EVALUATION OF INDUCED SIGN CHARACTERS AT A SPANNING SET OF THE HECKE ALGEBRA

Ryan Kaliszewski*, Justin Lambright†, and Mark Skandera*

*Lehigh University, †Anderson University

Outline

(1) Hecke algebra and its traces
(2) Wiring diagrams and products of Kazhdan-Lusztig basis elements
(3) Path tableaux and statistics
(4) Evaluation formula

The Hecke algebra $H_n(q)$

Generators over $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$: $T_{s_1}, \ldots, T_{s_{n-1}}$.

Relations:

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e && \text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} && \text{for } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} && \text{for } |i-j| \geq 2. \end{aligned}$$

Natural basis: $\{T_w \,|\, w \in S_n\}$,

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}, \qquad (w = s_{i_1} \cdots s_{i_\ell} \text{ reduced}); \qquad T_e = 1.$$

(Modified) Kazhdan-Lusztig basis: $\{\widetilde{C}_w(q) \,|\, w \in S_n\}$,

$$\widetilde{C}_w(q) = q^{\frac{\ell(w)}{2}} C'_w(q) = \sum_{v \leq w} P_{v,w}(q) T_v; \qquad \widetilde{C}_e(q) = 1.$$

$H_n(1) \cong \mathbb{Z}[\mathfrak{S}_n]$.

$H_n(q)$ **trace space**

Call linear functional $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ a *trace* if $\theta_q(gh) = \theta_q(hg)$ for all $g, h \in H_n(q)$.

Fact: dim (trace space) = # partitions of n. Three bases are

$\{\chi_q^\lambda \mid \lambda \vdash n\}$	irreducible characters,
$\{\epsilon_q^\lambda \mid \lambda \vdash n\}$	induced sign characters,
$\{\eta_q^\lambda \mid \lambda \vdash n\}$	induced trivial characters.

Problem: For some trace space basis $\{\theta_q^\lambda \mid \lambda \vdash n\}$ and some $H_n(q)$ basis $\{g_w \mid w \in \mathfrak{S}_n\}$, evaluate $\theta_q^\lambda(g_w) \in \mathbb{Z}[q]$ as

$$\sum_k (-1)^{|S(\lambda,w,k)|} |R(\lambda, w, k)| q^k,$$

where $S(\lambda, w, k)$, $R(\lambda, w, k)$ are sets.

evaluation	contained in semiring	cancellation-free set cardinality interpretation
$\chi_q^{\lambda}(T_w)$	$\mathbb{Z}[q]$	unknown
$\epsilon_q^{\lambda}(T_w)$	$\mathbb{Z}[q]$	unknown
$\eta_q^{\lambda}(T_w)$	$\mathbb{Z}[q]$	unknown
$\chi_q^{\lambda}(\widetilde{C}_w(q))$	$\mathbb{N}[q]$	unknown*
$\epsilon_q^{\lambda}(\widetilde{C}_w(q))$	$\mathbb{N}[q]$	unknown*
$\eta_q^{\lambda}(\widetilde{C}_w(q))$	$\mathbb{N}[q]$	unknown*

*known for w avoiding 3412, 4231.

$H_n(q)$ spanning set arising in total nonnegativity

Call $s_{[i,j]} = 1 \cdots (i-1) j \cdots i (j+1) \cdots n \in \mathfrak{S}_n$ a *reversal*.

Fact: For A TNN we have (S '91)

- $\mathrm{Imm}_{\chi^\lambda}(A) \underset{\mathrm{def}}{=} \sum_w \chi^\lambda(w) a_{1,w_1} \cdots a_{n,w_n} \geq 0$.
- Equivalently, $\chi^\lambda(\widetilde{C}_{s_{[i_1,j_1]}}(1) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(1)) \geq 0$.

Problem: Combinatorially interpret $\mathrm{Imm}_{\chi^\lambda}(A)$.

Fact: $\chi_q^\lambda(\widetilde{C}_{s_{[i_1,j_1]}}(q) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(q)) \in \mathbb{N}[q]$ (H '93).

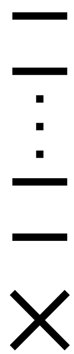
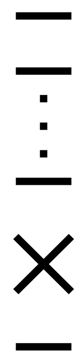
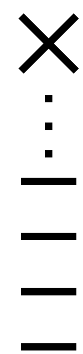
Fact: $\widetilde{C}_{s_{[i,i+1]}}(q) = \widetilde{C}_{s_i}(q) = 1 + T_{s_i}$.

