INEQUALITIES FOR COEFFICIENTS OF CHROMATIC SYMMETRIC FUNCTIONS

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Outline

(1) Three families of sets counted by Catalan numbers
(2) Chromatic symmetric functions, total nonnegativity, characters
(3) Inequalities related to induced sign characters
(4) Inequalities related to irreducible characters

Three sets counted by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$

$\mathcal{P}_n$ = unit interval orders on n elements,

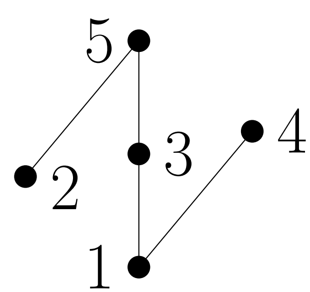
$\mathcal{A}_n$ = 0-1 matrices with 0 only above Dyck path above diagonal,

$\mathfrak{S}_n(312)$ = 312-avoiding permutations in $\mathfrak{S}_n$

are related by bijections $P \mapsto A = (a_{i,j}) \mapsto w_1 \cdots w_n$, where

$$a_{i,j} = \begin{cases} 0 & \text{if } i <_P j, \\ 1 & \text{otherwise,} \end{cases}$$

$$w_i = \max\{j \in [n] \smallsetminus \{w_1, \ldots, w_{i-1}\} \mid a_{i,j} = 1\}.$$



$$\mapsto \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \mapsto \quad 24351.$$

The chromatic symmetric function $\mathbf{X}_{\mathrm{inc}(\mathbf{P})}$

Given $\lambda \vdash n$, define $c_\lambda(P) :=$ # col.-strict P-tableaux of shape $\lambda^\top$,

$d_\lambda(P) :=$ # std. P-tableaux of shape λ,

$\epsilon^\lambda := \mathrm{sgn}\big\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} =$ induced sign character, $\quad \chi^\lambda :=$ irr. character,

$C'_w(1) :=$ Kazhdan–Lusztig basis element of $\mathbb{Z}[\mathfrak{S}_n]$ indexed by w.

For $\theta : \mathfrak{S}_n \to \mathbb{Z}$, define $\mathrm{Imm}_\theta(A) := \sum_{w \in \mathfrak{S}_n} \theta(w) a_{1,w_1} \cdots a_{n,w_n}$.

For corresponding P, A, w define $X_{\mathrm{inc}(P)} :=$

$$\sum_{\lambda \vdash n} c_\lambda(P) m_\lambda = \sum_{\lambda \vdash n} \mathrm{Imm}_{\epsilon^\lambda}(A) m_\lambda = \sum_{\lambda \vdash n} \epsilon^\lambda(C'_w(1)) m_\lambda$$

$$= \sum_{\lambda \vdash n} d_\lambda(P) s_{\lambda^\top} = \sum_{\lambda \vdash n} \mathrm{Imm}_{\chi^\lambda}(A) s_{\lambda^\top} = \sum_{\lambda \vdash n} \chi^\lambda(C'_w(1)) s_{\lambda^\top}.$$

Coefficient inequalities

Question: What inequalities hold for

$$\{c_\lambda(P) \mid \lambda \vdash n\}, \quad \{d_\lambda(P) \mid \lambda \vdash n\} \quad \forall P \in \mathcal{P}_n,$$
$$\{\mathrm{Imm}_{\epsilon^\lambda}(A) \mid \lambda \vdash n\}, \quad \{\mathrm{Imm}_{\chi^\lambda}(A) \mid \lambda \vdash n\} \quad \forall A \in \mathcal{A}_n,$$
$$\{\epsilon^\lambda(C'_w(1)) \mid \lambda \vdash n\}, \quad \{\chi^\lambda(C'_w(1)) \mid \lambda \vdash n\} \quad \forall w \in \mathfrak{S}_n(213)?$$

Or more generally, but no longer equivalently, for

- coefficients in X_G for G claw-free,
- immanants of $n \times n$ totally nonnegative (TNN) matrices,
- evaluations of characters at $\{C'_w(1) \mid w \in \mathfrak{S}_n\}$.

Results for $n \times n$ positive semidefinite (PSD) matrices involve character evaluations $\theta(e)$ and majorization $\preceq$ of partitions.

Barrett–Johnson inequalities

Thm: (BJ '93) The inequality

$$\frac{\mathrm{Imm}_{\epsilon^\lambda}(A)}{\epsilon^\lambda(e)} \geq \frac{\mathrm{Imm}_{\epsilon^\mu}(A)}{\epsilon^\mu(e)} \tag{1}$$

holds for all real PSD matrices A iff $\lambda \preceq \mu$.

Fact: $\epsilon^\lambda(e) = \#$ column-strict Young tableaux of shape $\lambda^\top$.

Thm: (SS '25) Ineq. (1) holds for all TNN matrices A iff $\lambda \preceq \mu$.

