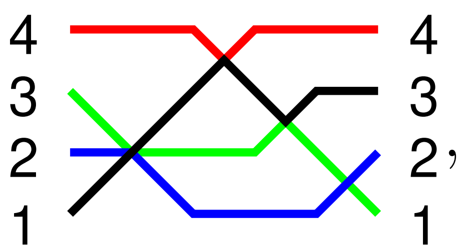
Example:

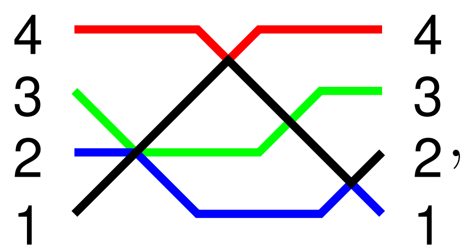
$\rho =$ 

$\sigma =$ 

$\text{type}(\rho) = 1234,$

$\text{type}(\sigma) = 1234,$

$\tau =$ 

$\omega =$ 

$\text{type}(\tau) = 3214,$

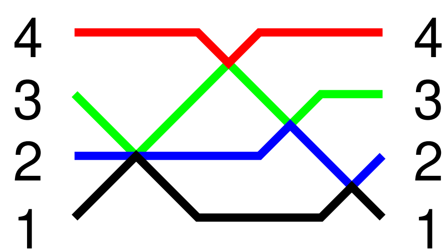
$\text{type}(\omega) = 2134.$

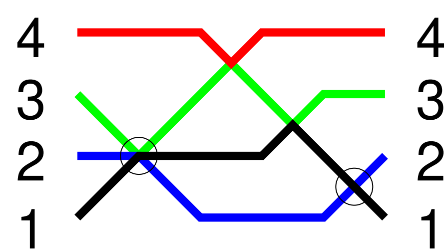
Defects in a path family

Call a point of intersection of two paths a *defect* if the paths have previously crossed an odd number of times.

Define $\mathrm{dfct}(\pi)$ to be the number of defects in path family π.

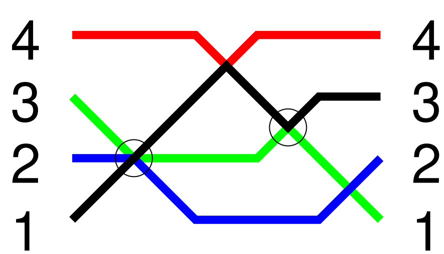
Example:

