GENERATING FUNCTIONS FOR MONOMIAL CHARACTERS OF WREATH PRODUCTS $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$

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Abstract. Let $\mathbb{Z}/d\mathbb{Z}\wr\mathfrak{S}_n$ denote the wreath product of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ with the symmetric group $\mathfrak{S}_n$. We define generating functions for monomial (induced one-dimensional) characters of $\mathbb{Z}/d\mathbb{Z}\wr\mathfrak{S}_n$ and express these in terms of determinants and permanents. This extends work of Littlewood (*The Theory of Group Characters and Representations of Groups*, 1940) and Merris and Watkins (*Linear Algebra Appl.*, **64**, 1985) on generating functions for the monomial characters of $\mathfrak{S}_n$.

1. Introduction

Let $z = (z_{i,j})$ be an $n \times n$ matrix of variables and let $\mathfrak{S}_n$ be the symmetric group. For each linear functional $\theta : \mathbb{C}[\mathfrak{S}_n] \to \mathbb{C}$, define the generating function

$$\mathrm{Imm}_\theta(z) := \sum_{w \in \mathfrak{S}_n} \theta(w) z_{1,w_1} \cdots z_{n,w_n} \in \mathbb{C}[z_{1,1}, \dots, z_{n,n}] \tag{1.1}$$

for θ, and call this the *θ-immanant*. Such functions appeared originally in [7, p. 81] for θ equal to irreducible $\mathfrak{S}_n$-characters χ^λ, and were extended in [14, §3] to general θ. As is the case with many functions, a simple formula for a generating function for θ can be as useful as a simple formula for the numbers $\{\theta(w) \,|\, w \in \mathfrak{S}_n\}$ themselves.

Particularly simple generating functions for the *monomial* (induced one-dimensional) characters of $\mathfrak{S}_n$ are expressed in terms of integer partitions, ordered set partitions, and submatrices of z. Call a nonnegative integer sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ satisfying $\lambda_1 + \cdots + \lambda_r = n$ a *weak composition of* n and write $|\lambda| = n$, $\ell(\lambda) = r$. If the components of λ are weakly decreasing and positive, call it an *(integer) partition of* n and write $\lambda \vdash n$. For any weak composition λ of n, call a sequence $(I_1, \dots, I_r)$ of pairwise disjoint subsets of $[n] := \{1, \dots, n\}$ an *ordered set partition of* $[n]$ *of type* λ if $|I_j| = \lambda_j$ for $j = 1, \dots, r$. (We remark that our nonstandard terminology allows empty sets in set partitions, whereas standard terminology [13, pp. 39, 73] does not.) Given subsets I, J of $[n]$, define the *(I,J)-submatrix* of z to be $z_{I,J} = (z_{i,j})_{i \in I, j \in J}$.

The class function space of $\mathfrak{S}_n$ has two standard bases consisting of monomial characters: the *induced trivial character* basis $\{\eta^\lambda = \mathrm{triv}\!\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \,|\, \lambda \vdash n\}$ and the *induced sign character* basis $\{\epsilon^\lambda = \mathrm{sgn}\!\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \,|\, \lambda \vdash n\}$, where $\mathfrak{S}_\lambda$ is the Young subgroup of $\mathfrak{S}_n$ indexed by λ. (See, e.g., [9].) Littlewood [7, §6.5] and Merris and Watkins [8] came close to expressing the η^λ- and

Date: July 16, 2020.

ϵ^λ-immanants as

$$\mathrm{Imm}_{\epsilon^\lambda}(z) = \sum_{(J_1,\dots,J_\ell)} \det(z_{J_1,J_1}) \cdots \det(z_{J_\ell,J_\ell}), \tag{1.2}$$

$$\mathrm{Imm}_{\eta^\lambda}(z) = \sum_{(J_1,\dots,J_\ell)} \mathrm{per}(z_{J_1,J_1}) \cdots \mathrm{per}(z_{J_\ell,J_\ell}), \tag{1.3}$$

where the sums are over all ordered set partitions $(J_1, \dots, J_\ell)$ of $[n]$ of type $\lambda = (\lambda_1, \dots, \lambda_\ell)$. For example, we have

$$\begin{aligned}\mathrm{Imm}_{\epsilon^{21}}(z) &= \det\begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2}\end{bmatrix} z_{3,3} + \det\begin{bmatrix} z_{1,1} & z_{1,3} \\ z_{3,1} & z_{3,3}\end{bmatrix} z_{2,2} + \det\begin{bmatrix} z_{2,2} & z_{2,3} \\ z_{3,2} & z_{3,3}\end{bmatrix} z_{1,1} \\ &= 3z_{1,1}z_{2,2}z_{3,3} - z_{1,2}z_{2,1}z_{3,3} - z_{1,3}z_{2,2}z_{3,1} - z_{1,1}z_{2,3}z_{3,2},\end{aligned}$$

and $\epsilon^{21}(123) = 3$, $\epsilon^{21}(213) = \epsilon^{21}(321) = \epsilon^{21}(132) = -1$, $\epsilon^{21}(312) = \epsilon^{21}(231) = 0$. While Littlewood, Merris, and Watkins may not have written Equations (1.2) – (1.3) explicitly, we call them the *Littlewood–Merris–Watkins identities*. These identities have played an important role in the evaluation of (type-A) Hecke algebra characters at Kazhdan–Lusztig basis elements [3], [4], [5], the formulation of a generating function for irreducible Hecke algebra characters [6], and the interpretation of coefficients of chromatic symmetric functions [3], [10]. The identity in our main result (Theorem 3.1) plays an important role in the evaluation of hyperoctahedral group characters at elements of the type-BC Kazhdan-Lusztig basis [11].

