

HOOK IMMANANTAL INEQUALITIES FOR TOTALLY NONNEGATIVE MATRICES

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ABSTRACT. Given a weakly decreasing positive integer sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ summing to n , let χ^λ denote the irreducible character of the symmetric group $\mathfrak{S}_n$ indexed by λ . This representation has dimension $\chi^\lambda(e)$, where e is the identity element of $\mathfrak{S}_n$. Let $\text{Imm}_{\chi^\lambda}$ denote the corresponding *irreducible character immanant*, the function on $n \times n$ matrices $A = (a_{i,j})$ defined by

$$\text{Imm}_{\chi^\lambda}(A) := \sum_{w \in \mathfrak{S}_n} \chi^\lambda(w) a_{1,w_1} \cdots a_{n,w_n}.$$

Merris conjectured [*Linear Multilinear Algebra* **14** (1983) pp. 21–35] and Heyfron proved [*Linear Multilinear Algebra* **24** (1988) pp. 65–78] that irreducible character immanants indexed by “hook” sequences $(k, 1, \dots, 1)$ satisfy the inequalities

$$\text{per}(A) = \frac{\text{Imm}_{\chi^n}(A)}{\chi^n(e)} \geq \frac{\text{Imm}_{\chi^{n-1,1}}(A)}{\chi^{n-1,1}(e)} \geq \frac{\text{Imm}_{\chi^{n-2,1,1}}(A)}{\chi^{n-2,1,1}(e)} \geq \cdots \geq \frac{\text{Imm}_{\chi^{1,\dots,1}}(A)}{\chi^{1,\dots,1}(e)} = \det(A),$$

whenever A is an $n \times n$ Hermitian positive semidefinite matrix. We prove that the same inequalities hold whenever A is an $n \times n$ totally nonnegative matrix.

1. INTRODUCTION

A matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$ is called *totally nonnegative* if each minor is nonnegative. That is, if for all $I, J \subseteq [n] := \{1, \dots, n\}$ with $|I| = |J|$, the submatrix $A_{I,J} := (a_{i,j})_{i \in I, j \in J}$ satisfies $\det(A_{I,J}) \geq 0$. A matrix $A \in \text{Mat}_{n \times n}(\mathbb{C})$ is called *Hermitian* if it satisfies $A^* = A$ where $*$ denotes conjugate transpose. Such a matrix is called *positive semidefinite* if we have $y^* A y \geq 0$ for all $y \in \mathbb{C}^n$. It is known that this property is equivalent to the condition that

$$(1.1) \quad \det(A_{I,I}) \geq 0 \quad \text{for all } I \subseteq [n].$$

For $A \in \text{Mat}_{n \times n}(\mathbb{R})$, the Hermitian property reduces to $y^\top A y \geq 0$ for all $y \in \mathbb{R}^n$.

Some inequalities satisfied by the entries of totally nonnegative matrices are also satisfied by the entries of Hermitian positive semidefinite matrices, with the latter inequalities typically being discovered first. (See [32, §1] for a short exposition.) A subset of these inequalities concern expressions of the form

$$(1.2) \quad \frac{\text{Imm}_\theta(A)}{\theta(e)},$$

where $\theta : \mathfrak{S}_n \rightarrow \mathbb{Z}$ is an $\mathfrak{S}_n$ -character, e is the identity element of $\mathfrak{S}_n$, and $\text{Imm}_\theta(A)$ is the θ -*immanant* defined by

$$(1.3) \quad \text{Imm}_\theta(A) = \sum_{w \in \mathfrak{S}_n} \theta(w) a_{1,w_1} \cdots a_{n,w_n}.$$

Since $\theta(e)$ is the dimension of the representation having character θ , the ratio (1.2) is clearly real when A is totally nonnegative. To see that it is also real when A is Hermitian positive semidefinite, let $\mathcal{T}_n$ be the real vector space spanned by all $\mathfrak{S}_n$ -characters, and recall that $\dim(\mathcal{T}_n)$ equals the number of *integer partitions* of n , weakly decreasing positive integer sequences λ summing to n . Let $\lambda \vdash n$ denote that λ is a partition of n , and consider the irreducible character basis $\{\chi^\lambda \mid \lambda \vdash n\}$ and induced sign character basis $\{\epsilon^\lambda \mid \lambda \vdash n\}$ of $\mathcal{T}_n$. Arbitrary $\mathfrak{S}_n$ -characters θ satisfy

$$(1.4) \quad \theta \in \text{span}_{\mathbb{N}}\{\chi^\lambda \mid \lambda \vdash n\} \subseteq \text{span}_{\mathbb{Z}}\{\epsilon^\lambda \mid \lambda \vdash n\},$$

with

$$(1.5) \quad \theta = \sum_{\lambda \vdash n} b_\lambda \epsilon^\lambda \iff \text{Imm}_\theta(A) = \sum_{\lambda \vdash n} b_\lambda \text{Imm}_{\epsilon^\lambda}(A),$$

and the Littlewood–Merris–Watkins identity [20], [24] asserts that

$$(1.6) \quad \text{Imm}_{\epsilon^\lambda}(A) = \sum \det(A_{I_1, I_1}) \cdots \det(A_{I_\ell, I_\ell}),$$

where $\ell = \ell(\lambda)$ is the number of nonzero components of λ , and the sum is over all sequences $(I_1, \dots, I_\ell)$ of disjoint subsets of $[n]$ satisfying $|I_j| = \lambda_j$. This number is real by (1.1), and therefore by (1.4) – (1.5) the number (1.2) is real as well.

The study of inequalities for the expressions (1.2) evaluated at Hermitian positive semidefinite matrices was originally motivated by work of Hadamard and Schur, and was later remotivated by generalizations of Marcus and Lieb. (See [39, §1].) The study of inequalities for the expressions evaluated at totally nonnegative matrices is motivated by Lusztig’s work on quantum groups and their connection to total nonnegativity [22]. (See also [30, §1].)

Barrett and Johnson [2] showed that expressions (1.2) with induced sign characters satisfy

$$(1.7) \quad \frac{\text{Imm}_{\epsilon^\lambda}(A)}{\epsilon^\lambda(e)} \geq \frac{\text{Imm}_{\epsilon^\mu}(A)}{\epsilon^\mu(e)}$$

for all real positive semidefinite matrices if and only if λ is *majorized by* μ , i.e., if and only if we have $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i . The author and Soskin [32] showed that the inequalities (1.7) hold for all $n \times n$ totally nonnegative matrices under the same conditions on λ and μ . The problem of characterizing the inequalities (1.7) which hold for all Hermitian positive semidefinite matrices remains open.

Borcea–Brändén proved inequalities involving products of expressions (1.2) when A is Hermitian positive semidefinite. In particular for $k = 2, \dots, n-1$ we have [5, Cor. 3.1(b)]

$$(1.8) \quad \left(\frac{\text{Imm}_{\epsilon^{k, n-k}}(A)}{\epsilon^{k, n-k}(e)} \right)^2 \geq \left(\frac{\text{Imm}_{\epsilon^{k+1, n-k-1}}(A)}{\epsilon^{k+1, n-k-1}(e)} \right) \left(\frac{\text{Imm}_{\epsilon^{k-1, n-k+1}}(A)}{\epsilon^{k-1, n-k+1}(e)} \right);$$

for $k = 1, \dots, n-1$ we have [5, Cor. 3.1(c)]

$$(1.9) \quad \left(\frac{\text{Imm}_{\epsilon^{k+1, n-k-1}}(A)}{\epsilon^{k+1, n-k-1}(e)} \right)^k \det(A) = \left(\frac{\text{Imm}_{\epsilon^{k+1, n-k-1}}(A)}{\epsilon^{k+1, n-k-1}(e)} \right)^k \frac{\text{Imm}_{\epsilon^n}(A)}{\epsilon^n(e)} \geq \left(\frac{\text{Imm}_{\epsilon^{k, n-k}}(A)}{\epsilon^{k, n-k}(e)} \right)^{k+1}.$$

Inequalities (1.8) – (1.9) are not known to hold for totally nonnegative matrices.