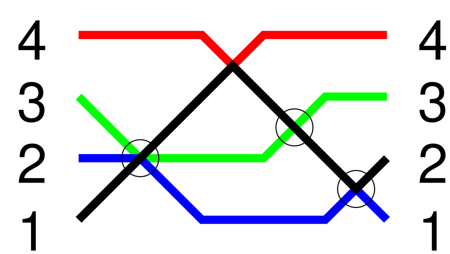
$\rho =$  ,

$\sigma =$  ,

$\mathrm{dfct}(\rho) = 0,$

$\mathrm{dfct}(\sigma) = 1,$

$\tau =$  ,

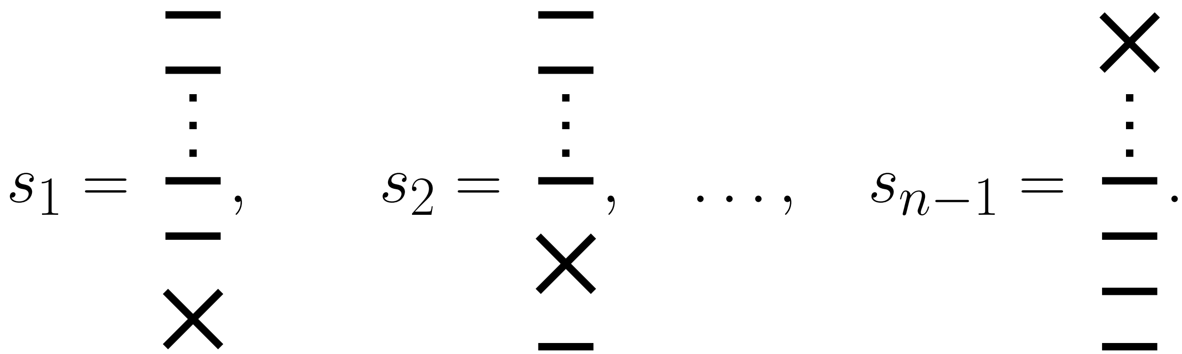
$\omega =$  ,

$\mathrm{dfct}(\tau) = 1,$

$\mathrm{dfct}(\omega) = 2.$

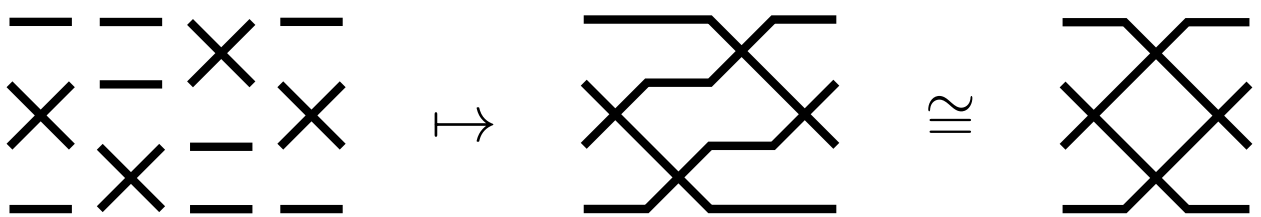
Wiring diagrams

To S_n generators $s_1, \ldots, s_{n-1}$, associate networks



(edges oriented left to right) and multiply by concatenation.

Example: To $s_2 s_1 s_3 s_2 \in S_4$, associate wiring diagram



At most two paths intersect at a vertex.

Wiring diagram theorem (Deodhar-Billey-Warrington)

Given planar network G define the element

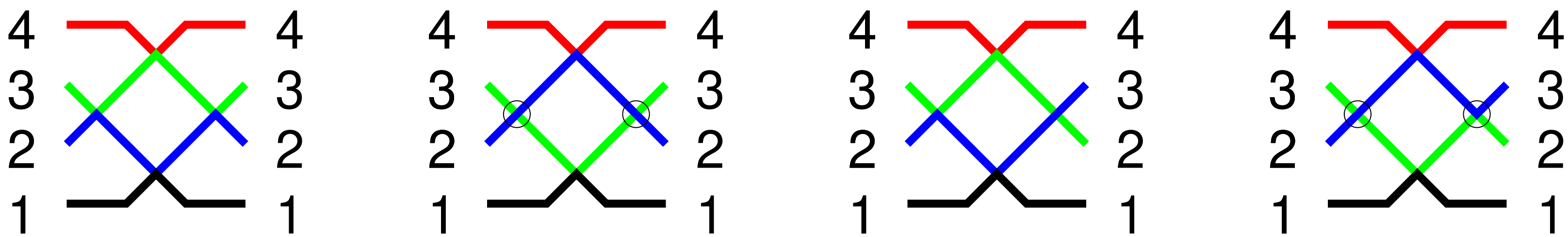
$$z = z(G) = \sum_{\pi} q^{\mathrm{dfct}(\pi)} T_{\mathrm{type}(\pi)} \in H_n(q),$$

where the sum is over all path families which cover G, and say that G *graphically represents* z.

Theorem: (Deodhar-Billey-Warrington) Let G be the wiring diagram of any reduced expression for the 321-hexagon-avoiding permutation $w \in S_n$. Then G graphically represents $\widetilde{C}_w(q)$.

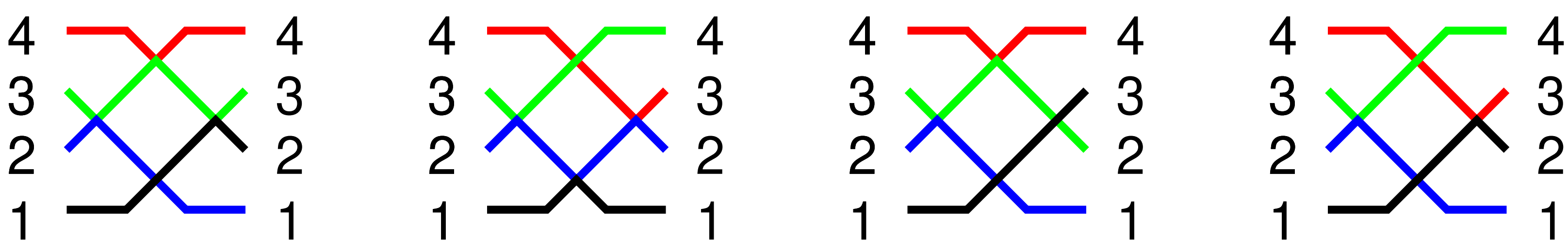
Example: To graphically represent $\widetilde{C}_{3412}(q)$, start with the reduced expression $s_2 s_1 s_3 s_2$ and its wiring diagram.