Let $\mathcal{G} = \mathcal{G}(n, d)$ be the wreath product $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$. Its class function space has 2^d standard bases consisting of monomial characters, and it is possible to use a matrix of dn^2 variables to construct generating functions analogous to (1.2) – (1.3) for the elements of these bases. In Section 2 we review $\mathcal{G}$ and its monomial characters; in Section 3 we present our generating functions for these.

2. $\mathcal{G}$ and its monomial characters

The group $\mathcal{G}$ is generated by n elements $s_1, \dots, s_{n-1}, t$ subject to the relations

$$\begin{aligned} s_i^2 &= e && \text{for } i = 1, \dots, n-1, \\ t^d &= e, && \\ ts_1ts_1 &= s_1ts_1t, && \\ s_is_j &= s_js_i && \text{for } |i-j| \geq 2, \\ ts_j &= s_jt && \text{for } j \geq 2, \\ s_is_js_i &= s_js_is_j && \text{for } |i-j| = 1. \end{aligned} \tag{2.1}$$

A one-line notation for elements of $\mathcal{G}$, analogous to that for elements of $\mathfrak{S}_n$, uses sequences of integer multiples of complex dth roots of unity. Let ζ be a primitive dth root of unity, and let S be the set of sequences

$$\{(\zeta^{\gamma_1}w_1, \dots, \zeta^{\gamma_n}w_n) \,|\, w_1 \cdots w_n \in \mathfrak{S}_n, (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}/d\mathbb{Z}^n\}. \tag{2.2}$$

We define an action of $\mathcal{G}$ on S by letting the generators act on a sequence $(a_1, \dots, a_n)$ as follows.

(1) $s_i \circ (a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_n)$,

(2) $t \circ (a_1, \ldots, a_n) = (\zeta a_1, a_2, \ldots, a_n)$.

A bijection between $\mathcal{G}$ and S is given by letting each element $g \in \mathcal{G}$ act on the sequence $(1, \ldots, n)$. If $g \circ (1, \ldots, n) = (\zeta^{\gamma_1} w_1, \ldots, \zeta^{\gamma_n} w_n)$, we define this second sequence to be the *one-line notation* of g, and we write $g = (\gamma, w)$, where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}/d\mathbb{Z}^n$, $w \in \mathfrak{S}_n$. In particular, the identity element e has one-line notation $1 \cdots n$.

Since $\mathcal{G}$ is a finite group, Brauer's Induced Character Theorem implies that the set of monomial characters of $\mathcal{G}$ spans *trace space* $\mathcal{T}(\mathcal{G})$ of $\mathcal{G}$, the set of all linear functionals $\theta : \mathbb{C}[\mathcal{G}] \to \mathbb{C}$ satisfying $\theta(gh) = \theta(hg)$ for all $g, h \in \mathcal{G}$. (See, e.g., [12].) This includes all $\mathcal{G}$-characters. $\mathcal{T}(\mathcal{G})$ has dimension equal to the number of conjugacy classes of $\mathcal{G}$, equivalently, to the number of sequences $\boldsymbol{\lambda} = (\lambda^0, \ldots, \lambda^{d-1})$ of d (possibly empty) integer partitions, with

$$|\lambda^0| + \cdots + |\lambda^{d-1}| = n.$$

We call such a sequence a *d-partition* of $[n]$ and write $\boldsymbol{\lambda} \vdash n$.

In order to describe natural bases of $\mathcal{T}(\mathcal{G})$, we introduce certain subgroups of $\mathcal{G}$ which are analogous to Young subgroups of $\mathfrak{S}_n$. Fix d-partition $\boldsymbol{\lambda} = (\lambda^0, \ldots, \lambda^{d-1}) \vdash n$, and define $r_k = \ell(\lambda^k)$ for $k = 0, \ldots, d-1$. We will say that an ordered set partition of $[n]$ of type

$$(\lambda^0_1, \ldots, \lambda^0_{r_0}, \lambda^1_1, \ldots, \lambda^1_{r_1}, \ldots, \lambda^{d-1}_1, \ldots, \lambda^{d-1}_{r_{d-1}}), \tag{2.3}$$

has *type* $\boldsymbol{\lambda}$. In particular, let $\mathbf{K}(\boldsymbol{\lambda}) = (K^0_1, \ldots, K^0_{r_0}, K^1_1, \ldots, K^1_{r_1}, \ldots, K^{d-1}_1, \ldots, K^{d-1}_{r_{d-1}})$ be the ordered set partition of $[n]$ of type $\boldsymbol{\lambda}$ whose blocks are the $r_0 + \cdots + r_{d-1}$ subintervals

$$K^0_1 = [1, \lambda^0_1], \quad K^0_2 = [\lambda^0_1 + 1, \lambda^0_1 + \lambda^0_2], \quad \ldots, \quad K^{d-1}_{r_{d-1}} = [n - \lambda^{d-1}_{r_{d-1}} + 1, n] \tag{2.4}$$

of $[n]$. For $1 \leq i \leq j \leq n$, define the element $t_i = s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1} \in \mathcal{G}$, and let $\mathcal{G}([i,j]) \cong \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_{j-i+1}$ be the subgroup of $\mathcal{G}$ generated by $\{t_i, s_i, \ldots, s_{j-1}\}$. For $k = 0, \ldots, d-1$, use (2.4) to define the subgroup

$$\mathcal{G}(\boldsymbol{\lambda}, k) := \mathcal{G}(K^k_1) \cdots \mathcal{G}(K^k_{r_k}) \cong \mathcal{G}(\lambda^k_1) \times \cdots \times \mathcal{G}(\lambda^k_{r_k}), \tag{2.5}$$

of $\mathcal{G}$, and finally define the *Young subgroup*

$$\mathcal{G}(\boldsymbol{\lambda}) := \mathcal{G}(\boldsymbol{\lambda}, 0) \cdots \mathcal{G}(\boldsymbol{\lambda}, d-1) \cong \prod_{k=0}^{d-1} \big(\mathcal{G}(\lambda^k_1) \times \cdots \times \mathcal{G}(\lambda^k_{r_k})\big) \tag{2.6}$$

of $\mathcal{G}$. Each element $y \in \mathcal{G}$ factors uniquely as $y_0 \cdots y_{d-1}$ with $y_k \in \mathcal{G}(\boldsymbol{\lambda}, k)$.

