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On the impossibility of parabolic factorization of certain Kazhdan–Lusztig basis elements

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Abstract. For w in the symmetric group $\mathfrak{S}_n$, let $\widetilde{C}_w$ be the corresponding modified, signless Kazhdan–Lusztig basis element of the type-A Hecke algebra $H_n(q)$. An extension [*Ann. Comb.* **25**, no. 3 (2021) pp. 757–787] of a result of Deodhar [*Geom. Dedicata* **36**, (1990) pp. 95–119] implies that any factorization of the form

$$\widetilde{C}_w = \frac{1}{f(q)}\widetilde{C}_{v^{(1)}} \cdots \widetilde{C}_{v^{(r)}},$$

with $v^{(1)}, \ldots, v^{(r)}$ maximal elements of parabolic subgroups of $\mathfrak{S}_n$ and $f(q) \in \mathbb{N}[q]$ depending on these, provides cancellation-free combinatorial interpretations of the polynomials $\{P_{v,w}(q) \,|\, v \in \mathfrak{S}_n\}$ appearing in the expansion $\sum_v P_{v,w}(q) T_v$ of $\widetilde{C}_w$ in terms of the natural basis $\{T_v \,|\, v \in \mathfrak{S}_n\}$ of $H_n(q)$. While the set of permutations $w \in \mathfrak{S}_n$ admitting such a factorization of $\widetilde{C}_w$ has not yet been characterized, we apply a result of Gaetz–Gao [*Adv. Math.* **457** (2024) Paper No. 109941] to describe a set for which such a factorization cannot exist.

Keywords: Hecke algebra, Kazhdan–Lusztig basis, planar network, factorization.

1 Introduction

The *Kazhdan–Lusztig polynomials* $\{P_{v,w}(q) \,|\, v, w \in \mathfrak{S}_n\} \subset \mathbb{N}[q]$ are entries of the change-of-basis matrix relating a certain *Kazhdan–Lusztig basis* of the Hecke algebra with another *natural basis.* First appearing in the study of representations of the Hecke algebra, they were given existential and recursive definitions in [19]. Appearances of the polynomials in other areas such as Lie Theory [1], [2], [9], quantum groups [14], and Schubert varieties [19], [20] have inspired a search for simpler descriptions. Ideally, such a description should interpret each coefficient of $P_{v,w}(q)$ as a set cardinality.

Some famous alternative formulas for the Kazhdan–Lusztig polynomials are due to Brenti and Deodhar. Brenti expressed $P_{v,w}(q)$ in two different ways as simple linear combinations of recursively defined polynomials in $\mathbb{Z}[q]$ having both positive and negative coefficients [7, §3], [8, §3]. Because of negative coefficients and recursive definitions, these formulas do not interpret coefficients in $P_{v,w}(q)$ as set cardinalities. Deodhar [13] developed an algorithm which takes any reduced expression for w an an input, and outputs a set $\mathscr{E}_{\min}$ of (not necessarily reduced) expressions for other permutations in $\mathfrak{S}_n$.

For each $v \in \mathfrak{S}_n$ and $k > 0$, the coefficient of q^k in $P_{v,w}(q)$ is equal to the cardinality of a certain subset of $\mathscr{E}_{\mathrm{min}}$. On the other hand, the algorithmic component of Deodhar's method makes it difficult to apply his combinatorial interpretation in practice.

Billey and Warrington showed [4, Thm. 1, Rmk. 6] that when w has certain properties, Deodhar's algorithm is trivial, and the output set $\mathscr{E}_{\mathrm{min}}$ of expressions can be replaced by a more visually appealing set of path families in a certain wiring diagram. Again for each v and k, the coefficient of q^k in $P_{v,w}(q)$ is equal to the cardinality of a subset of these path families. Clearwater and the second author [11, Cor. 5.3] then extended this result to permutations w for which the Kazhdan–Lusztig basis element $\widetilde{C}_w$ factors nicely, but did not solve the problem [23, Quest. 4.5] of characterizing such permutations w.

In Sections 2 – 3 we review basic facts about the symmetric group, planar networks, the Hecke algebra, and the Kazhdan–Lusztig basis and polynomials. In Section 4 we use the result [11, Cor. 5.3] to state properties of polynomials which arise in the natural expansion of products of certain Kazhdan–Lusztig basis elements of the Hecke algebra. This leads to a partial answer in Section 5 to the characterization question [23, Quest. 4.5]: a description of certain Kazhdan–Lusztig basis elements which do not factor as desired.