Sum contributions $q^{\mathrm{dfct}(\pi)} T_{\mathrm{type}(\pi)}$ for all path families.



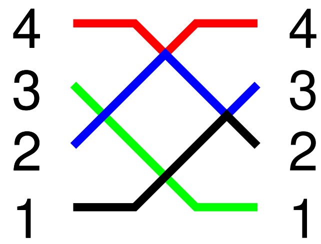
$(0, e)$ $(1, e)$ $(0, s_2)$ $(1, s_2)$

Contributions: $T_e + qT_e + T_{s_2} + qT_{s_2}$.

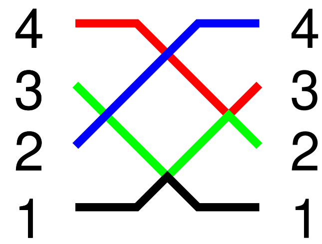


$(0, s_1)$ $(0, s_3)$ $(0, s_1 s_2)$ $(0, s_1 s_3)$

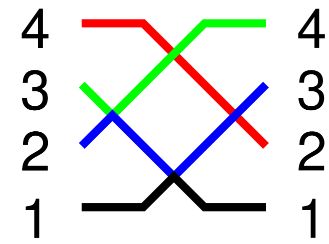
Contributions: $T_{s_1} + T_{s_3} + T_{s_1 s_2} + T_{s_1 s_3}$.



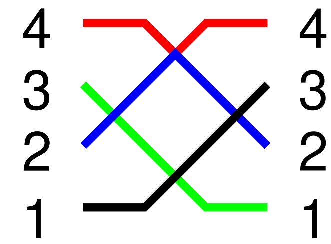
$(0, s_2 s_1)$



$(0, s_2 s_3)$

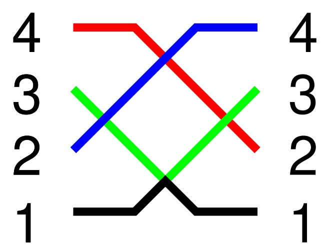


$(0, s_3 s_2)$

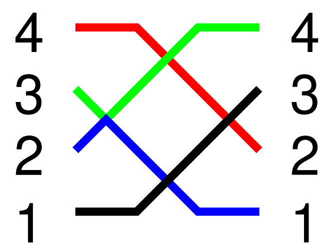


$(0, s_1 s_2 s_1)$

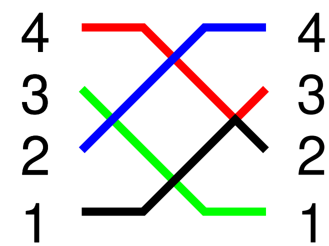
Contributions: $T_{s_2 s_1} + T_{s_2 s_3} + T_{s_3 s_2} + T_{s_1 s_2 s_1}$.