Several natural representations of $\mathcal{G}$ are defined by using symmetric group representations and induction from $\mathcal{G}(\boldsymbol{\lambda})$. First, observe that the subgroup of $\mathcal{G}$ generated by $s_1, \ldots, s_{n-1}$ is isomorphic to $\mathfrak{S}_n$, and that each r-dimensional $\mathfrak{S}_n$-representation ρ can trivially be extended to a r-dimensional $\mathcal{G}$-representation in at least d ways: by defining $\rho(t) = \zeta^k I$ for $k = 0, \ldots, d-1$. If the character of the $\mathfrak{S}_n$-representation is χ, call its extension $\delta_k \chi$. Thus the two one-dimensional $\mathfrak{S}_n$-representations

$$1 : s_i \mapsto 1 \qquad (w \mapsto 1 \text{ for all } w \in \mathfrak{S}_n),$$
$$\epsilon : s_i \mapsto -1 \qquad (w \mapsto (-1)^{\mathrm{inv}(w)} \text{ for all } w \in \mathfrak{S}_n)$$

yield $2d$ one-dimensional representations of $\mathcal{G}$:

$$\begin{aligned} \delta_k : (s_i, t) &\mapsto (1, \zeta^k), \qquad (g = (\gamma, w) \mapsto (\gamma_1 \cdots \gamma_n)^k \text{ for all } g \in \mathcal{G}), \\ \delta_k \epsilon : (s_i, t) &\mapsto (-1, \zeta^k), \qquad (g = (\gamma, w) \mapsto (-1)^{\mathrm{inv}(w)} (\gamma_1 \cdots \gamma_n)^k \text{ for all } g \in \mathcal{G}), \end{aligned} \tag{2.7}$$

for $k = 0, \dots, d-1$. Here, $\mathrm{inv}(w)$ denotes the Coxeter length of w. (See, e.g., [2, p. 15].) Next, observe that for any d-tuple $(H_0, \dots, H_{d-1})$ of subgroups of a group G which satisfy

$$H := H_0 \cdots H_{d-1} \cong H_0 \times \cdots \times H_{d-1}, \tag{2.8}$$

and characters $\theta_0, \dots, \theta_{d-1}$ of these, we have that the function $\theta = \theta_0 \otimes \cdots \otimes \theta_{d-1}$ defined by $\theta(h_0 \cdots h_{d-1}) = \theta_0(h_0) \cdots \theta_{d-1}(h_{d-1})$ is a character of H, and $\theta\!\uparrow_H^G$ is a character of G. In particular, the Young subgroup $\mathcal{G}(\boldsymbol{\lambda})$ has the form (2.8) with $H_k = \mathcal{G}(\boldsymbol{\lambda}, k)$. For every d-tuple $\boldsymbol{\beta} = (\beta_0, \dots, \beta_{d-1}) \in \{1, \epsilon\}^d$ of one-dimensional symmetric group characters we have the one-dimensional $\mathcal{G}(\boldsymbol{\lambda})$-character

$$\delta_0\beta_0 \otimes \cdots \otimes \delta_{d-1}\beta_{d-1}, \tag{2.9}$$

the corresponding monomial $\mathcal{G}$-character

$$\boldsymbol{\beta}^{\boldsymbol{\lambda}} := (\delta_0\beta_0 \otimes \cdots \otimes \delta_{d-1}\beta_{d-1})\!\uparrow_{\mathcal{G}(\boldsymbol{\lambda})}^{\mathcal{G}}, \tag{2.10}$$

and the basis $\{\boldsymbol{\beta}^{\boldsymbol{\lambda}} \,|\, \boldsymbol{\lambda} \vdash n\}$ of $\mathcal{T}(\mathcal{G})$. The irreducible character basis $\{\chi^{\boldsymbol{\lambda}} \,|\, \boldsymbol{\lambda} \vdash n\}$ of $\mathcal{T}(\mathcal{G})$ can be defined somewhat similarly. Given $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{d-1}) \vdash n$, define the d-partition $\boldsymbol{\lambda}^\bullet = (|\lambda^0|, \dots, |\lambda^{d-1}|)$, and the $\mathcal{G}(\boldsymbol{\lambda}^\bullet)$-character

$$\delta_0\chi^{\lambda^0} \otimes \cdots \otimes \delta_{d-1}\chi^{\lambda^{d-1}},$$

where χ^{λ^k} is the irreducible $\mathfrak{S}_{|\lambda^k|}$-character indexed by the partition λ^k. The corresponding induced characters

$$\chi^{\boldsymbol{\lambda}} = (\delta_0\chi^{\lambda^0} \otimes \cdots \otimes \delta_{d-1}\chi^{\lambda^{d-1}})\!\uparrow_{\mathcal{G}(\boldsymbol{\lambda}^\bullet)}^{\mathcal{G}} \tag{2.11}$$

are the irreducible characters of $\mathcal{G}$. (See, e.g., [1, p. 219].)