Schur [28] proved that for each $\mathfrak{S}_n$ -character θ , the inequality

$$(1.10) \quad \frac{\text{Imm}_{\chi^{1, \dots, 1}}(A)}{\chi^{1, \dots, 1}(e)} = \det(A) \leq \frac{\text{Imm}_\theta(A)}{\theta(e)}$$

holds for all Hermitian positive semidefinite matrices. Stembridge [36, Cor. 3.4] showed that it holds for totally nonnegative matrices as well. Johnson showed that for each $\mathfrak{S}_n$ -character θ , the permanental analog

$$(1.11) \quad \frac{\text{Imm}_{\chi^n}(A)}{\chi^n(e)} = \text{per}(A) \geq \frac{\text{Imm}_{\theta}(A)}{\theta(e)}$$

of (1.10) holds for totally nonnegative matrices (unpublished; see [37, p. 1088]). Lieb [18] conjectured the same for Hermitian positive semidefinite matrices. This statement, known as the *permanental dominance conjecture* is still open.

The problems of characterizing inequalities of the form

$$(1.12) \quad \frac{\text{Imm}_{\chi^\lambda}(A)}{\chi^\lambda(e)} \geq \frac{\text{Imm}_{\chi^\mu}(A)}{\chi^\mu(e)}$$

for Hermitian positive semidefinite matrices and totally nonnegative matrices are open, with preliminary work of James [14, Appendix] and Stembridge [37, §3], not suggesting any simple criterion with which to compare λ, μ . On the other hand, Merris conjectured [23, §4] and Heyfron proved [12, Thm. 1] a characterization of the subset of inequalities (1.12) which hold for Hermitian positive semidefinite matrices when λ and μ are *hook partitions*, i.e., have the form

$$k1^{n-k} := (k, \underbrace{1, \dots, 1}_{n-k}).$$

Theorem 1.1. *For A an $n \times n$ Hermitian positive semidefinite matrix we have*

$$\text{per}(A) = \frac{\text{Imm}_{\chi^n}(A)}{\chi^n(e)} \geq \frac{\text{Imm}_{\chi^{n-1,1}}(A)}{\chi^{n-1,1}(e)} \geq \frac{\text{Imm}_{\chi^{n-2,1,1}}(A)}{\chi^{n-2,1,1}(e)} \geq \dots \geq \frac{\text{Imm}_{\chi^{1,\dots,1}}(A)}{\chi^{1,\dots,1}(e)} = \det(A).$$

We will prove that the same inequalities hold whenever A is totally nonnegative. In Section 2 we review symmetric functions and traces. In Sections 3 – 4 we discuss chromatic symmetric functions, posets, and their applications to total nonnegativity. This leads to two proofs of our main theorem in Section 5.

2. SYMMETRIC FUNCTIONS AND $\mathfrak{S}_n$ -TRACES

Inside of $\mathcal{T}_n$, the $\mathbb{Z}$ -module $\text{span}_{\mathbb{Z}}\{\chi^\lambda \mid \lambda \vdash n\}$ of *virtual characters* is isomorphic to the $\mathbb{Z}$ -module Λ_n of homogeneous symmetric functions of degree n . Six standard bases of Λ_n consist of the monomial $\{m_\lambda \mid \lambda \vdash n\}$, elementary $\{e_\lambda \mid \lambda \vdash n\}$, (complete) homogenous $\{h_\lambda \mid \lambda \vdash n\}$, power sum $\{p_\lambda \mid \lambda \vdash n\}$, Schur $\{s_\lambda \mid \lambda \vdash n\}$, and forgotten $\{f_\lambda \mid \lambda \vdash n\}$ symmetric functions. (See, e.g., [34, Ch. 7] for definitions.) An involutive automorphism $\omega : \Lambda \rightarrow \Lambda$ defined by $\omega(e_k) = h_k$ for all k acts on these bases of Λ_n by

$$\omega(s_\lambda) = s_{\lambda^\top}, \quad \omega(m_\lambda) = f_\lambda, \quad \omega(e_\lambda) = h_\lambda, \quad \omega(p_\lambda) = (-1)^{n-\ell(\lambda)} p_\lambda,$$

where we define the *transpose* partition $\lambda^\top = (\lambda_1^\top, \dots, \lambda_{\lambda_1}^\top)$ of λ by

$$\lambda_i^\top = \#\{j \mid \lambda_j \geq i\}.$$

Linking these two $\mathbb{Z}$ -modules is the Frobenius isomorphism

$$(2.1) \quad \begin{aligned} \text{Frob} : \text{span}_{\mathbb{Z}}\{\chi^\lambda \mid \lambda \vdash n\} &\rightarrow \Lambda_n \\ \theta &\mapsto \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \theta(w) p_{\text{ctype}(w)}, \end{aligned}$$

where $\text{ctype}(w)$ is the cycle type of w . This maps the bases of irreducible characters, induced sign characters, and induced trivial characters of $\mathfrak{S}_n$ to the Schur, elementary, and homogeneous bases of Λ_n , respectively,

$$\chi^\lambda \mapsto s_\lambda, \quad \epsilon^\lambda := \text{sgn} \upharpoonright_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mapsto e_\lambda, \quad \eta^\lambda := \text{triv} \upharpoonright_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mapsto h_\lambda.$$

Here $\mathfrak{S}_\lambda$ is the Young subgroup of $\mathfrak{S}_n$ indexed by λ . (See, e.g., [27].) The power sum, monomial, and forgotten bases of Λ_n correspond to bases of $\mathcal{T}_n$ which are not characters. We call these the *power sum* $\{\psi^\lambda \mid \lambda \vdash n\}$, *monomial* $\{\phi^\lambda \mid \lambda \vdash n\}$, and *forgotten* $\{\gamma^\lambda \mid \lambda \vdash n\}$ traces of $\mathfrak{S}_n$, respectively. These are the virtual character bases related to the irreducible character bases by the same matrices of character evaluations and Kostka numbers that relate power sum, monomial, and forgotten symmetric functions to Schur functions,

$$(2.2) \quad \begin{aligned} p_\lambda &= \sum_{\mu} \chi^\mu(\lambda) s_\mu, & s_\lambda &= \sum_{\mu} K_{\lambda, \mu} m_\mu, & s_\lambda &= \sum_{\mu} K_{\lambda^\vee, \mu} f_\mu, \\ \psi^\lambda &= \sum_{\mu} \chi^\mu(\lambda) \chi^\mu, & \chi^\lambda &= \sum_{\mu} K_{\lambda, \mu} \phi^\mu, & \chi^\lambda &= \sum_{\mu} K_{\lambda^\vee, \mu} \gamma^\mu, \end{aligned}$$

where $\chi^\mu(\lambda) := \chi^\mu(w)$ for any $w \in \mathfrak{S}_n$ having cycle type λ . The power sum traces of $\mathcal{T}_n$ also have the natural definition

$$(2.3) \quad \psi^\lambda(w) := \begin{cases} z_\lambda & \text{if } \text{ctype}(w) = \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where $z_\lambda = \lambda_1 \cdots \lambda_\ell \alpha_1! \cdots \alpha_n!$ and α_i is the number of components of λ equal to i .