2 The symmetric group and planar networks

Let $\mathfrak{S}_n$ be the symmetric group, with standard generators $s_1, \ldots, s_{n-1}$, length function ℓ, and Bruhat order $\leq$. (See, e.g., [6] for definitions.) Given a word $u = u_1 \cdots u_k$ in $\mathfrak{S}_k$, and a word $y = y_1 \cdots y_k$ having k distinct letters, we say that *y matches the pattern u* if the letters of y appear in the same relative order as those of u; that is, if we have $u_i < u_j$ if and only if $y_i < y_j$ for all $i, j \in [k] := \{1, \ldots, k\}$. On the other hand, say that $w \in \mathfrak{S}_n$ *avoids the pattern u* if no subword of w matches the pattern u.

It is easy to see that for each subinterval $[a, b] := \{a, \ldots, b\}$ of $[n]$, the *reversal*

$$s_{[a,b]} := 1 \cdots (a-1) b \cdots a (b+1) \cdots n \in \mathfrak{S}_n \tag{2.1}$$

avoids the patterns 3412 and 4231. This element is the unique longest (maximum length) element of the subgroup of $\mathfrak{S}_n$ generated by $s_a, \ldots, s_{b-1}$. More generally, each *parabolic* subgroup of $\mathfrak{S}_n$ generated by a subset of generators has longest element equal to a product of reversals on disjoint intervals. Multiplication of reversals in $\mathfrak{S}_n$ or of related elements

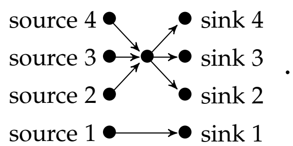
$$D_{[a,b]} := \sum_{v \leq s_{[a,b]}} v \tag{2.2}$$

in $\mathbb{Z}[\mathfrak{S}_n]$ can be performed graphically with certain planar networks.

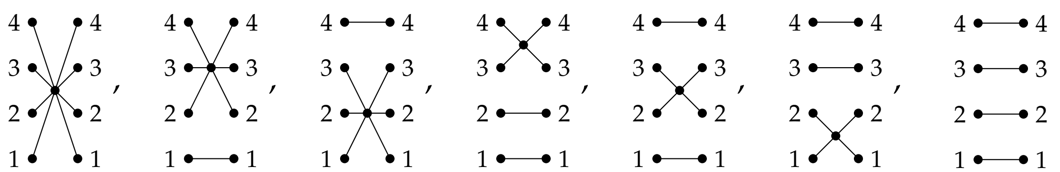
Define a *planar network of order n* to be a directed, planar, acyclic multigraph with $2n$ boundary vertices having n source vertices on the left and n sink vertices on the right, both labeled $1, \ldots, n$ from bottom to top. We will allow edges (x, y) to be marked by a positive integer multiplicity $m(x,y)$. Let $\mathcal{F}_n$ denote the set of such networks. For each subinterval $[a,b]$ of $[n]$ we define a *simple star network* $F_{[a,b]} \in \mathcal{F}_n$ by

1. an interior vertex x lies between the sources and sinks,
2. for $i \in [a,b]$ we have edges from source i to x and from x to sink i,
3. for $i \notin [a,b]$ we have edges from source i to sink i.

For example, the simple star network $F_{[2,4]} \in \mathcal{F}_4$ is



For economy, we will omit edge orientations and the words "source" and "sink" from figures. Thus the seven simple star networks in $\mathcal{F}_4$ are

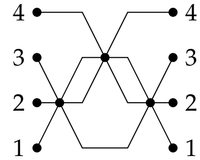
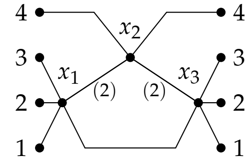


$F_{[1,4]}$ $F_{[2,4]}$ $F_{[1,3]}$ $F_{[3,4]}$ $F_{[2,3]}$ $F_{[1,2]}$ $F_{\emptyset}$ (2.3)

Given networks $E, F \in \mathcal{F}_n$, in which all sources have outdegree 1 and all sinks have indegree 1, define the concatenation $E \circ F$ of E and F as follows. For $i = 1, \ldots, n$, do

1. remove sink i of E and source i of F,
2. merge each edge $(x, \text{sink } i)$ in E with each edge $(\text{source } i, y)$ in F to form a single edge (x,y) in $E \circ F$.

