ON KAZHDAN–LUSZTIG BASIS ELEMENTS HAVING NO REVERSAL FACTORIZATION

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Outline

(1) The symmetric group and Hecke algebra
(2) Natural and Kazhdan–Lusztig bases
(3) Graphical representations
(4) Positive and negative factorization results

The symmetric group S_n

$S_n :=$ the $n!$ permutations $w = w_1 \cdots w_n$ of $e = 1 \cdots n$.

For $1 \leq a < b \leq n$ define the *reversal*

$$s_{[a,b]} := 1 \cdots (a-1) \underbrace{\underline{b \cdots a}}_{\text{(reversed)}} (b+1) \cdots n \in S_n.$$

Ex: In S_8 we have $s_{[3,5]} = 12\underline{543}678$ and

$$s_{[3,5]} w = w_1 w_2 \underline{w_5 w_4 w_3} w_6 w_7 w_8.$$

Fact: Every permutation can be written as $s_{[i_1, i_1+1]} \cdots s_{[i_\ell, i_\ell+1]}$.

Ex: $3412 = s_{[2,3]} s_{[1,2]} s_{[3,4]} s_{[2,3]}$.

The Hecke algebra $H_n(q)$ and its natural basis

$H_n(q) := \mathrm{span}_{\mathbb{Z}[q]}\{T_w \,|\, w \in S_n\}$ with multiplication

$$T_{s_{[i,i+1]}} T_w = \begin{cases} T_{s_{[i,i+1]}w} & \text{if } w_i < w_{i+1}, \\ q T_{s_{[i,i+1]}w} + (q-1)T_w & \text{if } w_i > w_{i+1}. \end{cases}$$

$H_n(q)$ has dimension $n!$, with *natural basis* $\{T_w \,|\, w \in S_n\}$.

Fact: Every natural basis element T_w can be written as

$$T_w = T_{s_{[i_1,i_1+1]}} \cdots T_{s_{[i_\ell,i_\ell+1]}}$$

if $s_{[i_1,i_1+1]} \cdots s_{[i_\ell,i_\ell+1]}$ is a minimum-length expression for w.

Ex: $T_{3412} = T_{s_{[2,3]}} T_{s_{[1,2]}} T_{s_{[3,4]}} T_{s_{[2,3]}}; \qquad T_e = 1.$

The Kazhdan–Lusztig basis of $\boldsymbol{H_n(q)}$

A second basis $\{\widetilde{C}_w \,|\, w \in S_n\}$ of $H_n(q)$ is defined existentially or recursively. The two bases are related by

$$\widetilde{C}_w = \sum_{v \in S_n} P_{v,w}(q) T_v, \qquad \{P_{v,w}(q) \,|\, v, w \in S_n\} \subseteq \mathbb{N}[q].$$

The coefficients are called *Kazhdan–Lusztig polynomials*.

Appl: alg. geom., repn. thy., topology, physics, TNN matrices.

Fact: No known simple formula computes $P_{v,w}(q)$ for all v, w.

Fact: For reversals $w = s_{[a,b]}$, we have $P_{v,w}(q) \in \{0, 1\}$.

Ex: $\widetilde{C}_{1432} = T_{1234} + T_{1243} + T_{1324} + T_{1342} + T_{1423} + T_{1432}$.

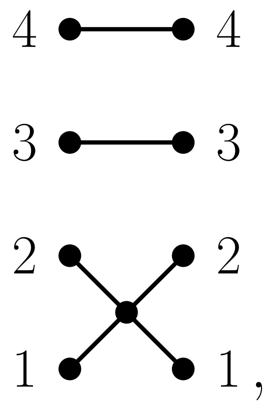
Fact: For some w, the element $\widetilde{C}_w$ has a reversal factorization:

Ex: $\widetilde{C}_{3412} = \widetilde{C}_{s_{[2,3]}} \widetilde{C}_{s_{[1,2]}} \widetilde{C}_{s_{[3,4]}} \widetilde{C}_{s_{[2,3]}}$, $\widetilde{C}_{4231} = \widetilde{C}_{s_{[1,2]}} \widetilde{C}_{s_{[2,4]}} \widetilde{C}_{s_{[1,2]}}$.