Fact: $H_n(q) = \mathrm{span}_{\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]} \{\widetilde{C}_{s_{i_1}}(q) \cdots \widetilde{C}_{s_{i_m}}(q)\}$.

Problem: Combinatorially interpret $\epsilon_q^\lambda(\widetilde{C}_{s_{i_1}}(q) \cdots \widetilde{C}_{s_{i_m}}(q))$.

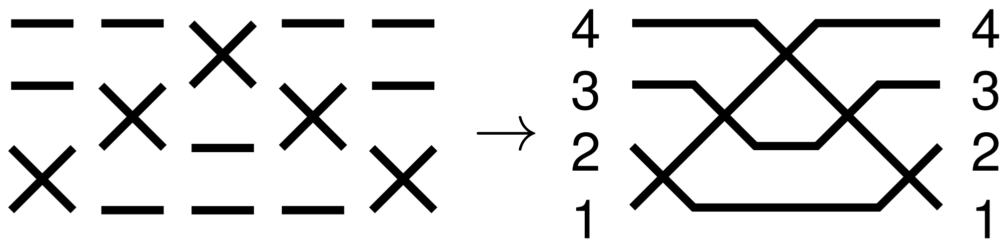
Wiring diagrams help evaluate traces

To $\mathfrak{S}_n$ generators $s_1, \ldots, s_{n-1}$, associate networks

$s_1 =$  , $s_2 =$  , $\ldots$, $s_{n-1} =$  .

(edges oriented left to right) and multiply by concatenation.

Example: To the $\mathfrak{S}_4$ expression $s_1 s_2 s_3 s_2 s_1$, associate the wiring diagram



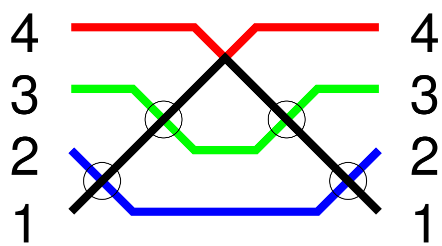
Path families in a wiring diagram G

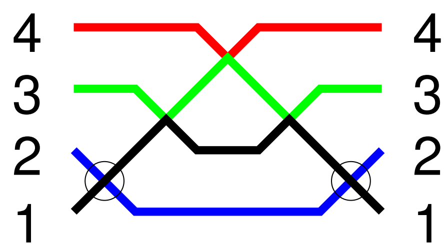
Call $\pi = (\pi_1, \ldots, \pi_n)$ a *path family in G* if

- π_i is a path from source i on left to sink i on right,
- π covers all edges of G.

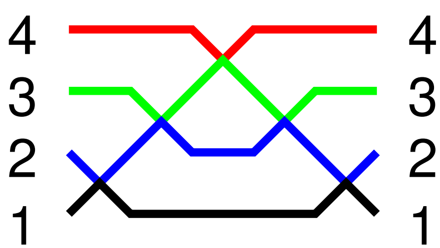
Let $\mathrm{C}(\pi) = \#$ crossings in π. This is always even.

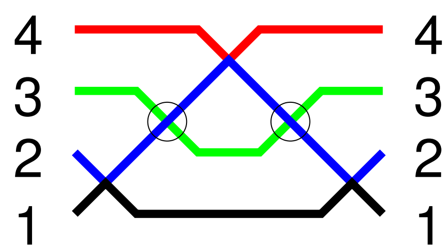
Example:

$\rho =$ 

, $\sigma =$ 

, $\mathrm{C}(\rho) = 4$, $\mathrm{C}(\sigma) = 2$,

$\tau =$ 

, $\omega =$ 

, $\mathrm{C}(\tau) = 0$, $\mathrm{C}(\omega) = 2$.

Path tableaux

Let path family π cover wiring diagram G.

A *π-tableau* (*G-tableau*) of shape $\lambda \vdash n$ is an arrangement of $\pi_1, \ldots, \pi_n$ into left-justified rows, with λ_i paths in row i.