Thus a concatenation of the form $F_{[a_1,b_1]} \circ \cdots \circ F_{[a_m,b_m]} \in \mathcal{F}_n$ has $2n+m$ edges: n sources inherited from $F_{[a_1,b_1]}$, n sinks inherited from $F_{[a_m,b_m]}$, and m internal vertices $x_1, \ldots, x_m$, where x_j is inherited from $F_{[a_j,b_j]}$. Sometimes in a concatenation $E \circ F$, there may exist internal vertices x in E, y in F with $m(x,y) > 1$ multiplicity-1 edges incident upon both. Define the *condensed concatenation* $E \bullet F$ to be the subdigraph of $E \circ F$ obtained by removing, for all such pairs (x,y), all but one of the $m(x,y)$ edges incident upon both, and by marking this edge with the multiplicity $m(x,y)$. For example, in $\mathcal{F}_4$ we have the graphs

$F_{[1,3]} \circ F_{[2,4]} \circ F_{[1,3]} =$  , $F_{[1,3]} \bullet F_{[2,4]} \bullet F_{[1,3]} =$ 

, (2.4)

in which the two multiplicity-2 edges (x_1, x_2), (x_2, x_3) of $F_{[1,3]} \bullet F_{[2,4]} \bullet F_{[1,3]}$ are the remnants of pairs of edges incident upon the same internal vertices in $F_{[1,3]} \circ F_{[2,4]} \circ F_{[1,3]}$.

Define a *star network* to be an element of $\mathcal{F}_n$ constructed by concatenation or condensed concatenation of simple star networks. Let $\mathcal{F}_n^\bullet$ denote the subset of $\mathcal{F}_n$ consisting of condensed concatenations of finitely many simple star networks. Call a sequence $\pi = (\pi_1, \ldots, \pi_n)$ of source-to-sink paths in a star network $F \in \mathcal{F}_n^\bullet$ a *path family of type* $v = v_1 \cdots v_n \in \mathfrak{S}_n$ if for all i, path π_i begins at source i and terminates at sink v_i. Say that π *covers* F if each edge (x_i, x_j) of F belongs to $m(x_i, x_j)$ of the paths in π, and define the sets

$$\begin{aligned} \Pi(F) &= \{\pi \,|\, \pi \text{ a path family covering } F\}, \\ \Pi_v(F) &= \{\pi \in \Pi(F) \,|\, \mathrm{type}(\pi) = v\}. \end{aligned} \tag{2.5}$$

In terms of the definitions (2.5), we may combinatorially interpret products of elements (2.2) quite simply. We say that F *graphically represents*

$$\sum_{v \in \mathfrak{S}_n} |\Pi_v(F)|\, v \tag{2.6}$$

as an element of $\mathbb{Z}[\mathfrak{S}_n]$. For $F = F_{[a_1,b_1]} \circ \cdots \circ F_{[a_m,b_m]}$, this element is $D_{[a_1,b_1]} \cdots D_{[a_m,b_m]}$; for $F = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_m,b_m]}$, it is $D_{[a_1,b_1]} \cdots D_{[a_m,b_m]}$ divided by the product, over all edges (x_i, x_j), of the numbers $m(x_i, x_j)!$.

3 The Hecke algebra and planar networks

Define the *(type-A Iwahori-) Hecke algebra* $H_n(q)$ to be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of its *natural basis* $\{T_w \,|\, w \in \mathfrak{S}_n\}$, with multiplication given by

$$T_{s_i} T_w = \begin{cases} T_{s_i w} & \text{if } s_i w > w, \\ (q-1)T_{s_i w} + qT_w & \text{if } s_i w < w. \end{cases}$$

Specializing at $q^{\frac{1}{2}} = 1$ we have $H_n(1) \cong \mathbb{Z}[\mathfrak{S}_n]$ with $T_w \mapsto w$.