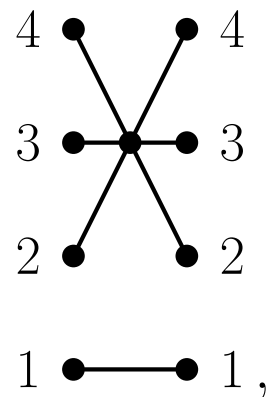
Graphical representation

$$\widetilde{C}_{4231} = (1+q)(T_{1234} + T_{2134} + T_{1243} + T_{2143}) + T_{1324} + T_{1423} + T_{1342} + T_{3124} + T_{2314} + T_{4123} + T_{1432} + T_{2413} + T_{3142} + T_{2341} + T_{3214} + T_{4132} + T_{4213} + T_{2431} + T_{3241} + T_{4231}.$$

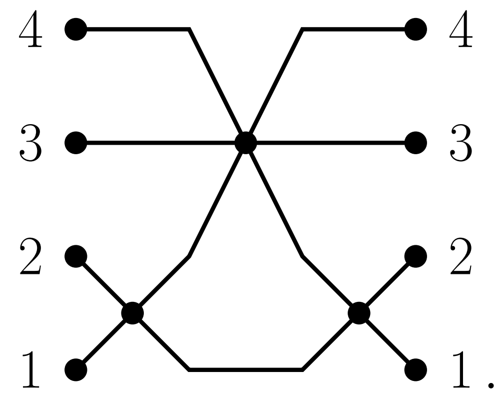
The reversal factorization $\widetilde{C}_{4231} = \widetilde{C}_{s_{[1,2]}}\widetilde{C}_{s_{[2,4]}}\widetilde{C}_{s_{[1,2]}}$ allows us to compute coefficients above with the star networks (CS, '21)