The character evaluations $\chi^\mu(\lambda)$ can be performed using the *Murnaghan–Nakayama rule*, or in the special case that $\lambda = 1^n$, the *hooklength formula*. (See, e.g., [34, §7.17, §7.21].) Alternatively, for $\mu = (\mu_1, \dots, \mu_r) \vdash n$, the number $\chi^\mu(1^n) = \chi^\mu(e)$ counts standard Young tableaux of shape μ . A *Young diagram of shape μ* consists of r left-justified rows of boxes with μ_i boxes in row i , and a *standard Young tableau of shape μ* is a filling of this with $1, \dots, n$, so that entries strictly increase in rows and columns. For example, in $\mathfrak{S}_4$ we have $\chi^{31}(e) = 3$:

$$\begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|c|} \hline 3 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|c|} \hline 2 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 4 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array}.$$

When μ is a hook partition, this leads to the following simple expression.

Observation 2.1. *We have $\chi^{k1^{n-k}}(e) = \binom{n-1}{k-1}$.*

Proof. Filling a Young diagram of shape $k1^{n-k}$ with $\{1, \dots, n\}$, we place 1 in position $(1, 1)$, we choose $k-1$ letters from $\{2, \dots, n\}$ to follow it in row 1 in increasing order, and place the remaining letters in column 1 in increasing order. $\square$

The Kostka number $K_{\lambda, \mu}$ equals the number of *semistandard Young tableaux of shape λ and content μ* i.e., Young diagrams of shape λ filled with μ_1 ones, μ_2 twos, etc., so that

entries strictly increase in columns and weakly increase in rows. For example, in $\mathfrak{S}_7$ we have $K_{411,2211} = K_{411,3111} = 3$:

$$\begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 4 \\ \hline \end{array}.$$

When λ is a hook partition $k1^{n-k}$, we have a simple formula for $K_{\lambda,\mu}$ which depends only upon k and the number $\ell(\mu)$ of components of μ . This leads to a simple expansion of the corresponding irreducible characters in terms of monomial traces.

Proposition 2.2. *For each hook partition $k1^{n-k} \vdash n$ and each arbitrary partition $\mu \vdash n$, we have*

$$(2.4) \quad K_{k1^{n-k}, \mu} = \binom{\ell(\mu) - 1}{n - k},$$

and therefore

$$(2.5) \quad \chi^{k1^{n-k}} = \sum_{\ell=n-k+1}^n \binom{\ell - 1}{n - k} \sum_{\substack{\mu \vdash n \\ \ell(\mu) = \ell}} \phi^\mu.$$

Proof. To see (2.4), observe that for $k = 1, \dots, n$, each semistandard Young tableau U of shape $k1^{n-k}$ and content μ has the entry 1 in position $(1, 1)$. The remaining $n - k$ entries of column 1 must be distinct, must be chosen from the set $\{2, \dots, \ell = \ell(\mu)\}$, and must appear in increasing order. There are $\binom{\ell-1}{n-k}$ ways to choose these. The remaining $k - 1$ entries of row 1 are then uniquely determined by completing the multiset $1^{\mu_1} \dots \ell^{\mu_\ell}$. These appear in row 1 in weakly increasing order. Now apply (2.2) to obtain (2.5). $\square$

It can be useful to record trace evaluations in a symmetric generating function. In particular, for $D \in \mathbb{Q}[\mathfrak{S}_n]$, we record induced sign character evaluations by defining

$$(2.6) \quad Y(D) := \sum_{\lambda \vdash n} \epsilon^\lambda(D) m_\lambda \in \mathbb{Q} \otimes \Lambda_n.$$

This symmetric generating function in fact gives all standard trace evaluations $\theta(D)$. (See [31, Prop. 2.1].)

Proposition 2.3. *The symmetric function $Y(D)$ is equal to*

$$\sum_{\lambda \vdash n} \eta^\lambda(D) f_\lambda = \sum_{\lambda \vdash n} \frac{(-1)^{n-\ell(\lambda)} \psi^\lambda(D)}{z_\lambda} p_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(D) s_\lambda = \sum_{\lambda \vdash n} \phi^\lambda(D) e_\lambda = \sum_{\lambda \vdash n} \gamma^\lambda(D) h_\lambda.$$

Furthermore, every element of Λ_n is a special case of this. (See [31, Prop. 2.3].)

Proposition 2.4. *Every symmetric function in Λ_n has the form $Y(D)$ for some element $D \in \mathbb{Q}[\mathfrak{S}_n]$.*

3. CHROMATIC SYMMETRIC FUNCTIONS AND POSETS

Certain symmetric functions which Stanley [33] associated to graphs are related by Proposition 2.4 to a subset of the Kazhdan–Lusztig basis $\{C'_w(1) \mid w \in \mathfrak{S}_n\}$ of $\mathbb{Z}[\mathfrak{S}_n]$ defined in [17]. Specifically, this subset is $\{C'_w(1) \mid w \in \mathfrak{S}_n \text{ avoids the pattern } 312\}$ [7, Thm. 7.1]. We make this relationship precise in Proposition 3.3 and then state some consequences.

Define a *proper coloring* of a (simple undirected) graph $G = (V, E)$ to be an assignment $\kappa : V \rightarrow \{1, 2, \dots\}$ of colors (positive integers) to V such that adjacent vertices have different colors. For G on $|V| = n$ vertices and any partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, say that a coloring κ of G has *type* λ if λ_i vertices have color i for $i = 1, \dots, \ell$. Let $c(G, \lambda)$ be the number of proper colorings of G of type λ . Define the *chromatic symmetric function* of G to be

$$(3.1) \quad X_G := \sum_{\lambda \vdash n} c(G, \lambda) m_\lambda.$$

By Proposition 2.4, we see that for each graph G on n vertices, there exists an element $D \in \mathbb{Q}[\mathfrak{S}_n]$ such that $X_G = Y(D)$. While G does not uniquely determine such an element D , it does uniquely determine all trace evaluations $\theta(D)$. (See [31, Obs. 3.1].)

Observation 3.1. *Let G be a graph on n vertices and let $D \in \mathbb{Q}[\mathfrak{S}_n]$ satisfy $Y(D) = X_G$. Then for each trace $\theta = \sum_{\lambda \vdash n} a_\lambda \epsilon^\lambda \in \mathcal{T}_n$, we have $\theta(D) = \sum_{\lambda \vdash n} a_\lambda c(G, \lambda)$.*

For every trace $\theta \in \mathcal{T}_n$, Proposition 2.4 and Observation 3.1 allow us to define

$$(3.2) \quad \theta(G) := \theta(D),$$

where D is any element in $\mathbb{Q}[\mathfrak{S}_n]$ satisfying $Y(D) = X_G$. By Proposition 2.3, we have that $c(G, \lambda) = \epsilon^\lambda(D)$ and furthermore that X_G is equal to

$$(3.3) \quad \sum_{\lambda \vdash n} \eta^\lambda(G) f_\lambda = \sum_{\lambda \vdash n} \frac{(-1)^{n-\ell(\lambda)} \psi^\lambda(G)}{z_\lambda} p_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(G) s_\lambda = \sum_{\lambda \vdash n} \phi^\lambda(G) e_\lambda = \sum_{\lambda \vdash n} \gamma^\lambda(G) h_\lambda.$$

Some conditions on graphs G and traces θ imply the numbers $\theta(G)$ to be positive and possibly to have nice algebraic and combinatorial interpretations. This is especially true when G is the incomparability graph of a poset which is a unit interval order. (See, e.g., [7], [29], [33].) Given a poset P , define its *incomparability graph* $\text{inc}(P)$ to be the graph having a vertex for each element of P and an edge $\{i, j\}$ for each incomparable pair of elements of P . Call the poset a *unit interval order* if it has no induced subposet isomorphic to the posets

$$(3+1) = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \bullet, \quad (2+2) = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}.$$

For example, the following unit interval order and graph

$$(3.4) \quad P = \begin{array}{ccccc} & 4 & 5 & & \\ \bullet & & \bullet & & \\ & 1 & 2 & & \\ \bullet & & \bullet & & \\ & & & \bullet & \\ & & & 3 & \end{array}, \quad G = \begin{array}{ccccc} 1 & 2 & 4 & 5 & \\ \bullet & \bullet & \bullet & \bullet & \\ & \bullet & \bullet & \bullet & \\ & & \bullet & & \\ & & 3 & & \end{array}$$

satisfy $\text{inc}(P) = G$.