A semilinear involution on $H_n(q)$, known as the *bar involution*, is defined by

$$\overline{q^{\frac{1}{2}}} = q^{\frac{-1}{2}}, \qquad \overline{T_w} = (T_{w^{-1}})^{-1}, \qquad \overline{\sum_{w \in \mathfrak{S}_n} B_w(q) T_w} = \sum_{w \in \mathfrak{S}_n} \overline{B_w(q)}\, \overline{T_w}.$$

Kazhdan and Lusztig showed [19] that $H_n(q)$ has a unique basis $\{C'_w \,|\, w \in \mathfrak{S}_n\}$ satisfying $\overline{C'_w} = C'_w$ for all w and

$$q^{\frac{\ell(w)}{2}} C'_w = \sum_{v \le w} P_{v,w}(q) T_w, \tag{3.1}$$

where coefficients $P_{v,w}(q) \in \mathbb{Z}[q]$, known as the *Kazhdan–Lusztig polynomials*, satisfy $\deg(P_{v,w}(q)) < \frac{\ell(w) - \ell(v) - 1}{2}$ for $v < w$, and $P_{w,w}(q) = 1$ for all w. It is known that these polynomials satisfy $P_{v,w}(q) \in \mathbb{N}[q]$, and $P_{v,w}(0) = 1$ for $v \leq w$. We also have [21] that if w avoids the patterns 3412 and 4231, then $P_{v,w}(q) = 1$ for all $v \leq w$. For convenience, we define

$$\widetilde{C}_w := q^{\frac{\ell(w)}{2}} C'_w. \tag{3.2}$$

Kazhdan–Lusztig basis elements and their products appear in various settings, including intersection homology [3], [25], algorithmic and combinatorial description of Kazhdan–Lusztig basis elements themselves [4], [13], Schubert varieties [4], total nonnegativity [16], [23], [27], [28], trace evaluations [10], [11], [17], [18], [24], and chromatic symmetric functions [10], [24].

Deodhar [13, Prop. 3.5] studied sequences $(s_{i_1}, \dots, s_{i_k})$ of generators of $\mathfrak{S}_n$, products of the corresponding Kazhdan–Lusztig basis elements $\widetilde{C}_{s_{i_j}} = T_e + T_{s_{i_j}}$ of $H_n(q)$, and their natural expansions

$$\widetilde{C}_{s_{i_1}} \cdots \widetilde{C}_{s_{i_m}} = \sum_{v \in \mathfrak{S}_n} A_v(q) T_v. \tag{3.3}$$

He described the coefficients $\{A_v(q) \,|\, v \in \mathfrak{S}_n\} \subset \mathbb{Z}[q]$ in terms of *subexpressions* of $(s_{i_1}, \dots, s_{i_m})$, sequences $\sigma = (\sigma_1, \dots, \sigma_m)$ with $\sigma_j \in \{e, s_{i_j}\}$ for $j = 1, \dots, m$. (Our treatment here differs slightly from that of [13] but is equivalent.) Call index j a *defect* of σ if

$$\sigma_1 \cdots \sigma_{j-1} s_{i_j} < \sigma_1 \cdots \sigma_{j-1} \tag{3.4}$$

and let $\mathrm{dfct}(\sigma)$ denote the number of defects of σ. (Observe that $j = 1$ cannot be a defect: we have $s_{i_1} > e$ always.) Each coefficient on the right-hand side of (3.3) is given by

$$A_v(q) = \sum_{\sigma} q^{\mathrm{dfct}(\sigma)}, \tag{3.5}$$

where the sum is over all subexpressions σ of $(s_{i_1}, \dots, s_{i_m})$ satisfying $\sigma_1 \cdots \sigma_m = v$.

Billey and Warrington observed [4, Rmk. 6] that the defect statistic has a simple graphical interpretation. Specifically, subexpressions of $(s_{i_1}, \dots, s_{i_m})$ correspond bijectively to path families covering

$$F = F_{[i_1, i_1+1]} \bullet \cdots \bullet F_{[i_m, i_m+1]} \tag{3.6}$$

in $\mathcal{F}_n^\bullet$ with $(\sigma_1, \dots, \sigma_m)$ corresponding to the family $\pi \in \Pi(F)$ constructed by prescribing

$$\text{the paths meeting at } x_j \begin{cases} \text{cross there} & \text{if } \sigma_j = s_{i_j}, \\ \text{do not cross there} & \text{if } \sigma_j = e. \end{cases}$$

By this bijection, index j is a defect of σ in the sense of (3.4) if and only if the paths meeting at x_j have previously crossed an odd number of times.