For the purpose of creating generating functions for characters $\boldsymbol{\beta}^{\boldsymbol{\lambda}}$, it will be convenient to realize each as the character of a submodule of $\mathbb{C}[\mathcal{G}]$, with $\mathcal{G}$ acting by left multiplication. To do this, we consider an arbitrary finite group G, a subgroup H, an H-character θ, and the element

$$T_H^\theta := \sum_{h \in H} \theta(h^{-1})h \in \mathbb{C}[G]. \tag{2.12}$$

Proposition 2.1. *Let H be a subgroup of a finite group G and let ρ be a one-dimensional complex representation of H with character θ (= ρ). Let $U = (u_1, \dots, u_r)$ be a transversal of representatives of cosets of H in G. Let G act by left multiplication on the submodule*

$$V := \mathrm{span}_{\mathbb{C}}\{u_i T_H^\theta \,|\, 1 \leq i \leq r\} \tag{2.13}$$

of $\mathbb{C}[G]$. Then V is a G-module with character $\theta\!\uparrow_H^G$.

Proof. To see that V is a G-module, consider the action of $g \in G$ on the jth element of the defining basis of V. Let u_iH be the unique coset satisfying $gu_jH = u_iH$, i.e., $u_i^{-1}gu_j \in H$. Then we have

$$\begin{aligned} gu_jT_H^\theta &= gu_j \sum_{h \in H} \theta(h^{-1})h = u_i \sum_{h \in H} \theta(h^{-1})u_i^{-1}gu_jh = u_i \sum_{h' \in H} \theta((h')^{-1}u_i^{-1}gu_j)h' \\ &= \theta(u_i^{-1}gu_j)u_iT_H^\theta, \end{aligned} \tag{2.14}$$

since $\theta = \rho$ is a homomorphism. It follows that in the jth column of the matrix representing g, all components are 0 except for the ith, which is $\theta(u_i^{-1}gu_j)$. But this is precisely the formula for entries of the matrix $\rho\!\uparrow_H^G(g)$. (See, e.g., [9, Defn. 1.12.2].) $\square$

For $\chi = \theta\!\uparrow_H^G$, Proposition 2.1 allows us to express T_G^χ as a sum of conjugates of T_H^θ.

Lemma 2.2. *Let groups G, H, transversal $U = (u_1, \dots, u_r)$, H-character θ, and G-module V be as in Proposition 2.1, and let $A = (a_{i,j})$ be the matrix of $g \in G$ with respect to the defining basis (2.13) of V. Then $a_{i,j}$ equals the coefficient of g^{-1} in $u_j T_H^\theta u_i^{-1}$. In particular if χ is the character of V, then we have the identity*

$$\sum_{i=1}^{r} u_i T_H^\theta u_i^{-1} = \sum_{g \in G} \chi(g) g^{-1}. \tag{2.15}$$

in $\mathbb{C}[G]$.

Proof. By the proof of Proposition 2.1, we have $a_{i,j} = \theta(u_i^{-1} g u_j)$ if some $h \in H$ satisfies $g = u_i h u_j^{-1}$, and is 0 otherwise. On the other hand, we have

$$u_j T_H^\theta u_i^{-1} = \sum_{h \in H} \theta(h^{-1}) u_j h u_i^{-1}. \tag{2.16}$$

If there is no $h \in H$ satisfying $g^{-1} = u_j h u_i^{-1}$, then the coefficient of g^{-1} in (2.16) is 0. Otherwise, the coefficient of g^{-1} is

$$\theta(h^{-1}) = \theta(u_i^{-1} g u_j).$$

It follows that $a_{i,j}$ is equal to the coefficient of g^{-1} in $u_j T_H^\theta u_i^{-1}$. Thus $\chi(g) = \sum_i a_{i,i}$ is equal to the coefficient of g^{-1} in $\sum_i u_i T_H^\theta u_i^{-1}$. □

For $G = \mathcal{G}$, $H = \mathcal{G}(\boldsymbol{\lambda})$, and θ as in (2.9), the module V (2.13) has a particularly nice form. The element T_H^θ factors as $T_{\mathcal{G}(\boldsymbol{\lambda},0)}^{\delta_0\beta_0} \cdots T_{\mathcal{G}(\boldsymbol{\lambda},d-1)}^{\delta_{d-1}\beta_{d-1}}$, and each coset $u\mathcal{G}(\boldsymbol{\lambda})$ of $\mathcal{G}(\boldsymbol{\lambda})$ has a unique representative $g = (\gamma, w)$ satisfying $\gamma_1 = \cdots = \gamma_n = 0$ and $w_i < w_{i+1}$ for $i, i+1$ belonging to the same block of $\mathbf{K}(\boldsymbol{\lambda})$, i.e.,

$$w_1 < \cdots < w_{\lambda_1^0}, \qquad w_{\lambda_1^0+1} < \cdots < w_{\lambda_1^0+\lambda_2^0}, \dots, \qquad w_{n-\lambda_{r_{d-1}}^{d-1}+1} < \cdots < w_n. \tag{2.17}$$

Letting $\mathcal{G}(\boldsymbol{\lambda})^-$ be the set of such coset representatives, we have

$$V = V(\boldsymbol{\lambda}, \boldsymbol{\beta}) = \mathrm{span}_{\mathbb{C}}\{u T_{\mathcal{G}(\boldsymbol{\lambda},0)}^{\delta_0\beta_0} \cdots T_{\mathcal{G}(\boldsymbol{\lambda},d-1)}^{\delta_{d-1}\beta_{d-1}} \mid u \in \mathcal{G}(\boldsymbol{\lambda})^-\},$$

and the following special case of Lemma 2.2.