Prop: Also equivalent to $\lambda \preceq \mu$ are the inequalities

$$\frac{\epsilon^\lambda(C'_w(1))}{\epsilon^\lambda(e)} \geq \frac{\epsilon^\mu(C'_w(1))}{\epsilon^\mu(e)} \quad \text{for all } w \in \mathfrak{S}_n(3412, 4231),$$

$$\frac{c_\lambda(P)}{\epsilon^\lambda(e)} \geq \frac{c_\mu(P)}{\epsilon^\mu(e)} \quad \text{for all } P \in \mathcal{P}_n.$$

Merris–Heyfron inequalities

Thm: (H '88) The hook immanant inequalities

$$\begin{aligned} \mathrm{per}(A) &= \frac{\mathrm{Imm}_{\chi^{n}}(A)}{\chi^{n}(e)} \geq \frac{\mathrm{Imm}_{\chi^{n-1,1}}(A)}{\chi^{n-1,1}(e)} \geq \frac{\mathrm{Imm}_{\chi^{n-2,1,1}}(A)}{\chi^{n-2,1,1}(e)} \\ &\geq \frac{\mathrm{Imm}_{\chi^{n-3,1,1,1}}(A)}{\chi^{n-3,1,1,1}(e)} \geq \cdots \geq \frac{\mathrm{Imm}_{\chi^{1,\ldots,1}}(A)}{\chi^{1,\ldots,1}(e)} = \det(A) \end{aligned} \tag{2}$$

hold for all Hermitian PSD matrices A.

Fact: $\chi^{\lambda}(e) = \#$ standard Young tableaux of shape λ.

Thm: (S '25) Inequalities (2) hold for all TNN matrices A.

Characterizing all pairs (λ, μ) for which

$$\frac{\mathrm{Imm}_{\chi^{\lambda}}(A)}{\chi^{\lambda}(e)} \geq \frac{\mathrm{Imm}_{\chi^{\mu}}(A)}{\chi^{\mu}(e)} \quad \text{for all } A \text{ HPSD/TNN}$$

will probably be hard (James, Stembridge).

Consequences of main result

Cor: For $w \in \mathfrak{S}_n(3412, 4231)$ we have

$$\#\{v \leq w\} = \frac{\chi^{n}(C'_w(1))}{\chi^{n}(e)} \geq \frac{\chi^{n-1,1}(C'_w(1))}{\chi^{n-1,1}(e)} \geq \frac{\chi^{n-2,1,1}(C'_w(1))}{\chi^{n-2,1,1}(e)}$$

$$\geq \frac{\chi^{n-3,1,1,1}(C'_w(1))}{\chi^{n-3,1,1,1}(e)} \geq \cdots \geq \frac{\chi^{1,\ldots,1}(C'_w(1))}{\chi^{1,\ldots,1}(e)} = \begin{cases} 1 & \text{if } w = e, \\ 0 & \text{otherwise.} \end{cases}$$

Cor: For each unit interval order $P \in \mathcal{P}_n$ we have

$$\#\{v \in \mathfrak{S}_n \mid \mathrm{des}_P(v) = 0\} = \frac{d_n(P)}{\chi^{n}(e)} \geq \frac{d_{n-1,1}(P)}{\chi^{n-1,1}(e)} \geq$$

$$\frac{d_{n-2,1,1}(P)}{\chi^{n-2,1,1}(e)} \geq \cdots \geq \frac{d_{1,\ldots,1}(P)}{\chi^{1,\ldots,1}(e)} = \begin{cases} 1 & \text{if } P \text{ is a chain,} \\ 0 & \text{otherwise.} \end{cases}$$

Defn: $\mathrm{des}_P(v) = \#\{i \mid v_i >_P v_{i+1}\}$.

Proof ideas and open problems

Defn: $\phi^\lambda = \sum_{\mu \vdash n} K^{-1}_{\mu,\lambda} \chi^\mu = \text{monomial trace}, \qquad \theta^\ell = \sum_{\substack{\mu \vdash n \\ \text{with } \ell \text{ parts}}} \phi^\mu.$

Fact: We have $\chi^{k1^{n-k}} = \sum_{\ell=n-k+1}^{n} \binom{\ell-1}{n-k} \theta^\ell.$

Conj: (Ha, '92) We have $\phi^\lambda(C'_w(1)) \geq 0$ for $w \in \mathfrak{S}_n$.

Thm: (Sta, '95) We have $\mathrm{Imm}_{\theta^\ell}(A) \geq 0$ for A TNN.
Cor: We have $\theta^\ell(C'_w(1)) \geq 0$ for $\ell \geq 0$, $w \in \mathfrak{S}_n(312)$.

Thm: (Hi, '24) We have $\phi^\lambda(C'_w(1)) \geq 0$ for $w \in \mathfrak{S}_n(312)$.

Question: Do we have $\theta^\ell(C'_w(1)) \geq 0$ for $\ell \geq 0$, $w \in \mathfrak{S}_n$?