Clearwater–Skandera extended this result [11, Cor. 5.3] to products of the form

$$\widetilde{C}_{s_{[a_1,b_1]}} \cdots \widetilde{C}_{s_{[a_m,b_m]}} = \sum_{v \in \mathfrak{S}_n} A_v(q) T_v, \tag{3.7}$$

where each factor satisfies

$$\widetilde{C}_{s_{[a_j,b_j]}} = \sum_{u \leq s_{[a_j,b_j]}} T_u,$$

since reversals avoid the patterns 3412 and 4231. This extension requires a more general definition of defects. While the intersection of two paths in (3.6) is a union of vertices, the intersection of two paths in

$$F = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_m,b_m]} \tag{3.8}$$

is a subgraph of F whose connected components are vertices or paths of the form

$$(x_k, \dots, x_\ell) \tag{3.9}$$

for some $k < \ell$. For each initial vertex x_k in a component (3.9) of the intersection of two paths, we will say that the paths *meet* at x_k. Our embedding of star networks in the plane naturally allows us to declare an edge entering (exiting) a vertex x_k to be above or below another edge entering (exiting) x_k. We will call a component (3.9) in the intersection of paths π_i, π_j, a *crossing* of π_i and π_j if the two paths enter x_k and exit x_ℓ in different orders. Extending the Billey–Warrington definition of defect to accommodate three or more paths passing through a vertex, we have the following [11, §5].

Definition 3.1. Given a path family π covering $F = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_m,b_m]}$, define a *defect* of π at x_k to be a triple (π_i, π_j, k) with $i < j$ and π_i and π_j meeting at x_k after having crossed an odd number of times. Define $\mathrm{dfct}(\pi)$ to be the number of defects of π.

Extending the interpretation (2.6) of a planar network, we define the set

$$\Pi_{v,d}(F) = \{\pi \in \Pi_v(F) \mid \mathrm{dfct}(\pi) = d\} \tag{3.10}$$

and we say that F *graphically represents the element*

$$\sum_{v \in \mathfrak{S}_n} \sum_{d \geq 0} |\Pi_{v,d}(F)| q^d T_v = \sum_{\pi \in \Pi(F)} q^{\mathrm{dfct}(\pi)} T_{\mathrm{type}(\pi)} \tag{3.11}$$

as an element of $H_n(q)$. For $F = F_{[a_1,b_1]} \circ \cdots \circ F_{[a_m,b_m]}$, this element is $\widetilde{C}_{s_{[a_1,b_1]}} \cdots \widetilde{C}_{s_{[a_m,b_m]}}$; for $F = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_m,b_m]}$, it is $\widetilde{C}_{s_{[a_1,b_1]}} \cdots \widetilde{C}_{s_{[a_m,b_m]}}$, divided by the product, over all edges (x_i, x_j), of the q-factorial polynomials $m(x_i, x_j)_q!$. (See, e.g., [26].)

The denominator above can also be expressed in terms of intervals appearing in reversals. Given a sequence of m intervals

$$\mathcal{A} = ([a_1, b_1], \dots, [a_m, b_m]), \tag{3.12}$$

define $\binom{m}{2}$ more intervals $\{I_{i,j} \mid i < j\}$ by

$$I_{i,j} = [a_i, b_i] \cap [a_j, b_j] \smallsetminus ([a_{i+1}, b_{i+1}] \cup \cdots \cup [a_{j-1}, b_{j-1}]). \tag{3.13}$$

Let $f_{\mathcal{A}}(q)$ be the product of the q-factorials of the cardinalities of the intervals (3.13),

$$f_{\mathcal{A}}(q) := \prod_{i<j} |I_{i,j}|_q!. \tag{3.14}$$

Say that a Kazhdan–Lusztig basis element $\widetilde{C}_w$ has a *parabolic factorization* if there is a sequence (3.12) of intervals satisfying

$$\widetilde{C}_w = \frac{1}{f_{\mathcal{A}}(q)} \widetilde{C}_{s_{[a_1,b_1]}} \cdots \widetilde{C}_{s_{[a_m,b_m]}}. \tag{3.15}$$

Two results on parabolic factorization are the following [4, Thm. 1], [23, Thm. 4.3].

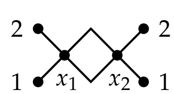
Theorem 3.2. *If $w \in \mathfrak{S}_n$ avoids the patterns* 321, 56781234, 46781235, 56718234, 46718235, *and if $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for w, then we have $\widetilde{C}_w = \widetilde{C}_{s_{i_1}} \cdots \widetilde{C}_{s_{i_\ell}}$.*

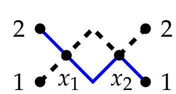
Theorem 3.3. *If $w \in \mathfrak{S}_n$ avoids the patterns* 3412 *and* 4231, *then there exists a sequence* (3.12) *of intervals such that we have the factorization* (3.15).