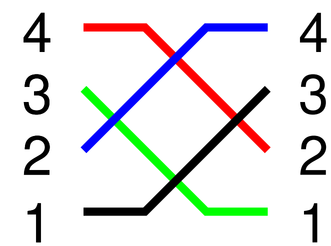
$(0, s_2 s_3 s_2)$



$(0, s_1 s_3 s_2)$



$(0, s_2 s_1 s_3)$



$(0, s_2 s_1 s_3 s_2)$

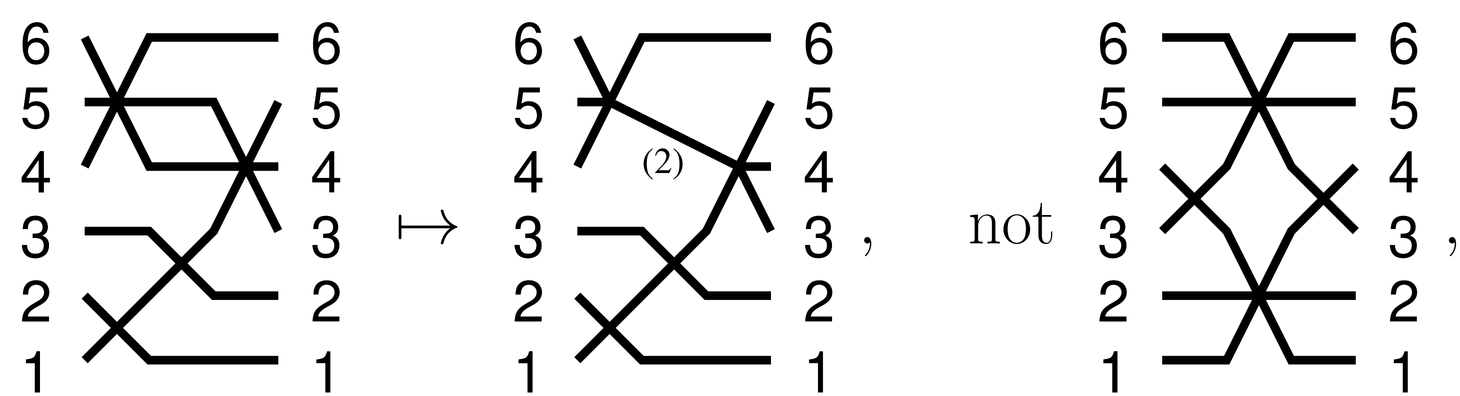
Contributions: $T_{s_2 s_3 s_2} + T_{s_1 s_3 s_2} + T_{s_2 s_1 s_3} + T_{s_2 s_1 s_3 s_2}$.

$$\begin{aligned}\widetilde{C}_{3412}(q) =& (1+q)T_e + T_{s_1} + (1+q)T_{s_2} + T_{s_3} \\ &+ T_{s_1 s_2} + T_{s_1 s_3} + T_{s_2 s_1} + T_{s_2 s_3} + T_{s_3 s_2} \\ &+ T_{s_1 s_2 s_1} + T_{s_2 s_3 s_2} + T_{s_1 s_3 s_2} + T_{s_2 s_1 s_3} + T_{s_2 s_1 s_3 s_2}.\end{aligned}$$

Zig-zag network theorem (S)

Planar networks called *star networks* are concatenations of stars in which two *or more* paths intersect at a vertex.

Certain star networks called *zig-zag networks* correspond bijectively to permutations avoiding the patterns 3412 and 4231.



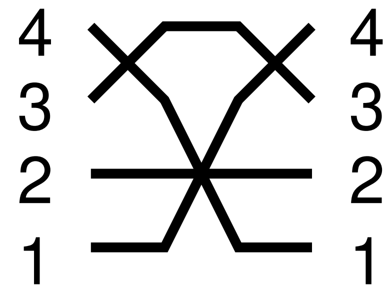
Theorem: Let zig-zag network G correspond to $w \in S_n$ avoiding the patterns 3412 and 4231. Then G graphically represents $\widetilde{C}_w(q)$.

A generalization

Theorem: If $w \in S_n$ avoids 321 and the hexagon patterns, *and* $3412, 4231$, then its zig-zag network and wiring diagram are equal.

Facts:

- The permutation $4231 \in S_4$ contains the pattern 321.
- It also contains the pattern 4231.
- Neither theorem applies to $\widetilde{C}_{4231}(q) \in H_4(q)$.
- Nevertheless, the star network



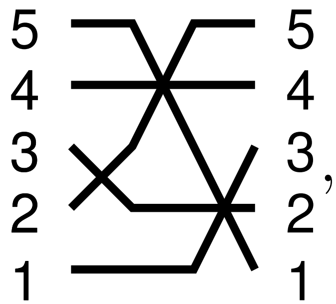
graphically represents $\widetilde{C}_{4231}(q)$.

Question: For which w can we graphically represent $\widetilde{C}_w(q)$?

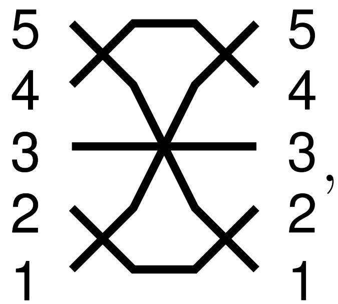
Restriction to S_5

Theorem: For all $w \in S_5$ except 45312, there is a planar network G which graphically represents $\widetilde{C}_w(q)$.