Corollary 2.3. *Fix a d-partition $\boldsymbol{\lambda} \vdash n$. For each one-dimensional $\mathcal{G}(\boldsymbol{\lambda})$-character θ of the form (2.9), the monomial $\mathcal{G}$-character $\boldsymbol{\beta}^{\boldsymbol{\lambda}} = \theta\!\uparrow_{\mathcal{G}(\boldsymbol{\lambda})}^{\mathcal{G}}$ satisfies*

$$\sum_{u \in \mathcal{G}(\boldsymbol{\lambda})^-} u T_{\mathcal{G}(\boldsymbol{\lambda})}^\theta u^{-1} = \sum_{g \in \mathcal{G}} \boldsymbol{\beta}^{\boldsymbol{\lambda}}(g^{-1}) g. \tag{2.18}$$

For $d = 1, 2$, the group $\mathcal{G}$ (equal to the symmetric group or the hyperoctahedral group) has real-valued irreducible characters. Therefore each group element is conjugate to its inverse, and the final sum of (2.18) may be expressed as $\sum_{g \in \mathcal{G}} \boldsymbol{\beta}^{\boldsymbol{\lambda}}(g) g$.

3. Main result

A generalization of the generating functions (1.2) – (1.3) to monomial characters of $\mathcal{G}$ requires a polynomial ring and a $|\mathcal{G}| = d^n n!$-dimensional subspace analogous to the $n!$-dimensional span of the functions (1.1). Let $C_d = \{\zeta^k \,|\, k \in \mathbb{Z}/d\mathbb{Z}\}$ be the subgroup of $\mathbb{C}$ consisting of dth roots of unity, and for any subset $M \subseteq [n]$, let C_dM be the complex numbers of the form $\{\zeta^k m \,|\, k \in \mathbb{Z}/d\mathbb{Z}, m \in M\}$, and define the set $x = \{x_{i,j} \,|\, i \in [n], j \in C_d[n]\}$ of dn^2 variables. One can think of x as a collection of d matrices of n^2 variables. For example when $n = 2$ and $d = 3$, the variables are

$$\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}, \qquad \begin{bmatrix} x_{1,\dot{1}} & x_{1,\dot{2}} \\ x_{2,\dot{1}} & x_{2,\dot{2}} \end{bmatrix}, \qquad \begin{bmatrix} x_{1,\ddot{1}} & x_{1,\ddot{2}} \\ x_{2,\ddot{1}} & x_{2,\ddot{2}} \end{bmatrix}, \tag{3.1}$$

where we define $\dot{m} := \zeta m$, $\ddot{m} := \zeta^2 m$ for variable subscripts $m = 1, 2$, e.g., $x_{2,\ddot{1}} = x_{2,\zeta^2}$.

For $u \in \mathfrak{S}_n$, $g \in \mathcal{G}$, write

$$x^{u,g} := x_{u_1,g_1} \cdots x_{u_n,g_n},$$

and define the $\mathcal{G}$-*immanant subspace* of $\mathbb{C}[x]$ to be

$$\mathrm{span}_{\mathbb{C}}\{x^{e,g} = x_{1,g_1} \cdots x_{n,g_n} \,|\, g \in \mathcal{G}\}.$$

It is easy to see that these monomials satisfy

$$x^{u,g} = x^{e,u^{-1}g} \tag{3.2}$$

for all $u \in \mathfrak{S}_n$, $g \in \mathcal{G}$. Thus for any fixed $u \in \mathfrak{S}_n$, the $\mathcal{G}$-immanant subspace of $\mathbb{C}[x]$ may also be expressed as $\mathrm{span}_{\mathbb{C}}\{x^{u,g} \,|\, g \in \mathcal{G}\}$. The left- and right-regular representations of $\mathcal{G}$ define left- and right-actions of $\mathcal{G}$ on the $\mathcal{G}$-immanant space,

$$h_1 \circ x^{e,g} \circ h_2 = x^{e,h_1 g h_2}, \tag{3.3}$$

for $g, h_1, h_2 \in \mathcal{G}$. For any function $\theta : \mathcal{G} \to \mathbb{C}$, define the *type-*$\mathcal{G}$ θ*-immanant* to be the generating function

$$\mathrm{Imm}_\theta^{\mathcal{G}}(x) = \sum_{g \in \mathcal{G}} \theta(g^{-1}) x^{e,g} \tag{3.4}$$

for evaluations of θ. Our counterintuitive use of g^{-1} in place of g is necessitated by Proposition 2.1 – Corollary 2.3. (See also [15, Eq. (1)].) By the comment following Corollary 2.3, symmetric group and hyperoctahedral group ($\mathfrak{B}_n \cong \mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$) immanants can be written

$$\mathrm{Imm}_\theta^{\mathfrak{S}_n}(x) = \sum_{w \in \mathfrak{S}_n} \theta(w) x^{e,w}, \qquad \mathrm{Imm}_\theta^{\mathfrak{B}_n}(x) = \sum_{w \in \mathfrak{B}_n} \theta(w) x^{e,w}. \tag{3.5}$$

For economy, we will generally supress $\mathfrak{S}_n$ from the notation of symmetric group immanants.