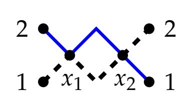
The combination of Theorems 3.2, 3.3 is not the strongest result possible. Indeed, the known parabolic factorization $\widetilde{C}_{4231} = \widetilde{C}_{s_{[1,2]}} \widetilde{C}_{s_{[2,4]}} \widetilde{C}_{s_{[1,2]}}$ is guaranteed by neither theorem.

4 Reduction of defects

Our defect reduction theorem asserts that if a star network can be covered by a path family having d defects, then it can also be covered by a path family of the same type having $d-1$ defects. In certain simple cases, we can easily produce a $(d-1)$-defect family from a d-defect family by swapping a pair of subpaths. For example, consider the star network and path families

$F_{[1,2]} \circ F_{[1,2]} =$ 

, $\pi =$ 

, $\sigma =$ 

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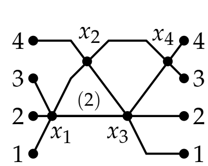
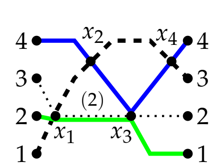
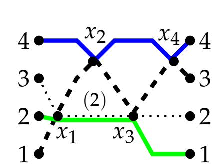
with $\mathrm{dfct}(\pi) = 1$, σ constructed from π by swapping the two x_1-to-x_2 subpaths of π, and $\mathrm{dfct}(\sigma) = 0$. On the other hand, this simple procedure does not always reduce defects by 1. Consider the star network and path families of type 3124

$F_{[1,3]} \bullet F_{[3,4]} \bullet F_{[1,3]} \bullet F_{[3,4]} =$ 

, $\pi =$ 

, $\sigma =$ 

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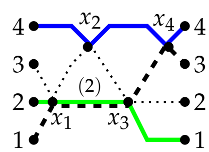
with $\mathrm{dfct}(\pi) = 1$, and σ constructed from π by swapping the x_2-to-x_4 subpaths of π_1 and π_4. The swap eliminates the defect $(\pi_1, \pi_4, 4)$, but introduces two more: $(\sigma_1, \sigma_2, 3)$, $(\sigma_1, \sigma_3, 3)$. Thus we have $\mathrm{dfct}(\sigma) = 2$. There is in fact a path family

$\tau =$ 

of type 3124 satisfying $\mathrm{dfct}(\tau) = \mathrm{dfct}(\pi) - 1 = 0$, but the naive approach above does not produce it from π.

To describe the process of reducing defects in a path family by exactly 1, we begin by stating a map which modifies a path family by removing a defect at a specified vertex, possibly creating earlier defects. For $F = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_m,b_m]}$ and $k \in \{2, \ldots, m\}$, define

$$\begin{aligned} \phi_k : \{\pi \in \Pi(F) \mid \pi \text{ has a defect at } x_k\} &\to \Pi(F) \\ \pi &\mapsto \hat{\pi} \end{aligned} \tag{4.1}$$

by the following algorithm.

Algorithm 4.1. Given star network $F = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_m,b_m]} \in \mathcal{F}_n^{\bullet}$, path family $\pi \in \Pi(F)$, and index k such that π has a defect at x_k, do

1. Let (r, t) be the lexicographically least pair such that (π_r, π_t, k) is a defect.
2. Let s be the largest index such that π_s enters vertex x_k through the same edge as π_r and (π_s, π_t, k) is a defect.
3. Let x_l be the final vertex in the unique crossing of π_s and π_t prior to x_k.
4. Create a new path family $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_n)$ by
   (a) $\hat{\pi}_i = \pi_i$ if $i \notin \{s, t\}$.
   (b) $\hat{\pi}_s$ is π_s with the x_l-to-x_k subpath replaced by that of π_t.
   (c) $\hat{\pi}_t$ is π_t with the x_l-to-x_k subpath replaced by that of π_s.