Given an n -element unit interval order P , it is easy to explicitly describe an element $D = D(P) \in \mathfrak{S}_n$ satisfying $Y(D) = X_{\text{inc}(P)}$ [31, §3].

Algorithm 3.2. *Given unit interval order P , do*

- (1) *For each element $y \in P$, compute $\beta(y) := \#\{x \in P \mid x \leq_P y\} - \#\{z \in P \mid z \geq_P y\}$.*
- (2) *Label the poset elements $1, \dots, n$ so that we have $\beta(1) \leq \dots \leq \beta(n)$.*
- (3) *Define $w = w(P) = w_1 \cdots w_n$ by $w_j = \max(\{i \mid i \not\leq_P j\} \setminus \{w_1, \dots, w_{j-1}\})$.*
- (4) *Define $D = \sum_{v \leq w} v$, where $\leq$ is the Bruhat order on $\mathfrak{S}_n$.*

The element D produced by Algorithm 3.2 is usually written $C'_w(1)$ and is called the *Kazhdan–Lusztig basis element* indexed by w [17]. (See [4] for more information on this basis and the Bruhat order.) The map $P \mapsto w(P)$ defined by Steps 1–3 of Algorithm 3.2 is a bijection from n -element unit interval orders to the $\frac{1}{n+1} \binom{2n}{n}$ 312-avoiding permutations in $\mathfrak{S}_n$, and gives us the following result [7, Cor. 7.5]. (See [3] for more information on pattern avoidance.)

Proposition 3.3. *Let P be an n -element unit interval order and $w = w(P)$ be the corresponding 312-avoiding permutation in $\mathfrak{S}_n$. Then we have $Y(C'_w(1)) = X_{\text{inc}(P)}$.*

For example, elements of the poset P in (3.4) are already labeled as in Step 2 of Algorithm 3.2: $(\beta(1), \beta(2), \beta(3), \beta(4), \beta(5)) = (-2, -1, 0, 1, 2)$. Thus we compute

$$w(P) = 34521, \quad D = C'_{34521}(1) = \sum_{v \leq 34521} v,$$

and we have $Y(C'_{34521}(1)) = X_{\text{inc}(P)}$.

The labeling in Step 2 of Algorithm 3.2 [9, p. 33], [38, §8.2] also associates a totally nonnegative matrix to a unit interval order P . Define the *antiadjacency matrix* of a labeled poset P to be the matrix $A = (a_{i,j})$ with entries

$$(3.5) \quad a_{i,j} = \begin{cases} 0 & \text{if } i <_P j, \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 3.4. *For P a unit interval order labeled as in Algorithm 3.2, the antiadjacency matrix $A = A(P)$ is totally nonnegative.*

Proof. Entries of A equal to 0 form a Ferrers diagram in the upper right of the matrix. Thus each submatrix of A has repeated rows and columns, or is unitriangular. $\square$

Combinatorial interpretations of numbers $\theta(\text{inc}(P))$ often involve generalizations of Young tableaux called P -tableaux, and statistics on these. Define a P -tableau of shape $\lambda \vdash |P|$ to be a filling of a Young diagram of shape λ with the elements of P , one per box. For example we have the poset P and P -tableaux

$$(3.6) \quad P = \begin{array}{c} 5 \\ \bullet \\ 3 \\ \bullet \\ 1 \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} 4 \\ \bullet \\ 2 \end{array}, \quad T = \begin{array}{|c|c|} \hline 5 & \\ \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}, \quad U = \begin{array}{|c|c|c|} \hline 5 & 4 & \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \quad V = \begin{array}{|c|c|c|} \hline 5 & 4 & \\ \hline 3 & 1 & 2 \\ \hline \end{array},$$

$$W = \begin{array}{|c|c|c|c|c|} \hline 5 & 4 & 1 & 2 & 3 \\ \hline \end{array}.$$

The statistics we apply to P -tableaux are P -analogs of traditional permutation statistics. Given a P -tableau U , let U_i be the i th row (from the bottom) of U , and let $U_{i,j}$ be the j th entry in row i . Call a position (i, j) in U a *descent* if $U_{i,j} >_P U_{i,j+1}$, and a *record* if $U_{i,1}, \dots, U_{i,j-1} <_P U_{i,j}$. Define $\text{des}_P(U)$ and $\text{rec}_P(U)$ to be the numbers of descents and records in U , respectively, and call U

- (1) *descent-free* or *row-semistrict* if $\text{des}_P(U) = 0$,
- (2) *column-strict* if the entries of each column satisfy $U_{i,j} <_P U_{i+1,j}$,
- (3) *standard* if it is column-strict and row-semistrict.

For example, the tableaux in (3.6) satisfy $\text{des}_P(T) = \text{des}_P(U) = 0$, $\text{des}_P(V) = \text{des}_P(W) = 1$, and $\text{rec}_P(W) = 1$, $\text{rec}_P(U) = \text{rec}_P(V) = 2$, $\text{rec}_P(T) = 5$. Tableaux T, U , are row-semistrict, U, V, W are column-strict, and U is standard.

In terms of the above definitions, we have the following combinatorial interpretations of trace evaluations. Interpretations of induced sign character evaluations are the simplest. (See, e.g., [31, Eqn. (3.11)].)

Theorem 3.5. *Let G be any (simple) graph on n vertices, and let P be any poset on n elements. For all $\lambda \vdash n$ we have*

- (1) $\epsilon^\lambda(G)$ is $c(G, \lambda)$, the number of proper colorings of G of type λ .
- (2) $\epsilon^\lambda(\text{inc}(P))$ is the number of column-strict P -tableaux of shape $\lambda^\top$.

Induced trivial character evaluations are also rather simple [7, Thm. 4.7 (ii-b)], [31, Thm. 3.7].

Theorem 3.6. *Let G be any (simple) graph on n vertices, and let P be any poset on n elements. For all $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ we have*

- (1) $\eta^\lambda(G)$ is the number of sequences $(O_1, \dots, O_r)$ of acyclic orientations of induced subgraphs of G on pairwise disjoint vertex subsets of cardinalities $\lambda_1, \dots, \lambda_r$.
- (2) $\eta^\lambda(\text{inc}(P))$ is the number of row-semistrict P -tableaux of shape λ .

While $\chi^\lambda(G)$ is negative for some graphs G , even for some incomparability graphs $\text{inc}(P)$, Stanley and Stembridge [35, Conj. 5.1] conjectured $\chi^\lambda(\text{inc}(P))$ to be nonnegative for $(\mathbf{3} + \mathbf{1})$ -free posets P . Gasharov [10] proved this, and Kaliszewski [15, Prop. 4.3] showed that when λ is a hook partition, the evaluation $\chi^\lambda(\text{inc}(P))$ is nonnegative for all posets P . Combining these results into one statement, we have the following.

Theorem 3.7. *Let P be an n -element poset.*

- (1) For $\lambda \vdash n$ a hook shape, $\chi^\lambda(\text{inc}(P))$ is the number of standard P -tableaux of shape λ .
- (2) If P is $(\mathbf{3} + \mathbf{1})$ -free then for any $\lambda \vdash n$, $\chi^\lambda(\text{inc}(P))$ is the number of standard P -tableaux of shape λ .

Observation 2.1 and Theorem 3.7 imply that when P is an n -element chain, the hook irreducible character evaluations $\chi^{k1^{n-k}}(\text{inc}(P)) = \binom{n-1}{k-1}$ and $\chi^{(k-1)1^{n-k+1}}(\text{inc}(P)) = \binom{n-1}{k-2}$ are related by

$$(3.7) \quad (k-1)\chi^{k1^{n-k}}(\text{inc}(P)) = (n-k+1)\chi^{(k-1)1^{n-k+1}}(\text{inc}(P))$$

for $k = 2, \dots, n$. For an arbitrary poset P no analogous equality exists, but we do have similar inequalities.