Define the d $n \times n$ matrices $Q_0(x), \dots, Q_{d-1}(x)$ by $Q_k(x) = (q_{i,j,k}(x))_{i,j \in [n]}$, where

$$q_{i,j,k}(x) = x_{i,j} + \zeta^{-k} x_{i,\zeta j} + \zeta^{-2k} x_{i,\zeta^2 j} + \cdots + \zeta^{-(d-1)k} x_{i,\zeta^{(d-1)} j}. \tag{3.6}$$

The permanent and determinant of these matrices are equal to $\mathcal{G}$-immanants for the one-dimensional characters $\delta_0, \dots, \delta_{d-1}$, $\delta_0\epsilon, \dots, \delta_{d-1}\epsilon$ of $\mathcal{G}$. Specifically, we have

$$\begin{aligned} \mathrm{per}(Q_k(x)) &= \sum_{g=(\gamma,w)\in\mathcal{G}} (\gamma_1 \cdots \gamma_n)^{-k} x^{e,g} = \mathrm{Imm}^{\mathcal{G}}_{\delta_k}(x), \\ \det(Q_k(x)) &= \sum_{g=(\gamma,w)\in\mathcal{G}} (-1)^{\mathrm{inv}(w)} (\gamma_1 \cdots \gamma_n)^{-k} x^{e,g} = \mathrm{Imm}^{\mathcal{G}}_{\delta_k\epsilon}(x). \end{aligned} \tag{3.7}$$

More generally, we obtain $\mathcal{G}$-analogs of the Littlewood-Merris-Watkins generating functions (1.2) – (1.3) by taking sums of products of immanants $\mathrm{Imm}^{\mathfrak{S}_m}_{\eta^\mu}$, $\mathrm{Imm}^{\mathfrak{S}_m}_{\epsilon^\mu}$ of submatrices of $Q_0(x), \dots, Q_{d-1}(x)$, where $\eta^\mu = 1 \big\uparrow^{\mathfrak{S}_m}_{\mathfrak{S}_\mu}$ and $\epsilon^\mu = \epsilon \big\uparrow^{\mathfrak{S}_m}_{\mathfrak{S}_\mu}$ are monomial characters of $\mathfrak{S}_m$, for $m \leq n$.

Theorem 3.1. *Fix d-partition $\boldsymbol{\lambda} = (\lambda^0, \dots, \lambda^{d-1}) \vdash n$, and let $a_k = |\lambda^k|$, $r_k = \ell(\lambda^k)$. Fix character sequence $\boldsymbol{\beta} = (\beta_0, \dots, \beta_{d-1}) \in \{1, \epsilon\}^d$ and define*

$$\beta_k^{\lambda^k} = \beta_k \big\uparrow^{\mathfrak{S}_{a_k}}_{\mathfrak{S}_{\lambda^k}} \in \{\epsilon^{\lambda^k}, \eta^{\lambda^k}\}, \quad k = 0, \dots, d-1.$$

Then we have

$$\mathrm{Imm}^{\mathcal{G}}_{\boldsymbol{\beta}^{\boldsymbol{\lambda}}}(x) = \sum_{(I_0,\dots,I_{d-1})} \mathrm{Imm}_{\beta_0^{\lambda^0}}(Q_0(x)_{I_0,I_0}) \cdots \mathrm{Imm}_{\beta_{d-1}^{\lambda^{d-1}}}(Q_{d-1}(x)_{I_{d-1},I_{d-1}}), \tag{3.8}$$

where the sum is over all ordered set partitions of $[n]$ of type $\boldsymbol{\lambda}^\bullet = (a_0, \dots, a_{d-1})$.

Proof. Define the $\mathcal{G}(\boldsymbol{\lambda})$-character $\theta = \delta_0\beta_0 \otimes \cdots \otimes \delta_{d-1}\beta_{d-1}$ and let $\boldsymbol{\beta}^{\boldsymbol{\lambda}} = \theta \big\uparrow^{\mathcal{G}}_{\mathcal{G}(\boldsymbol{\lambda})}$. By Corollary 2.3, (3.3), and (3.4), we can express the left-hand side of (3.8) as

$$\sum_{g\in\mathcal{G}} \boldsymbol{\beta}^{\boldsymbol{\lambda}}(g^{-1}) \circ x^{e,g} = \sum_{g\in\mathcal{G}} \boldsymbol{\beta}^{\boldsymbol{\lambda}}(g^{-1}) g \circ x^{e,e} = \sum_{u\in\mathcal{G}(\boldsymbol{\lambda})^-} u T^{\theta}_{\mathcal{G}(\boldsymbol{\lambda})} u^{-1} \circ x^{e,e}. \tag{3.9}$$

Now consider the right-hand side of (3.8). By (1.2) – (1.3), we may rewrite this as a sum of products of permanents and determinants,

$$\sum_{\mathbf{J}} \Big(\prod_{i=0}^{r_0} \mathrm{Imm}_{\beta_0}(Q_0(x)_{J_i^0,J_i^0}) \Big) \cdots \Big(\prod_{i=0}^{r_{d-1}} \mathrm{Imm}_{\beta_{d-1}}(Q_{d-1}(x)_{J_i^{d-1},J_i^{d-1}}) \Big), \tag{3.10}$$

where the sum is over all ordered set partitions $\mathbf{J} = (J_1^0, \dots, J_{r_0}^0, \dots, J_1^{d-1}, \dots, J_{r_{d-1}}^{d-1})$ of $[n]$ of type $\boldsymbol{\lambda}$, and where $\mathrm{Imm}_\epsilon = \det$, $\mathrm{Imm}_1 = \mathrm{per}$. For all i, k, the variables that appear in $Q_k(x)_{J_i^k,J_i^k}$ are $x_{J_i^k,C_dJ_i^k}$. By (3.7), we may again rewrite (3.10) as a sum

