GENERATING FUNCTIONS FOR MONOMIAL CHARACTERS OF THE HYPEROCTAHEDRAL GROUP

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Monomial characters of $\mathfrak{S}_n$

One-dimensional characters of $\mathfrak{S}_n$: $1 \underset{\text{def}}{=} \text{triv}$, $\epsilon \underset{\text{def}}{=} \text{sgn}$.

Given Young subgroup $\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_\ell} \subset \mathfrak{S}_n$, define the *induced trivial* and *induced sign* characters by

$$1^\lambda = 1\!\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}, \qquad \epsilon^\lambda = \epsilon\!\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}.$$

Generating functions in variables $x = (x_{i,j})_{i,j=1}^n$,

$$\mathrm{Imm}_\chi(x) \underset{\text{def}}{=} \sum_{w \in \mathfrak{S}_n} \chi(w) x_{1,w_1} \cdots x_{n,w_n}$$

may be stated in terms of ordered set partitions $(I_1, \ldots, I_\ell)$,

$$I_1 \uplus \cdots \uplus I_\ell = [n] \underset{\text{def}}{=} \{1, \ldots, n\},$$

of *type* λ, i.e., $(|I_1|, \ldots, |I_\ell|) = (\lambda_1, \ldots, \lambda_\ell)$.

Littlewood-Merris-Watkins identities

Define $x_{J,J} = (x_{i,j})_{i,j \in J} = (J, J)$-*submatrix* of x.

Theorem: (L '40, M-W '85) For $\lambda = (\lambda_1, \dots, \lambda_\ell)$,

$$\mathrm{Imm}_{\epsilon^\lambda}(x) = \sum_{(I_1,\dots,I_\ell)} \det(x_{I_1,I_1}) \cdots \det(x_{I_\ell,I_\ell}),$$

$$\mathrm{Imm}_{1^\lambda}(x) = \sum_{(I_1,\dots,I_\ell)} \mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_\ell,I_\ell}),$$

summed over all ordered set partitions of type λ.

Example: $\mathrm{Imm}_{\epsilon^{21}}(x) =$

$$\det\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} x_{3,3} + \det\begin{bmatrix} x_{1,1} & x_{1,3} \\ x_{3,1} & x_{3,3} \end{bmatrix} x_{2,2} + \det\begin{bmatrix} x_{2,2} & x_{2,3} \\ x_{3,2} & x_{3,3} \end{bmatrix} x_{1,1}.$$

The Hyperoctahedral group $\mathfrak{B}_n$

Generated by $t = s_0, s_1, \ldots, s_{n-1}$, subject to relations

$$\begin{aligned}
s_i^2 &= 1 && \text{for } i = 1, \ldots, n-1,\\
s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j| = 1, i, j \geq 1,\\
s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0, &&\\
s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2.
\end{aligned}$$

One-line notation: permutations of $\{1, \ldots, n\}$ with bars over some letters. Generators act on $w_1 \cdots w_n$ by

- s_i swaps w_i and w_{i+1}, for $i \geq 1$,
- t replaces w_1 with $\overline{w_1}$. ($\overline{\overline{a}} = a$)

$$s_1 t s_1 s_2(123) = s_1 t s_1(132) = s_1 t(312) = s_1(\overline{3}12) = 1\overline{3}2.$$

Characters of $\mathfrak{B}_n$

\# conjugacy classes of $\mathfrak{B}_n$ = # partition pairs (λ, μ) with

$$\lambda_1 + \cdots + \lambda_\ell + \mu_1 + \cdots + \mu_m = n.$$

Write $(\lambda, \mu) \vdash n$.

This is also the dimension of the *trace space* of $\mathfrak{B}_n$,

$$\mathcal{T}_n = \{\theta : \mathfrak{B}_n \to \mathbb{C} \,|\, \theta(gh) = \theta(hg)\}.$$

The *irreducible character basis* $\{\chi^{\lambda,\mu} \,|\, (\lambda, \mu) \vdash n\}$ of $\mathcal{T}_n$ includes four one-dimensional characters:

$$\begin{aligned}
1 : &\quad (t, s_i) \mapsto (1, 1),\\
\delta : &\quad (t, s_i) \mapsto (-1, 1),\\
\epsilon : &\quad (t, s_i) \mapsto (1, -1),\\
\delta\epsilon : &\quad (t, s_i) \mapsto (-1, -1).
\end{aligned}$$