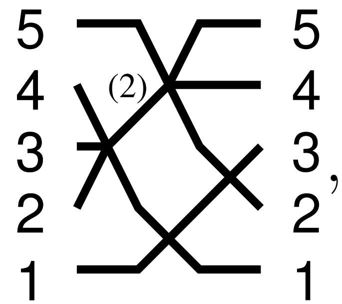
Examples:



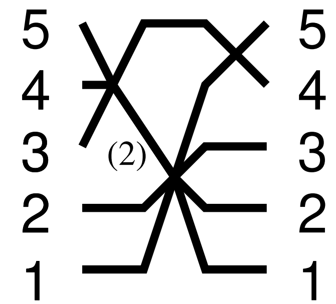
$\widetilde{C}_{35241}(q)$



$\widetilde{C}_{52341}(q)$



$\widetilde{C}_{35412}(q)$



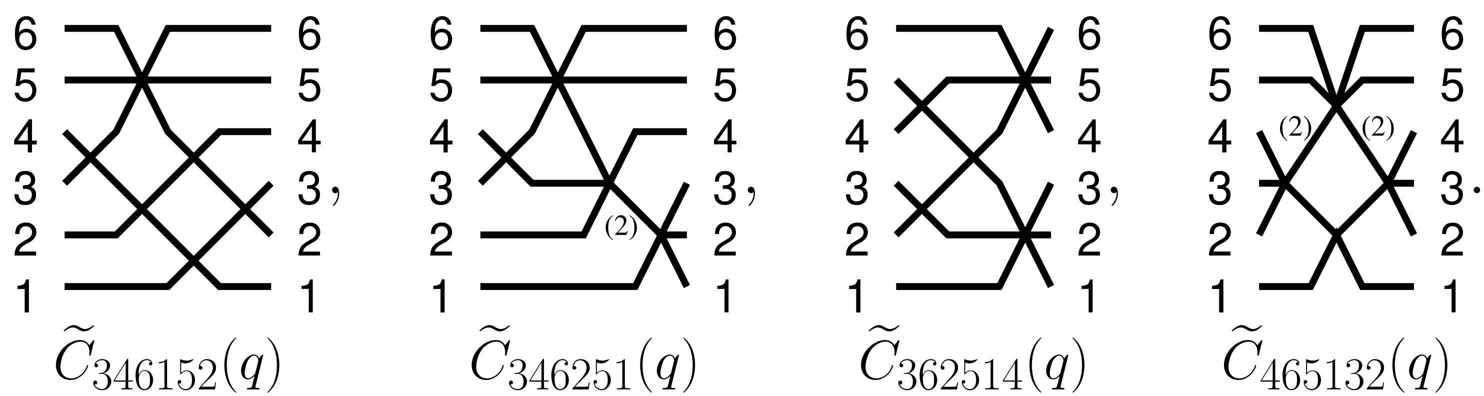
$\widetilde{C}_{53421}(q)$

There are 119 such permutations.

Restriction to S_6

Theorem: For all $w \in S_6$ avoiding the pattern 45312, there is a planar network G which graphically represents $\widetilde{C}_w(q)$.

Examples:



There are 694 such permutations.

Question: For all 45312-hexagon avoiding $w \in S_n$, does there exist a planar network G which graphically represents $\widetilde{C}_w(q)$?

Useful facts

Fact: If G and H are graphical representations of $\widetilde{C}_v(q)$, $\widetilde{C}_w(q)$, respectively, then the concatenation $G \circ H$ is a graphical representation of the product $\widetilde{C}_v(q)\widetilde{C}_w(q)$.

Fact: If $s_i w > w$ in the Bruhat order then

$$\widetilde{C}_{s_i}(q)\widetilde{C}_w(q) = \widetilde{C}_{s_i w}(q) + \sum_{\substack{v<w \\ s_i v<v}} \mu(v,w)\widetilde{C}_v(q),$$

where $\mu(v,w)$ is the coefficient of

$$q^{\frac{\ell(w)-\ell(v)-1}{2}}$$

in the Kazhdan-Lusztig polynomial $P_{v,w}(q)$. This is 0 unless $\ell(w) - \ell(v)$ is odd.