$$\sum_{\mathbf{J}} \Big(\prod_{i=1}^{r_0} \mathrm{Imm}^{\mathcal{G}(\lambda_i^0)}_{\delta_0\beta_0}(x_{J_i^0,C_dJ_i^0}) \Big) \cdots \Big(\prod_{i=1}^{r_{d-1}} \mathrm{Imm}^{\mathcal{G}(\lambda_i^{d-1})}_{\delta_{d-1}\beta_{d-1}}(x_{J_i^{d-1},C_dJ_i^{d-1}}) \Big) \tag{3.11}$$

in which each factor of each term has the form

$$\mathrm{Imm}^{\mathcal{G}(\lambda_i^k)}_{\delta_k\beta_k}(x_{J_i^k,C_dJ_i^k}) = \begin{cases} \displaystyle\sum_{g=(\gamma,w)\in\mathcal{G}(J_i^k)} (\gamma_1 \cdots \gamma_n)^{d-k} (x_{J_i^k,C_dJ_i^k})^{e,g} & \text{if } \beta_k = 1, \\ \displaystyle\sum_{g=(\gamma,w)\in\mathcal{G}(J_i^k)} (\gamma_1 \cdots \gamma_n)^{d-k} (-1)^{\ell(w)} (x_{J_i^k,C_dJ_i^k})^{e,g} & \text{if } \beta_k = \epsilon. \end{cases}$$

Define the set partition $\mathbf{K} = (K_1^0, \dots, K_{r_0}^0, \dots, K_1^{d-1}, \dots, K_{r_{d-1}}^{d-1})$ of type $\boldsymbol{\lambda}$ as in (2.4), and for each ordered set partition $\mathbf{J}$ of type $\boldsymbol{\lambda}$ define $u = u(\mathbf{J}) \in \mathcal{G}(\boldsymbol{\lambda})^-$ to be the element whose one-line notation has the λ_i^k consecutive letters K_i^k in positions J_i^k, for $k = 0, \dots, d-1$ and $i = 1, \dots, r_k$. In particular, u^{-1} is the element in $\mathfrak{S}_n \subset \mathcal{G}$ whose one-line notation contains the increasing rearrangement of J_i^k in the consecutive positions K_i^k for $k = 0, \dots, d-1$ and $i = 1, \dots, r_k$. By (2.17), the map $\mathbf{J} \mapsto u(\mathbf{J})$ defines a bijective correspondence between ordered set partitions of type $\boldsymbol{\lambda}$ and $\mathcal{G}(\boldsymbol{\lambda})^-$. Thus in the expansion of the product (3.11), the monomials which appear are precisely the set $\{x^{u^{-1},yu^{-1}} \,|\, y \in \mathcal{G}(\boldsymbol{\lambda})\}$. Factoring $y = y_0 \cdots y_{d-1}$ with $y_k \in \mathcal{G}(\boldsymbol{\lambda}, k)$, we may express the coefficient of each such monomial as

$$\delta_0\beta_0(y_0^{-1}) \cdots \delta_{d-1}\beta_{d-1}(y_{d-1}^{-1}) = \theta(y^{-1}). \tag{3.12}$$

Using these facts and (3.2), (3.3), we may rewrite (3.10) as

$$\sum_{u\in\mathcal{G}(\boldsymbol{\lambda})^-} \sum_{y\in\mathcal{G}(\boldsymbol{\lambda})} \theta(y^{-1}) x^{u^{-1},yu^{-1}} = \sum_{u\in\mathcal{G}(\boldsymbol{\lambda})^-} \sum_{y\in\mathcal{G}(\boldsymbol{\lambda})} \theta(y^{-1}) uyu^{-1} \circ x^{e,e} = \sum_{u\in\mathcal{G}(\boldsymbol{\lambda})^-} uT^{\theta}_{\mathcal{G}(\boldsymbol{\lambda})}u^{-1} \circ x^{e,e}$$

to see that it is equal to (3.9). □

We illustrate with an example. Consider the group $\mathcal{G}(6,3) = \mathbb{Z}/3\mathbb{Z} \wr \mathfrak{S}_6$. It trace space $\mathcal{T}(\mathcal{G}(6,3))$ has dimension equal to the number of 3-partitions of 6, and its immanant space

$$\mathrm{span}_{\mathbb{C}}\{x_{1,g_1} \cdots x_{6,g_6} \,|\, (g_1, \dots, g_6) \in \mathcal{G}(6,3)\}$$

requires the $6^2 \cdot 3 = 108$ variables $\{x_{i,\zeta^k j} \,|\, i,j = 1,\dots,6; k = 0,\dots,2\}$, where $\zeta = e^{2\pi i/3}$. To economize notation, we define $\dot{m} := \zeta m$, $\ddot{m} := \zeta^2 m$, as in (3.1). The $2^3 = 8$ monomial character bases correspond to the triples of one-dimensional symmetric group characters $(1,1,1), (1,1,\epsilon), (1,\epsilon,1), \dots, (\epsilon,\epsilon,\epsilon)$, so that the basis corresponding to $(\epsilon,\epsilon,1)$ is

$$\big\{(\epsilon,\epsilon,1)^{\boldsymbol{\lambda}} = (\epsilon \otimes \delta_1\epsilon \otimes \delta_2)\big\uparrow^{\mathcal{G}(6,3)}_{\mathcal{G}(\boldsymbol{\lambda})} \,\big|\, \boldsymbol{\lambda} \vdash 6\big\}. \tag{3.13}$$