Lemma 3.8. *For each naturally labeled n -element poset P and $k = 2, \dots, n$ we have*

$$(3.8) \quad (k-1)\chi^{k1^{n-k}}(\text{inc}(P)) \geq (n-k+1)\chi^{k-1,1^{n-k+1}}(\text{inc}(P)).$$

Furthermore, the difference $(k-1)\chi^{k1^{n-k}}(\text{inc}(P)) - (n-k+1)\chi^{k-1,1^{n-k+1}}(\text{inc}(P))$ equals the number of standard P -tableaux of shape $k1^{n-k}$ in which one entry of the first row is marked and is incomparable in P to an entry in an earlier column of the tableau.

Proof. Let $\mathcal{C}_k = \mathcal{C}_k(P)$ be the set of standard P -tableaux of shape $k1^{n-k}$ in which one entry of column 1 other than that in position $(1, 1)$ is marked. Let $\mathcal{R}_k = \mathcal{R}_k(P)$ be the set of standard P -tableaux of shape $k1^{n-k}$ in which one entry of row 1 other than that in position $(1, 1)$ is marked. The inequality (3.8) asserts that $|\mathcal{R}_k| \geq |\mathcal{C}_k|$. To see this, we define a family of maps $\{f_k : \mathcal{C}_{k-1} \rightarrow \mathcal{R}_k \mid k = 2, \dots, n\}$, by letting $f_k(U)$ be the tableau constructed from U by removing the marked element from the first column of U , and reinserting it as far as possible to the right in row 1 subject to the requirement that it be a record.

To see that the map f_k is well-defined, let $m \in [n]$ be the marked element in column 1 of U . Since U is standard, m must be greater than the element in position $(1, 1)$. Thus it will certainly be inserted into positions $(1, 2), \dots, (1, k-1)$ or at the end of row 1, creating a tableau of shape $k1^{n-k}$ with one marked element in the first row. We claim that the map f_k is injective. To see this, consider a tableau $V \in \mathcal{R}_k$ which satisfies $V = f_k(U)$ for some $U \in \mathcal{C}_{k-1}$. A marked element i in row 1 of V will necessarily be greater in P than all elements to its left in row 1, and will be comparable in P to all elements in column 1. To recover U , remove i from row 1 of V and insert it into the unique position of column 1 so that entries there increase. The resulting tableau will still be P -semistrict because if the element preceding i in V , say h , and the element following i in V , say j , satisfy $h >_P j$, then we have $i >_P h >_P j$, contradicting the semistrictness of row 1 of V .

To see that the difference $(k-1)\chi^{k1^{n-k}}(P) - (n-k+1)\chi^{k-1,1^{n-k+1}}(P)$ has the claimed interpretation, consider the elements of $\mathcal{R}_k \setminus f(\mathcal{C}_{k-1})$. These are standard P -tableaux V which contain a marked element i in row 1 which is not a record in row 1, or which is incomparable to some element of column 1. We claim that if i is not a record in row 1, then it must be incomparable to an element to its left in row 1. Assume otherwise: assume that i follows h in row 1 with $h <_P i$ and that some element $j >_P i$ appears earlier than h in row 1. Assume that j is the rightmost such element with these properties. Then we have $j >_P h$. Let g be the element immediately following j in row 1. By our choice of j we cannot have $g >_P j$, and since V is standard we cannot have $j >_P g$. Thus j must be incomparable to g . By our choice of j we cannot have $g >_P i$, and our assumption on i we cannot have g incomparable to i . Thus we have $g <_P i$. But then we have $g <_P i <_P j$, a contradiction. $\square$

For example, define $\mathcal{C}_k$ and $\mathcal{R}_k$ as in the proof of 3.8, and consider the poset P in (3.6). The set $\mathcal{C}_3$ consists of two tableaux. Applying f_4 to these we have

$$f_4\left(\begin{array}{|c|c|c|c|} \hline 5 \\ \hline \textcircled{3} \\ \hline 1 & 2 & 4 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|} \hline 5 \\ \hline 1 & \textcircled{3} & 2 & 4 \\ \hline \end{array}, \quad f_4\left(\begin{array}{|c|c|c|c|} \hline \textcircled{5} \\ \hline 3 \\ \hline 1 & 2 & 4 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|} \hline 3 \\ \hline 1 & 2 & \textcircled{5} & 4 \\ \hline \end{array},$$

and the difference $3\chi^{41}(P) - 2\chi^{311}(P)$ counts elements of $\mathcal{R}_4$ whose first row contains a marked element which is not a record in that row, or which is incomparable to some element

of the first column,

$$\begin{array}{|c|} \hline 5 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 3 & \textcircled{2} & 4 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 5 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 3 & 2 & \textcircled{4} \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 5 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & \textcircled{4} & 3 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 5 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 4 & \textcircled{3} & 2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 5 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 1 & 4 & 3 & \textcircled{2} \\ \hline \end{array}, \dots$$

The statement preceding Theorem 3.7 implies that $\phi^\lambda(G)$ is negative for some graphs G , even for some incomparability graphs $G = \text{inc}(P)$. Hikita [13, Thm.3] showed that it is positive for $G = \text{inc}(P)$ and P a $(\mathbf{3} + \mathbf{1})$ -free poset, settling the Stanley–Stembridge conjecture [35, Conj. 5.5].

Theorem 3.9. *For P a $(\mathbf{3} + \mathbf{1})$ -free poset and $\lambda \vdash n$, we have $\phi^\lambda(\text{inc}(P)) \geq 0$.*

Hikita’s proof unfortunately does not provide a combinatorial interpretation of $\phi^\lambda(\text{inc}(P))$.

Problem 3.10. *Find a combinatorial interpretation for $\phi^\lambda(\text{inc}(P))$ which holds for all $\lambda \vdash n$ and for n -element posets P avoiding $\mathbf{3} + \mathbf{1}$.*

A related result for monomial traces [1, Lem. 4.1], [33, Thm. 3.3] concerns sums of the form

$$(3.9) \quad \theta^\ell = \sum_{\substack{\mu \vdash n \\ \ell(\mu) = \ell}} \phi^\mu.$$

Proposition 3.11. *Let G be any (simple) graph on n vertices, and let P be any poset on n elements. The traces $\{\theta^\ell \mid 1 \leq \ell \leq n\}$ satisfy*

- (1) $\theta^\ell(G)$ is the number of acyclic orientations of G having ℓ sources,
- (2) $\theta^\ell(\text{inc}(P))$ is the number of descent-free P -tableaux of shape n having ℓ records.

4. APPLICATIONS TO TOTAL NONNEGATIVITY

Nonnegative expansions of chromatic symmetric functions in the standard bases are closely related to the immanants defined in (1.3) and to certain directed planar graphs. We will make these relationships precise in Proposition 4.4 and state some immanantal analogs of results from Section 3.