Monomial characters of $\mathfrak{B}_n$

For each $(\lambda, \mu) \vdash n$, define a Young subgroup

$$\mathfrak{B}_{\lambda,\mu} \cong \mathfrak{B}_{\lambda_1} \times \cdots \times \mathfrak{B}_{\lambda_\ell} \times \mathfrak{B}_{\mu_1} \times \cdots \times \mathfrak{B}_{\mu_m}.$$

One-dimensional characters $\theta_1 \otimes \theta_2$ of Young subgroups

$$(\theta_1 \otimes \theta_2)(g_1 \cdots g_\ell h_1 \cdots h_m) \underset{\text{def}}{=} \theta_1(g_1 \cdots g_\ell)\theta_2(h_1 \cdots h_m)$$

define four monomial character bases:

$$\begin{aligned}
&\{1^\lambda 1^\mu \,|\, (\lambda, \mu) \vdash n\}, && 1^\lambda 1^\mu = (1 \otimes \delta)\big\uparrow_{\mathfrak{B}_{\lambda,\mu}}^{\mathfrak{B}_n}, \\
&\{1^\lambda \epsilon^\mu \,|\, (\lambda, \mu) \vdash n\}, && 1^\lambda \epsilon^\mu = (1 \otimes \delta\epsilon)\big\uparrow_{\mathfrak{B}_{\lambda,\mu}}^{\mathfrak{B}_n}, \\
&\{\epsilon^\lambda 1^\mu \,|\, (\lambda, \mu) \vdash n\}, && \epsilon^\lambda 1^\mu = (\epsilon \otimes \delta)\big\uparrow_{\mathfrak{B}_{\lambda,\mu}}^{\mathfrak{B}_n}, \\
&\{\epsilon^\lambda \epsilon^\mu \,|\, (\lambda, \mu) \vdash n\}, && \epsilon^\lambda \epsilon^\mu = (\epsilon \otimes \delta\epsilon)\big\uparrow_{\mathfrak{B}_{\lambda,\mu}}^{\mathfrak{B}_n}.
\end{aligned}$$

Generating functions for $\mathfrak{B}_n$-characters

Let $x = (x_{1,1}, x_{1,2}, \dots, x_{n,n}, x_{1,\bar{1}}, x_{1,\bar{2}}, \dots, x_{n,\bar{n}})$,
$\mathbb{C}[x] = \mathbb{C}[x_{1,1}, x_{1,2}, \dots, x_{n,n}, x_{1,\bar{1}}, x_{1,\bar{2}}, \dots, x_{n,\bar{n}}]$.

Given function $\chi : \mathfrak{B}_n \to \mathbb{C}$, define $\mathrm{Imm}_\chi^{\mathfrak{B}_n}(x) \in \mathbb{C}[x]$ by

$$\mathrm{Imm}_\chi^{\mathfrak{B}_n}(x) \underset{\mathrm{def}}{=} \sum_{w \in \mathfrak{B}_n} \chi(w) x_{1,w_1} \cdots x_{n,w_n}.$$

Define matrices

$P = (x_{i,j} + x_{i,\bar{j}})$,

$D = (x_{i,j} - x_{i,\bar{j}})$.

Ordered set partitions of type (λ, μ)

Formulas for generating functions of monomial characters of $\mathfrak{B}_n$ again involve ordered set partitions.

Given $(\lambda, \mu) = ((\lambda_1, \ldots, \lambda_\ell), (\mu_1, \ldots, \mu_m)) \vdash n$, call a sequence $(I, J) = (I_1, \ldots, I_\ell, J_1, \ldots, J_m)$ of subsets of $[n]$ an *ordered set partition of* $[n]$ *of type* (λ, μ) if

$$I_1 \uplus \cdots \uplus I_\ell \uplus J_1 \uplus \cdots \uplus J_m = [n],$$
$$(|I_1|, \ldots, |I_\ell|) = (\lambda_1, \ldots, \lambda_\ell),$$
$$(|J_1|, \ldots, |J_m|) = (\mu_1, \ldots, \mu_m).$$

Example: An ordered set partition of 8 of type $(21, 32)$ is $(36, 4, 178, 25)$.