Proposition 4.2. *Algorithm 4.1 produces a path family $\phi_k(\pi) = \hat{\pi}$ that satisfies*

1. $\mathrm{type}(\hat{\pi}) = \mathrm{type}(\pi)$,
2. *for each $p > k$, we have $\{(i,j) \mid (\hat{\pi}_i, \hat{\pi}_j, p)$ defective$\} = \{(i,j) \mid (\pi_i, \pi_j, p)$ defective$\}$,*
3. *$\#\{(i,j) \mid (\hat{\pi}_i, \hat{\pi}_j, k)$ defective$\} = \#\{(i,j) \mid (\pi_i, \pi_j, k)$ defective$\} - 1$.*

Proof. Omitted. □

We may further modify the path family $\hat{\pi}$ to produce a path family having no defects at $x_1, \ldots, x_{k-1}$.

Proposition 4.3. *Fix $F = F_{[a_1,b_1]} \bullet \cdots \bullet F_{[a_m,b_m]} \in \mathcal{F}_n^\bullet$, and path family $\pi \in \Pi_v(F)$ having earliest defect at x_k. Then there exists a path family $\sigma \in \Pi_v(F)$ which satisfies*

1. *for all $p \neq k$, we have $\{(i,j) \mid (\sigma_i, \sigma_j, p)$ defective$\} = \{(i,j) \mid (\pi_i, \pi_j, p)$ defective$\}$,*
2. *for $p = k$, we have $\#\{(i,j) \mid (\sigma_i, \sigma_j, p)$ defective$\} = \#\{(i,j) \mid (\pi_i, \pi_j, p)$ defective$\} - 1$.*

Proof. By Proposition 4.2, $\hat{\pi} = \phi_k(\pi)$ has one fewer defect at x_k than π has, with defects at $\{x_p \mid p > k\}$ matching those of π. Now let d be the number of defects of $\hat{\pi}$ at x_{k-1} and let $\sigma^{(k-1)} = \phi_{k-1}^d(\hat{\pi})$. Then $\sigma^{(k-1)}$, like π, has no defects at x_{k-1} and belongs to $\Pi_v(F)$. Thus it satisfies (1), (2) for $p \geq k-1$. Similarly applying $\phi_{k-2}, \ldots, \phi_2$, we construct path families $\sigma^{(k-2)}, \ldots, \sigma^{(2)}$ in $\Pi_w(F)$ having no defects at $x_{k-2}, \ldots, x_2$. By definition, no defect can occur at x_1. Thus $\sigma = \sigma^{(2)}$ satisfies (1), (2) for all p. □

By Proposition 4.3, we see that a path family having type v and d defects (i.e., in $\Pi_{v,d}(F)$) implies the existence of other path families of type v having $d-1, \ldots, 0$ defects.

Theorem 4.4. *Fix star network $F \in \mathcal{F}_n^\bullet$. If for some $v \in \mathfrak{S}_n$ and $d \geq 1$ the set $\Pi_{v,d}(F)$ is nonempty, then the sets $\Pi_{v,d-1}(F), \ldots, \Pi_{v,0}(F)$ are also nonempty, and $|\Pi_{v,0}(F)| = 1$.*

Proof. Omitted. □

Corollary 4.5. *For $v \in \mathfrak{S}_n$ and $F \in \mathcal{F}_n^\bullet$, the number of path families of type v and having* 0 *defects is at most one. If $v = e$, then this number is exactly one.*

Proof. Omitted. □

Corollary 4.6. *Fix star network $F \in \mathcal{F}_n^\bullet$. If $\Pi_e(F)$ contains more than one path family, then it contains a path family having exactly one defect.*

Proof. Suppose $\Pi_e(F)$ contains at least two path families. By Corollary 4.5, one element of $\Pi_e(F)$ is the unique element of $\Pi_{e,0}(F)$. Choose another path family in $\Pi_e(F)$ and let $d \geq 1$ be the number of its defects. By Theorem 4.4, the set $\Pi_{e,1}(F)$ is nonempty. □

5 Main Result

Every polynomial in $\mathbb{N}[q]$ with constant term 1 arises as a Kazhdan–Lusztig polynomial [22]. Gaetz–Gao [15] studied the sequences of coefficients in these polynomials, especially coefficients equal to 0 between other nonzero coefficients. Define a function singdeg : $\mathfrak{S}_n \to \mathbb{N} \cup \{\infty\}$ by

$$\mathrm{singdeg}(w) = \begin{cases} \infty & \text{if } P_{e,w}(q) = 1, \\ \min\{k > 0 \,|\, \text{coefficient of } q^k \text{ in } P_{e,w}(q) \text{ is nonzero}\} & \text{if } P_{e,w}(q) \neq 1. \end{cases} \tag{5.1}$$