Define a (nonnegative weighted) *planar network of order n* to be a directed, planar, acyclic digraph $F = (V, E)$ which can be embedded in a disc so that $2n$ distinguished vertices labeled clockwise as $s_1, \dots, s_n, t_n, \dots, t_1$ lie on the boundary of the disc, with a nonnegative real *weight* $c_{u,v}$ assigned to each edge $(u, v) \in E$. We may assume that $s_1, \dots, s_n$, called *sources*, have indegree 0 and that $t_n, \dots, t_1$, called *sinks*, have outdegree 0. To every source-to-sink path, we associate a weight equal to the product of its edge weights, and we define the *path matrix* $A = A(F) = (a_{i,j})_{i,j \in [n]}$ by setting $a_{i,j}$ equal to the sum of weights of all paths from s_i to t_j . For example, one planar network F of order 3, with edges weighted by positive numbers $1, a, \dots, h$, and its path matrix A are

$$(4.1) \quad F = \begin{array}{c} \begin{array}{ccccc} s_3 & & & & t_3 \\ & \nearrow d & & \nearrow g & \\ & & & & \\ s_2 & \xrightarrow{1} & & \xrightarrow{1} & t_2 \\ & \nearrow a & \nearrow b & \nearrow c & \\ & & & & \\ s_1 & \xrightarrow{1} & & \xrightarrow{1} & t_1 \end{array} \end{array}, \quad A = \begin{bmatrix} 1 & b+c & bh \\ a & ab+ac+e+f & abh+fh+eh+eg \\ 0 & e+f & dh+eh+fh+eg+dg \end{bmatrix}.$$

A result often attributed to Lindström [19] but proved earlier by Karlin and McGregor [16] asserts the total nonnegativity of such a matrix.

Theorem 4.1. *The path matrix A of a nonnegative weighted planar network F of order n is totally nonnegative. Moreover, the nonnegative number $\det(A)$ equals*

$$\sum_{\pi} \text{wgt}(\pi),$$

where the sum is over all families $\pi = (\pi_1, \dots, \pi_n)$ of pairwise nonintersecting paths in F , with π_i a path from s_i to t_i for $i = 1, \dots, n$, and where

$$(4.2) \quad \text{wgt}(\pi) := \text{wgt}(\pi_1) \cdots \text{wgt}(\pi_n).$$

Thus by inspection of the network F in (4.1), its path matrix A satisfies $\det(A) = fdg$.

The converse of Theorem 4.1 is true as well. That is, path matrices are essentially the only examples of totally nonnegative matrices [6], [8], [21], [41].

Theorem 4.2. *For each $n \times n$ totally nonnegative matrix A , there exists a nonnegative weighted planar network of order n whose path matrix is A .*

Also belonging to the subject of total nonnegativity are polynomial functions

$$f(x) := f(x_{1,1}, x_{1,2}, \dots, x_{n,n}) \in \mathbb{Z}[x_{1,1}, x_{1,2}, \dots, x_{n,n}]$$

having the property that

$$f(A) := f(a_{1,1}, a_{1,2}, \dots, a_{n,n}) \geq 0$$

for all totally nonnegative matrices $A = (a_{i,j})$. Interest in such polynomials comes from the fact that elements of a certain *dual canonical basis* of $\mathbb{Z}[x_{1,1}, x_{1,2}, \dots, x_{n,n}]$ have this property [22]. (See also [26].) Certainly subtraction-free polynomials such as $\text{Imm}_{\psi^\lambda}(x)$ are totally nonnegative. (See (2.3).) Sums of products of minors such as $\text{Imm}_{\epsilon^\lambda}(x)$ in (1.6) are as well, as are the analogous sums of products of permanents [20], [24]

$$(4.3) \quad \text{Imm}_{\eta^\lambda}(A) = \sum \text{per}(A_{I_1, I_1}) \cdots \text{per}(A_{I_\ell, I_\ell}).$$

The total nonnegativity of other polynomials is less obvious. For instance, Stembridge showed that all character immanants are totally nonnegative [36, Cor. 3.3].

Theorem 4.3. *For $\lambda \vdash n$ the polynomial $\text{Imm}_{\chi^\lambda}(x)$ is totally nonnegative.*

For some totally nonnegative polynomials $\text{Imm}_\theta(x)$, one can combinatorially interpret the evaluation $\text{Imm}_\theta(A)$ when A is a totally nonnegative matrix. Such an interpretation typically employs a planar network F having path matrix A , guaranteed to exist by Theorem 4.2, and families of paths in F from all sources to all sinks. In particular, for a multiset K of edges of F , let $\Pi_e(K)$ denote the set of all path families $\pi = (\pi_1, \dots, \pi_n)$ with π_i a path from source i to sink i , whose multiset of edges is K . Call K a *bijective skeleton in F* if $\Pi_e(K)$ is nonempty. Define $\text{wgt}(K)$ to be the product of weights of edges in K , with multiplicities, so that $\text{wgt}(\pi) = \text{wgt}(K)$ for all $\pi \in \Pi_e(K)$. For each path family $\pi \in \Pi_e(K)$, define the poset $P = P(\pi)$ by declaring $\pi_i < \pi_j$ if $i < j$ as integers and π_i does not intersect π_j . We will refer to the union of $P(\pi)$ -tableaux, over all path families π covering a bijective skeleton of F ,

$$(4.4) \quad \{U \text{ a } P(\pi)\text{-tableau} \mid \pi \in \Pi_e(K), K \text{ a bijective skeleton in } F\}$$

as the set of F -tableaux. These are fillings of Young diagrams with path families in F . For F -tableau U containing path family $\pi \in \Pi_e(K)$, we define $\text{wgt}(U) := \text{wgt}(K)$.

For example, consider the network F in (4.1) and three multisets of edges

$$(4.5) \quad K_1 = \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagup \quad \diagdown \\ \hline \end{array}, \quad K_2 = \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagup \quad \diagdown \\ \hline \end{array}, \quad K_3 = \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagup \quad \diagdown \\ \hline \end{array},$$

where the marked edge in K_3 has multiplicity 2. The multisets have weights $\text{wgt}(K_1) = feh$, $\text{wgt}(K_2) = abfh$, $\text{wgt}(K_3) = f^2h$. The path families π, ρ, σ, τ defined by

$$(4.6) \quad \begin{array}{c} \pi_3 \\ \pi_2 \\ \pi_1 \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagup \quad \diagdown \\ \hline \end{array}, \quad \begin{array}{c} \rho_3 \\ \rho_2 \\ \rho_1 \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagup \quad \diagdown \\ \hline \end{array}, \quad \begin{array}{c} \sigma_3 \\ \sigma_2 \\ \sigma_1 \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagup \quad \diagdown \\ \hline \end{array}, \quad \begin{array}{c} \tau_3 \\ \tau_2 \\ \tau_1 \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \hline \diagup \quad \diagdown \\ \hline \end{array},$$

satisfy $\Pi_e(K_1) = \{\pi, \rho\}$, $\Pi_e(K_2) = \{\sigma\}$, $\Pi_e(K_3) = \{\tau\}$, and have posets

$$P(\pi) = \begin{array}{c} \pi_2 \quad \pi_3 \\ \diagdown \quad \diagup \\ \pi_1 \end{array}, \quad P(\rho) = \begin{array}{c} \rho_2 \quad \rho_3 \\ \diagdown \quad \diagup \\ \rho_1 \end{array}, \quad P(\sigma) = \begin{array}{c} \sigma_3 \\ \diagdown \\ \sigma_1 \end{array} \bullet \sigma_2, \quad P(\tau) = \begin{array}{c} \tau_2 \quad \tau_3 \\ \diagdown \quad \diagup \\ \tau_1 \end{array}.$$

The standard F -tableaux

$$\begin{array}{|c|} \hline \pi_3 \\ \hline \pi_1 \pi_2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \rho_3 \\ \hline \rho_1 \rho_2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \sigma_3 \\ \hline \sigma_1 \sigma_2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \tau_3 \\ \hline \tau_1 \tau_2 \\ \hline \end{array},$$

have weights feh , feh , $abfh$, f^2e , respectively.

For an $n \times n$ totally nonnegative matrix A and a trace $\theta \in \mathcal{T}_n$, we may compute $\text{Imm}_\theta(A)$ by considering a planar network F having path matrix A , the union over bijective skeletons K of path families $\pi \in \Pi_e(K)$, and the corresponding chromatic symmetric functions $X_{\text{inc}(P(\pi))}$. Specifically, we have the following [31, Cor. 4.6].