Consider the basis element $(\epsilon,\epsilon,1)^{(21,1,2)}$. To evaluate $(\epsilon,\epsilon,1)^{(21,1,2)}(g)$ for all $g \in \mathcal{G}$, we write its immanant $\mathrm{Imm}^{\mathcal{G}(6,3)}_{(\epsilon,\epsilon,1)^{(21,1,2)}}(x)$ as a sum of 60 terms

$$\begin{aligned}
&\mathrm{Imm}_{\epsilon^{21}}(Q_0(x)_{123,123})\mathrm{Imm}_{\epsilon^1}(Q_1(x)_{4,4})\mathrm{Imm}_{\eta^2}(Q_2(x)_{56,56})\\
&+\mathrm{Imm}_{\epsilon^{21}}(Q_0(x)_{123,123})\mathrm{Imm}_{\epsilon^1}(Q_1(x)_{5,5})\mathrm{Imm}_{\eta^2}(Q_2(x)_{46,46})\\
&+\mathrm{Imm}_{\epsilon^{21}}(Q_0(x)_{123,123})\mathrm{Imm}_{\epsilon^1}(Q_1(x)_{6,6})\mathrm{Imm}_{\eta^2}(Q_2(x)_{45,45})\\
&+\mathrm{Imm}_{\epsilon^{21}}(Q_0(x)_{124,124})\mathrm{Imm}_{\epsilon^1}(Q_1(x)_{3,3})\mathrm{Imm}_{\eta^2}(Q_2(x)_{56,56})\\
&\qquad\vdots\\
&+\mathrm{Imm}_{\epsilon^{21}}(Q_0(x)_{456,456})\mathrm{Imm}_{\epsilon^1}(Q_1(x)_{3,3})\mathrm{Imm}_{\eta^2}(Q_2(x)_{12,12}),
\end{aligned} \tag{3.14}$$

each corresponding to an ordered set partition of $[6]$ of type $(3,1,2)$. Consider the term corresponding to the ordered set partition $(136,4,25)$. It is a product of the three factors

$$
\begin{aligned}
\mathrm{Imm}_{\epsilon^{21}}(Q_0(x)_{136,136}) &= \det \begin{bmatrix} x_{1,1} + x_{1,\dot{1}} + x_{1,\ddot{1}} & x_{1,3} + x_{1,\dot{3}} + x_{1,\ddot{3}} \\ x_{3,1} + x_{3,\dot{1}} + x_{3,\ddot{1}} & x_{3,3} + x_{3,\dot{3}} + x_{3,\ddot{3}} \end{bmatrix} (x_{6,6} + x_{6,\dot{6}} + x_{6,\ddot{6}}) \\
&\quad + \det \begin{bmatrix} x_{1,1} + x_{1,\dot{1}} + x_{1,\ddot{1}} & x_{1,6} + x_{1,\dot{6}} + x_{1,\ddot{6}} \\ x_{6,1} + x_{6,\dot{1}} + x_{6,\ddot{1}} & x_{6,6} + x_{6,\dot{6}} + x_{6,\ddot{6}} \end{bmatrix} (x_{3,3} + x_{3,\dot{3}} + x_{3,\ddot{3}}) \\
&\quad + \det \begin{bmatrix} x_{3,3} + x_{3,\dot{3}} + x_{3,\ddot{3}} & x_{3,6} + x_{3,\dot{6}} + x_{3,\ddot{6}} \\ x_{6,3} + x_{6,\dot{3}} + x_{6,\ddot{3}} & x_{6,6} + x_{6,\dot{6}} + x_{6,\ddot{6}} \end{bmatrix} (x_{1,1} + x_{1,\dot{1}} + x_{1,\ddot{1}}), \qquad (3.15) \\
\mathrm{Imm}_{\epsilon^{1}}(Q_1(x)_{4,4}) &= x_{4,4} + \zeta^2 x_{4,\dot{4}} + \zeta x_{4,\ddot{4}}, \\
\mathrm{Imm}_{\eta^{2}}(Q_2(x)_{25,25}) &= \mathrm{per} \begin{bmatrix} x_{2,2} + \zeta x_{2,\dot{2}} + \zeta^2 x_{2,\ddot{2}} & x_{2,5} + \zeta x_{2,\dot{5}} + \zeta^2 x_{2,\ddot{5}} \\ x_{5,2} + \zeta x_{5,\dot{2}} + \zeta^2 x_{5,\ddot{2}} & x_{5,5} + \zeta x_{5,\dot{5}} + \zeta^2 x_{5,\ddot{5}} \end{bmatrix}.
\end{aligned}
$$

It is easy to see that this term, like all others in (3.14), contributes 3 to the coefficient of $x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,6}$. Thus we have $(\epsilon, \epsilon, 1)^{(21,1,2)}(123456) = 180$. Now consider the computation of $(\epsilon, \epsilon, 1)^{(21,1,2)}(623\dot{4}5\dot{1})$. The term (3.15) contributes $-\zeta^2$ to the coefficient of $x_{1,6}x_{2,2}x_{3,3}x_{4,\dot{4}}x_{5,5}x_{6,\dot{1}}$, as do the terms in (3.14) corresponding to the other two ordered set partitions $(1a6, 4, bc)$. The term corresponding to the ordered set partition $(235, 4, 16)$ contributes $3\zeta^2\zeta = 3$, and the three terms corresponding to the ordered set partitions $(ab4, c, 16)$ contribute 3ζ. Terms corresponding to all other ordered set partitions contribute 0. Thus we have

$$(\epsilon, \epsilon, 1)^{(21,1,2)}(623\dot{4}5\dot{1}) = 3(1 + \zeta - \zeta^2).$$

It would be interesting to extend Theorem 3.1 to obtain a generating function for the monomial characters of Hecke algebras of wreath products [1], as was done for monomial characters of the Hecke algebra of $\mathfrak{S}_n$ in [6, Thm. 2.1].

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