$F_{[1,2]}$

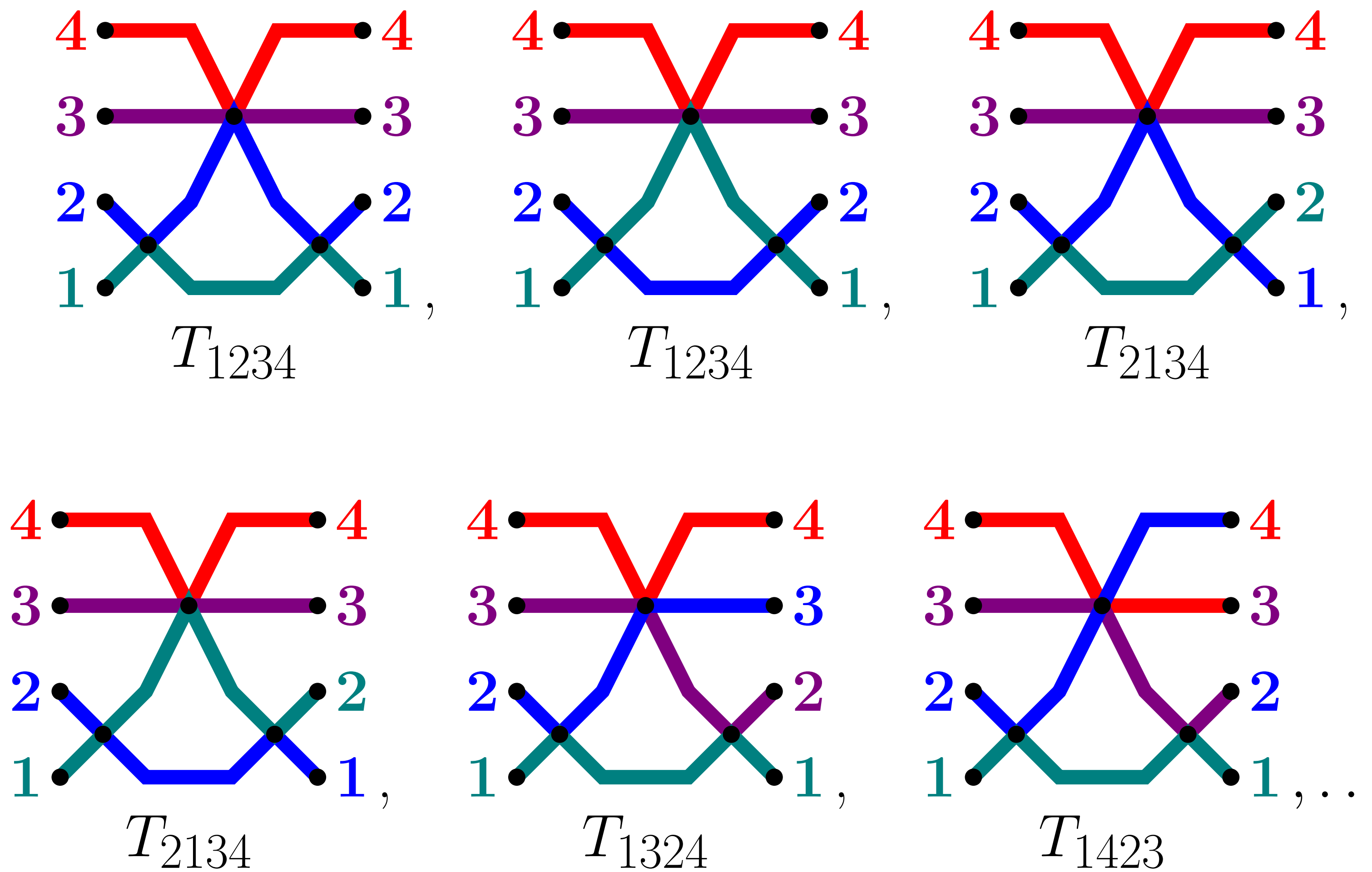


$F_{[2,4]}$

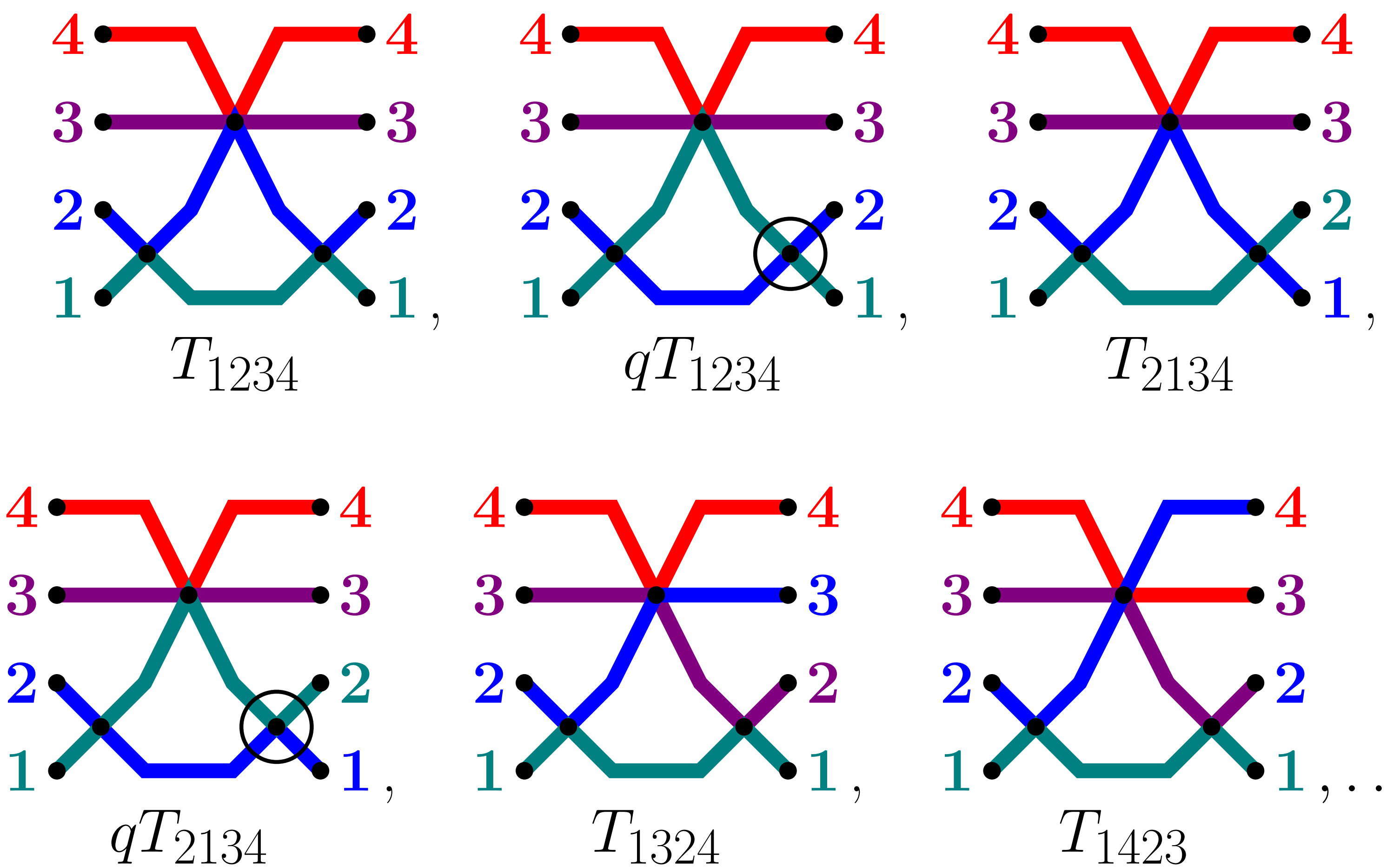


$F_{[1,2]}F_{[2,4]}F_{[1,2]}$

Contributing to T_w are path families of type w:



Defn: Call a meeting point of already-crossed paths a *defect*. Defects each contribute q.



Reversal factorization theorem and question

Thm: (CS) If $\widetilde{C}_w = \sum_v P_{v,w}(q) T_v$ has a reversal factorization producing star network F, then the coefficient of q^d in $P_{v,w}(q)$ is # path families covering F having type v and d defects.

Q: For which $w \in S_n$ does $\widetilde{C}_w$ have a reversal factorization?

Three theorems (partially) answer in terms of pattern avoidance.

Ex: 45312 contains the pattern 321 four times:

45312, 45312, 45312, 45312.

It contains the pattern 3412 once: 45312.

It avoids the pattern 4231. No subword matches this pattern.

Positive and negative results

Thm: (BW '01) If w avoids the patterns 321, 56781234, 46781235, 56718234, 46718235, then it has a reversal factorization.

Thm: (S '08) If w avoids the patterns 3412, 4231, then it has a reversal factorization.

Fact: Factorization of $\widetilde{C}_{4231}$ is not guaranteed by either theorem.

Defn: For w containing the pattern 3412, define

$\mathrm{gap}_{3412}(w) = \min\{w_{i_1} - w_{i_4} \mid w_{i_1}w_{i_2}w_{i_3}w_{i_4} \text{ matches } 3412\}$.