Proposition 4.4. *For F a planar network having path matrix A and $\theta \in \mathcal{T}_n$, we have*

$$(4.7) \quad \text{Imm}_\theta(A) = \sum_K \text{wgt}(K) \sum_{\pi \in \Pi_e(K)} \theta(\text{inc}(P(\pi))),$$

where K varies over all bijective skeletons in F . If for all posets P , $\theta(\text{inc}(P))$ counts P -tableaux having a particular property, then we have

$$(4.8) \quad \text{Imm}_\theta(A) = \sum_U \text{wgt}(U),$$

where the sum is over F -tableaux U having the property.

Proposition 4.4 has the following consequence [31, Cor. 4.7].

Corollary 4.5. *If $\theta \in \mathcal{T}_n$ satisfies $\theta(\text{inc}(P)) \geq 0$ for all posets P , then the polynomial $\text{Imm}_\theta(x)$ is totally nonnegative.*

By Theorems 3.5 – 3.7, Corollary 4.5 applies to induced sign character immanants, induced trivial character immanants, and irreducible character immanants indexed by hook partitions. It does not apply to irreducible character immanants in general, because we have $\chi^\lambda(\text{inc}(P)) < 0$ for some λ, P .

Theorem 4.6. *For $k \leq n$, the polynomial $\text{Imm}_{\chi_{k1^{n-k}}}(x)$ is totally nonnegative. In particular, for A the path matrix of planar network F , we have*

$$\text{Imm}_{\chi_{k1^{n-k}}}(A) = \sum_U \text{wgt}(U),$$

where the sum is over all standard F -tableaux of shape $k1^{n-k}$.

Problem 4.7. *Combinatorially interpret the numbers $\text{Imm}_{\chi^\lambda}(A)$ in Theorem 4.3.*

Corollary 4.5 also does not apply to monomial traces, which satisfy $\phi^\lambda(\text{inc}(P)) < 0$ for some λ, P . Nevertheless, Stembridge conjectured that monomial trace immanants are totally nonnegative [37, Conj. 2.1].

Conjecture 4.8. *For $\lambda \vdash n$ the polynomial $\text{Imm}_{\phi^\lambda}(x)$ is totally nonnegative.*

Some evidence for Conjecture 4.8 follows from recent work of Hikita [13].

Proposition 4.9. *If A is the antiadjacency matrix of a unit interval order labeled as in Step 2 of Algorithm 3.2, then for all $\lambda \vdash n$ we have that $\text{Imm}_{\phi^\lambda}(A) \geq 0$.*

Proof. Let A be the antiadjacency matrix of unit interval order P , labeled as in Step 2 of Algorithm 3.2, and let w be the 312-avoiding permutation associated to P by Step 3 of Algorithm 3.2. It is well known that the entries of A which are equal to 1 form a Ferrers shape, and that we have

$$a_{1,v_1} \cdots a_{n,v_n} = \begin{cases} 1 & \text{if } v \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

(See, e.g., [40, Prop. 19, Prop. 22].) Thus we have

$$\text{Imm}_{\phi^\lambda}(A) = \sum_{v \in \mathfrak{S}_n} \phi^\lambda(v) a_{1,v_1} \cdots a_{n,v_n} = \sum_{v \leq w} \phi^\lambda(v) = \phi^\lambda\left(\sum_{v \leq w} v\right) = \phi^\lambda(C'_w(1)).$$

By Proposition 3.3 this number is $\phi^\lambda(\text{inc}(P))$, and by Theorem 3.9 it is nonnegative. $\square$

More evidence for Conjecture 4.8 follows from work of Stanley [33].

Proposition 4.10. *For $\ell = 1, \dots, n$, the sum*

$$(4.9) \quad \sum_{\substack{\mu \vdash n \\ \ell(\mu) = \ell}} \text{Imm}_{\phi^\mu}(x)$$

of monomial immanants is a totally nonnegative polynomial. In particular, for A the path matrix of planar network F , we have

$$(4.10) \quad \sum_{\substack{\mu \vdash n \\ \ell(\mu) = \ell}} \text{Imm}_{\phi^\mu}(A) = \sum_U \text{wgt}(U),$$

where the sum is over row-semistrict F -tableaux U of shape n having ℓ records.

Proof. Fix a totally nonnegative matrix A and define the trace

$$(4.11) \quad \theta^\ell = \sum_{\substack{\mu \vdash n \\ \ell(\mu) = \ell}} \phi^\mu$$

so that the left-hand side of (4.10) is $\text{Imm}_{\theta^\ell}(A)$. By Proposition 4.4 we have that

$$\text{Imm}_{\theta^\ell}(A) = \sum_K \text{wgt}(K) \sum_{\pi \in \Pi_\ell(K)} \theta^\ell(\text{inc}(P(\pi))),$$

where the first sum is over all bijective skeletons K in F . By Proposition 3.11, the inner sum equals the number of descent-free $P(\pi)$ -tableaux of shape n having ℓ records. Equivalently, it equals the number of row-semistrict F -tableaux of shape n having ℓ records. Thus the evaluation (4.10) has the desired interpretation. Since each such F -tableau has weight $\text{wgt}(K) \geq 0$, the polynomial (4.9) is totally nonnegative. $\square$

For example, let F, A be as in (4.1). The path families π, ρ, σ , and τ in (4.6) contribute 7 to $\text{Imm}_{\theta^2}(A)$, with each of the tableaux

$$\boxed{\pi_1 \pi_2 \pi_3}, \quad \boxed{\pi_1 \pi_3 \pi_2}, \quad \boxed{\rho_1 \rho_2 \rho_3}, \quad \boxed{\rho_1 \rho_3 \rho_2}, \quad \boxed{\sigma_1 \sigma_3 \sigma_2}, \quad \boxed{\tau_1 \tau_2 \tau_3}, \quad \boxed{\tau_1 \tau_3 \tau_2}$$

having records in positions 1 and 2.

5. MAIN RESULT AND OPEN PROBLEMS

We now show that Heyfron's inequalities (Theorem 1.1) hold not only for Hermitian positive semidefinite matrices, but also for totally nonnegative matrices.

Theorem 5.1. *For each $n \times n$ totally nonnegative matrix A we have*

$$\text{per}(A) = \frac{\text{Imm}_{\chi^n}(A)}{\chi^n(e)} \geq \frac{\text{Imm}_{\chi^{n-1,1}}(A)}{\chi^{n-1,1}(e)} \geq \frac{\text{Imm}_{\chi^{n-2,1,1}}(A)}{\chi^{n-2,1,1}(e)} \geq \dots \geq \frac{\text{Imm}_{\chi^{1,\dots,1}}(A)}{\chi^{1,\dots,1}(e)} = \det(A).$$

Equivalently, for $k = 2, \dots, n$, the difference

$$(5.1) \quad \frac{\text{Imm}_{\chi_{k1^{n-k}}}(x)}{\chi^{k1^{n-k}}(e)} - \frac{\text{Imm}_{\chi_{(k-1)1^{n-k+1}}}(x)}{\chi^{(k-1)1^{n-k+1}}(e)} = \frac{\text{Imm}_{\chi_{k1^{n-k}}}(x)}{\binom{n-1}{k-1}} - \frac{\text{Imm}_{\chi_{(k-1)1^{n-k+1}}}(x)}{\binom{n-1}{k-2}}$$

is a totally nonnegative polynomial.