This is a lower bound on degrees for which Poincaré duality fails in the Schubert variety X_w, and can be computed in terms of patterns in w and a related definition. Specifically, given $w \in \mathfrak{S}_n$ not avoiding the pattern 3412, define the *3412-gap* of w by

$$\mathrm{gap}_{3412}(w) = \min\{w_{i_1} - w_{i_4} \,|\, \text{subword } w_{i_1}w_{i_2}w_{i_3}w_{i_4} \text{ matches the pattern 3412}\}. \tag{5.2}$$

For w avoiding the patterns 3412 and 4231, we have $\mathrm{singdeg}(w) = \infty$. Otherwise, we can compute $\mathrm{singdeg}(w)$ in terms of $\mathrm{gap}_{3412}(w)$ as follows [15, Thm. 1.6].

Theorem 5.1. *For w not avoiding the patterns* 3412 *and* 4231 *we have*

$$\mathrm{singdeg}(w) = \begin{cases} \mathrm{gap}_{3412}(w) & \textit{if } w \textit{ avoids the pattern } 4231, \\ 1 & \textit{otherwise.} \end{cases}$$

For example, consider the permutation $45312 \in \mathfrak{S}_5$, which avoids the pattern 4231 and has only the subword 4512 matching the pattern 3412. Since $\mathrm{gap}_{3412}(45312) = 2$, Theorem 5.1 implies that the coefficient of q in $P_{e,45312}(q)$ is 0 and that the coefficient of q^2 is not. This is consistent with the fact that $P_{e,45312}(q) = 1 + q^2$. (See [5, p. 75].)

For each permutation w having $\mathrm{singdeg}(w) > 1$, the Kazhdan–Lusztig basis element $\widetilde{C}_w$ has no parabolic factorization (3.15) and therefore is not graphically representable by a star network, in the sense of (3.11).

Theorem 5.2. *For $w \in \mathfrak{S}_n$ avoiding the pattern* 4231*, not avoiding the pattern* 3412*, and having $\mathrm{gap}_{3412}(w) > 1$, the Kazhdan–Lusztig basis element $\widetilde{C}_w$ has no parabolic factorization.*

Proof. Fix w as above with $k = \mathrm{gap}_{3412}(w)$ and suppose that the star network F graphically represents $\widetilde{C}_w$ as an element of $H_n(q)$,

$$\widetilde{C}_w = \sum_{v \in \mathfrak{S}_n} \sum_{d \geq 0} |\Pi_{v,d}(F)| q^d T_v = \sum_{v \leq w} P_{v,w}(q) T_v. \tag{5.3}$$

Since the constant term of $P_{e,w}(q)$ is 1, we have $|\Pi_{e,0}(F)| = 1$. By our definition of k and Theorem 5.1, the coefficients of $q, \ldots, q^{k-1}$ in $P_{e,w}(q)$ are 0, while the coefficient of q^k is positive. In particular, we have the cardinalities $|\Pi_{e,1}(F)| = \cdots = |\Pi_{e,k-1}(F)| = 0$ and $|\Pi_{e,k}(F)| > 0$, which contradict Theorem 4.4. □

By Theorem 4.4, parabolic factorization of $\widetilde{C}_w$ implies that *none* of the Kazhdan–Lusztig polynomials $P_{v,w}(q)$ has internal coefficients equal to zero.

Theorem 5.3. *If $\widetilde{C}_w$ has a parabolic factorization, then for every $v < w$ there exists $k = k(v)$ in $\mathbb{N}$ such that we have $P_{v,w}(q) = 1 + a_1 q + \cdots + a_k q^k$ with $a_1, \ldots, a_k > 0$.*

It is easy to show that the inequality $\mathrm{gap}_{3412}(w) > 1$ implies that w does not avoid the pattern 45312. It is also easy to show that no star network graphically represents $\widetilde{C}_{453129786}$, even though the subword 9786 matches the pattern 4231. Indeed, some limited experimentation [12] suggests that avoidance of the pattern 45312 is important and avoidance of the pattern 4231 is unimportant in the classification of permutations w for which $\widetilde{C}_w$ is graphically representable by a star network. We conjecture the following partial answer to [23, Quest. 4.5].

Conjecture 5.4. *If $w \in \mathfrak{S}_n$ does not avoid the pattern* 45312*, then the Kazhdan–Lusztig basis element $\widetilde{C}_w$ has no parabolic factorization.*

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