Call a π-tableau *column-strict* (CS) if

$$\begin{array}{|c|}\hline \pi_j \\ \hline \pi_i \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{c} \pi_i \text{ lies entirely below } \pi_j \\ \text{(no shared vertex.)} \end{array}$$

Example: There are four CS G-tableaux of shape 22.

$$\underset{U_1}{\begin{array}{|c|c|}\hline \sigma_3 & \sigma_4 \\ \hline \sigma_2 & \sigma_1 \\ \hline \end{array}}, \quad \underset{U_2}{\begin{array}{|c|c|}\hline \sigma_4 & \sigma_3 \\ \hline \sigma_1 & \sigma_2 \\ \hline \end{array}}, \quad \underset{U_3}{\begin{array}{|c|c|}\hline \tau_3 & \tau_4 \\ \hline \tau_1 & \tau_2 \\ \hline \end{array}}, \quad \underset{U_4}{\begin{array}{|c|c|}\hline \tau_4 & \tau_3 \\ \hline \tau_2 & \tau_1 \\ \hline \end{array}}.$$

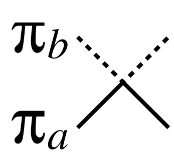
None is a ρ-tableau or an ω-tableau.

A statistic on path tableaux

Let π cover wiring diagram G and let U be a π-tableau. If

- π_a, π_b share a vertex of G but do not cross there,
- π_b enters and exits above π_a,
- π_b appears in an earlier column of U than π_a,

then call the vertex an *inverted noncrossing in* U:



,

$$\boxed{\pi_b} \quad \cdots \quad \boxed{\pi_a}\,.$$

Let $\text{INVNC}(U)$ be the number of inverted noncrossings in U.

Example:

$\text{INVNC}(U_1) = 2 \ (\sigma_3, \sigma_1)$, $\quad \text{INVNC}(U_3) = 2 \ (\tau_3, \tau_2)$,

$\text{INVNC}(U_2) = 1 \ (\sigma_4, \sigma_3)$, $\quad \text{INVNC}(U_4) = 3 \ (\tau_4, \tau_3), \ (\tau_2, \tau_1)$.

Main result

Theorem: (KLS '17) Let G be the wiring diagram of $s_{i_1} \cdots s_{i_m}$. Then

$$\epsilon_q^{\lambda}(\widetilde{C}_{s_{i_1}}(q) \cdots \widetilde{C}_{s_{i_m}}(q)) = \sum_{\pi} q^{\mathrm{C}(\pi)/2} \sum_{U} q^{\mathrm{INVNC}(U)},$$

where the first sum is over path families in G and the second sum is over all CS π-tableaux of shape $\lambda^{\top}$.

Example: To compute $\epsilon_q^{22}(\widetilde{C}_{s_1}(q)\widetilde{C}_{s_2}(q)\widetilde{C}_{s_3}(q)\widetilde{C}_{s_2}(q)\widetilde{C}_{s_1}(q))$, we observe that $22^{\top} = 22$. Thus, this is

$$\begin{aligned} &q^{\mathrm{C}(\sigma)/2}(q^{\mathrm{INVNC}(U_1)} + q^{\mathrm{INVNC}(U_2)}) \\ &\qquad + q^{\mathrm{C}(\tau)/2}(q^{\mathrm{INVNC}(U_3)} + q^{\mathrm{INVNC}(U_4)}) \\ &= q^{2/2}(q^2 + q^1) + q^{0/2}(q^2 + q^3) \\ &= 2q^2 + 2q^3. \end{aligned}$$

Corollary and open problems

Fact: $\widetilde{C}_w(q)$ factors as $\widetilde{C}_{s_{i_1}}(q) \cdots \widetilde{C}_{s_{i_1}}(q)$ if and only if w avoids patterns 321, 56781234, 56718234, 46781235, 46718235 (BW '01). Thus, Theorem applies to $\epsilon_q^\lambda(\widetilde{C}_w(q))$ for these permutations.

Problem: Extend Theorem to characters χ_q^λ, η_q^λ.

Problem: Extend Theorem to evaluations of ϵ_q^λ at elements $\widetilde{C}_{s_{[i_1,j_1]}}(q) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(q)$. (Work in progress.)

Question: Which Kazhdan-Lusztig basis elements $\widetilde{C}_w(q)$ factor as $\widetilde{C}_{s_{[i_1,j_1]}}(q) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(q)$?
(At least all $\widetilde{C}_w(q)$ with $w \in \mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5 \smallsetminus \{45312\}$.)