Monomial character immanant theorem for $\mathfrak{B}_n$

$\mathrm{Imm}_{\chi}^{\mathfrak{B}_n}(x)$ for $\chi = 1^{\lambda}1^{\mu}, \quad 1^{\lambda}\epsilon^{\mu}, \quad \epsilon^{\lambda}1^{\mu}, \quad \epsilon^{\lambda}\epsilon^{\mu}$:

$$\sum_{(I,J)} \mathrm{per}(P_{I_1,I_1}) \cdots \mathrm{per}(P_{I_\ell,I_\ell})\mathrm{per}(D_{J_1,J_1}) \cdots \mathrm{per}(D_{J_m,J_m}),$$

$$\sum_{(I,J)} \mathrm{per}(P_{I_1,I_1}) \cdots \mathrm{per}(P_{I_\ell,I_\ell}) \det(D_{J_1,J_1}) \cdots \det(D_{J_m,J_m}),$$

$$\sum_{(I,J)} \det(P_{I_1,I_1}) \cdots \det(P_{I_\ell,I_\ell})\mathrm{per}(D_{J_1,J_1}) \cdots \mathrm{per}(D_{J_m,J_m}),$$

$$\sum_{(I,J)} \det(P_{I_1,I_1}) \cdots \det(P_{I_\ell,I_\ell}) \det(D_{J_1,J_1}) \cdots \det(D_{J_m,J_m}),$$

summed over ordered set partitions of $[n]$ of type (λ, μ).

Example: $\mathrm{Imm}^{\mathfrak{B}_n}_{\epsilon^2 1^{11}}(x) = \mathrm{Imm}^{\mathfrak{B}_n}_{\epsilon^2 \epsilon^{11}}(x)$

Using ordered set partitions $(12|3|4, 12|4|3, 13|2|4, 13|4|2, \ldots,)$,

$$\det \begin{bmatrix} x_{1,1} + x_{1,\overline{1}} & x_{1,2} + x_{1,\overline{2}} \\ x_{2,1} + x_{2,\overline{1}} & x_{2,2} + x_{2,\overline{2}} \end{bmatrix} (x_{3,3} - x_{3,\overline{3}})(x_{4,4} - x_{4,\overline{4}}) +$$

$$\det \begin{bmatrix} x_{1,1} + x_{1,\overline{1}} & x_{1,2} + x_{1,\overline{2}} \\ x_{2,1} + x_{2,\overline{1}} & x_{2,2} + x_{2,\overline{2}} \end{bmatrix} (x_{4,4} - x_{4,\overline{4}})(x_{3,3} - x_{3,\overline{3}}) +$$

$$\det \begin{bmatrix} x_{1,1} + x_{1,\overline{1}} & x_{1,3} + x_{1,\overline{3}} \\ x_{3,1} + x_{3,\overline{1}} & x_{3,3} + x_{3\overline{3}} \end{bmatrix} (x_{2,2} - x_{2,\overline{2}})(x_{4,4} - x_{4,\overline{4}}) +$$

$$\det \begin{bmatrix} x_{1,1} + x_{1,\overline{1}} & x_{1,3} + x_{1,\overline{3}} \\ x_{3,1} + x_{3,\overline{1}} & x_{3,3} + x_{3,\overline{3}} \end{bmatrix} (x_{2,2} - x_{2,\overline{2}})(x_{4,4} - x_{4,\overline{4}}) + \cdots$$

An application

Let $\{q^{\frac{\ell(w)}{2}} C'_w \mid w \in \mathfrak{S}_n\}$ = Kazhdan-Lusztig basis of $H_n(q)$.

Theorem(s): (KS '11, CHSS '16, KLS '18, CS '19)
For w avoiding certain patterns, we have

$$1_q^{\lambda}(q^{\frac{\ell(w)}{2}} C'_w) = \sum_{T \in \mathcal{U}_1} q^{\mathrm{inv}(T)}, \quad \epsilon_q^{\lambda}(q^{\frac{\ell(w)}{2}} C'_w) = \sum_{T \in \mathcal{U}_2} q^{\mathrm{inv}(T)},$$

$$\chi_q^{\lambda}(q^{\frac{\ell(w)}{2}} C'_w) = \sum_{T \in \mathcal{U}_3} q^{\mathrm{inv}(T)}.$$

Question: Can we do this in type B?