Ex:

w	$\mathrm{gap}_{3412}(w)$
34512	$3-2=1$
45312	$4-2=2$
563124	$5-4=1$
564312	$5-2=3$

Main Theorem: (PSSW, '26) If w avoids the pattern 4231, contains the pattern 3412, and has $\mathrm{gap}_{3412}(w) > 1$, then $\widetilde{C}_w$ has no reversal factorization.

Proof idea: Combine the following results.

Thm: (PSSW, '26) If a path family covers star network F and has d defects, then other path families have $d-1$, $d-2, \ldots, 0$ defects.

Cor: (PSSW, '26) If F graphically represents $\widetilde{C}_w$, then $P_{e,w}(q)$ has no "missing" powers of q.

Thm: (GG, '24) If $w \in S_n$ avoids the pattern 4231, then
missing powers of q after $q^0 = 1$ in $P_{e,w}(q)$ is $\mathrm{gap}_{3412}(w) - 1$.

Example and conjecture

Fact: 45312 avoids the pattern 4231, contains the pattern 3412, and has $\mathrm{gap}_{3412}(45312) = 2$: 45312.
Fact: We have $P_{e,45312}(q) = 1 + q^2$; the coefficient of q is 0.
Consequence: $\widetilde{C}_{45312}$ has no reversal factorization.

Fact: (DS, GASCom '22) For all $w \in S_5, S_6, S_7$, the Kazhdan–Lusztig basis element $\widetilde{C}_w$ has a reversal factorization if and only if w avoids the pattern 45312.

Conj: (DS, GASCom '22) If $w \in S_n$ does not avoid the pattern 45312, then $\widetilde{C}_w$ has no reversal factorization.