First proof. Multiplying (5.1) by $\frac{(n-1)!}{(n-k)!(k-2)!}$, we have

$$(5.2) \quad (k-1)\text{Imm}_{\chi_{k1^{n-k}}}(x) - (n-k+1)\text{Imm}_{\chi_{(k-1)1^{n-k+1}}}(x).$$

Let A be an $n \times n$ totally nonnegative matrix. By Theorem 4.2, we may choose a planar network F whose path matrix is A . By Theorem 4.6, the evaluation of (5.2) at A equals

$$(5.3) \quad \begin{aligned} & (k-1) \sum_{\pi} \text{wgt}(\pi) d_k(\pi) - (n-k+1) \sum_{\pi} \text{wgt}(\pi) d_{k-1}(\pi) \\ &= \sum_{\pi} \text{wgt}(\pi) [(k-1)d_k(\pi) - (n-k+1)d_{k-1}(\pi)], \end{aligned}$$

where $d_k(\pi)$ is the number of standard π -tableaux of shape $k1^{n-k}$, and the sums are over

$$\pi \in \bigcup_K \Pi_e(K)$$

with K varying over all bijective skeletons in F . By Proposition 3.8 the difference in square brackets in the last sum equals the number of standard $P(\pi)$ -tableaux of shape $k1^{n-k}$ with one marked entry in columns $2, \dots, k$ which is incomparable to at least one entry in an earlier column. Since this number and $\text{wgt}(\pi)$ are nonnegative, the evaluation (5.3) is nonnegative. Thus the polynomial (5.2) is totally nonnegative and so is the polynomial (5.1). $\square$

Second proof. Define the traces $\theta^1, \dots, \theta^n$ as in (3.9). By Proposition 2.2 we have that the hook irreducible character immanant indexed by $k1^{n-k}$ belongs to the n -dimensional space spanned by $\text{Imm}_{\theta^1}(x), \dots, \text{Imm}_{\theta^n}(x)$. Specifically,

$$(5.4) \quad \text{Imm}_{\chi_{k1^{n-k}}}(x) = \sum_{\ell=n-k+1}^n \binom{\ell-1}{n-k} \text{Imm}_{\theta^\ell}(x).$$

It follows that for $k = 2, \dots, n$, the difference (5.1) expands in the basis of θ^ℓ -immanants as $c_{k,1}\text{Imm}_{\theta^1}(x) + \dots + c_{k,n}\text{Imm}_{\theta^n}(x)$ with nonnegative coefficients

$$c_{k,\ell} = \frac{\binom{\ell-1}{n-k}}{\binom{n-1}{k-1}} - \frac{\binom{\ell-1}{n-k+1}}{\binom{n-1}{k-2}} = \begin{cases} 0 & \text{if } \ell = 1, \dots, n-k, \\ \frac{1}{\binom{n-1}{k-1}} & \text{if } \ell = n-k+1, \\ \frac{n-\ell}{\ell-(n-k+1)} & \text{if } \ell = n-k+2, \dots, n. \end{cases}$$

By Proposition 4.10, each immanant $\text{Imm}_{\theta^\ell}(x)$ is a totally nonnegative polynomial. Thus for $k = 2, \dots, n$ the difference (5.1) is as well. $\square$

Theorem 5.1 thus provides some progress on the problem of understanding (1.12).

Problem 5.2. *Characterization the pairs (λ, μ) of partitions satisfying*

$$(5.5) \quad \frac{\text{Imm}_{\chi^\lambda}(A)}{\chi^\lambda(e)} \geq \frac{\text{Imm}_{\chi^\mu}(A)}{\chi^\mu(e)}$$

for all totally nonnegative or Hermitian positive semidefinite matrices.

Generalizing Heyfron's inequalities, Pate [25] proved that if λ, μ are two partitions of n , with k the multiplicity of λ_1 in $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and

$$(5.6) \quad \mu = (\lambda_1 - 1, \dots, \lambda_k - 1, \lambda_{k+1}, \dots, \lambda_\ell, \underbrace{1, \dots, 1}_k),$$

then we have (5.5) for all Hermitian positive semidefinite matrices. In other words, the Young diagram of μ is obtained by removing the rightmost column of the Young diagram of λ and by appending this to the first column, for example

$$\lambda = (4, 4, 3, 2) = \begin{array}{|c|c|c|c|} \hline \square & \square & & \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \quad \mu = (3, 3, 3, 2, 1, 1) = \begin{array}{|c|c|c|c|} \hline \square & & & \\ \hline \square & & & \\ \hline \square & \square & & \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}.$$

Perhaps this inequality holds for totally nonnegative matrices as well.

Problem 5.3. *Decide if Pate's inequality, i.e., (5.5) for λ, μ satisfying (5.6), holds for all totally nonnegative matrices.*

Theorem 5.1 and its proofs suggest several other open problems. Haiman [11, Conj.2.1] has conjectured that certain q -analogs ϕ_q^λ and $C'_w(q)$ of the monomial trace ϕ^λ and Kazhdan–Lusztig basis element $C'_w(1)$ satisfy

$$\phi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q)) \in \mathbb{N}[q]$$

for all λ and all w . (See [11] for definitions.) Perhaps the following weaker statement would be easier to prove.

Problem 5.4. *Show that for $\ell = 1, \dots, n$ and all $w \in \mathfrak{S}_n$ we have $\theta_q^\ell(q^{\frac{\ell(w)}{2}} C'_w(q)) \in \mathbb{N}[q]$, where*

$$\theta_q^\ell = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = \ell}} \phi_q^\lambda.$$

This is known to be true for w avoiding the patterns 3412 and 4231. (See, e.g., [7, Thm. 5.6], [31, Prop. 5.4].)

It would also be interesting to find an analog of the Littlewood–Merris–Watkins identities (1.6), (4.3) for the immanants $\{\text{Imm}_{\theta^\ell}(x) \mid \ell \in [n]\}$.

Problem 5.5. *For $\ell = 1, \dots, n$, find an expression for the immanant $\text{Imm}_{\theta^\ell}(x)$ which makes its total nonnegativity apparent.*

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REFERENCES

- [1] C. ATHANASIADIS. Power sum expansion of chromatic quasisymmetric functions. *Electron. J. Combin.*, **22**, 2 (2015). Paper 2.7, 9 pages.
- [2] W. BARRETT AND C. JOHNSON. Majorization monotonicity of symmetrized Fischer products. *Linear Multilinear Algebra*, **34**, 1 (1993) pp. 67–74.
- [3] S. BILLEY AND V. LAKSHMIBAI. *Singular loci of Schubert varieties*, vol. 182 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA (2000).
- [4] A. BJÖRNER AND F. BRENTI. *Combinatorics of Coxeter groups*, vol. 231 of *Graduate Texts in Mathematics*. Springer, New York (2005).
- [5] J. BORCEA AND P. BRÄNDÉN. Applications of stable polynomials to mixed determinants: Johnson’s conjectures, unimodality, and symmetrized Fischer products. *Duke Math. J.*, **143**, 2 (2008) pp. 205–223.
- [6] F. BRENTI. Combinatorics and total positivity. *J. Combin. Theory Ser. A*, **71**, 2 (1995) pp. 175–218.
- [7] S. CLEARMAN, M. HYATT, B. SHELTON, AND M. SKANDERA. Evaluations of Hecke algebra traces at Kazhdan–Lusztig basis elements. *Electron. J. Combin.*, **23**, 2 (2016). Paper 2.7, 56 pages.
- [8] C. W. CRYER. Some properties of totally positive matrices. *Linear Algebra Appl.*, **15** (1976) pp. 1–25.
- [9] P. C. FISHBURN. *Interval Graphs and Interval Orders*. Wiley, New York (1985).
- [10] V. GASHAROV. Incomparability graphs of $(\mathbf{3} + \mathbf{1})$ -free posets are s -positive. *Discrete Math.*, **157** (1996) pp. 211–215.
- [11] M. HAIMAN. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, **6**, 3 (1993) pp. 569–595.
- [12] P. HEYFRON. Immanant dominance orderings for hook partitions. *Linear Multilinear Algebra*, **24**, 1 (1988) pp. 65–78.

- [13] T. HIKITA. A proof of the Stanley–Stembridge conjecture (2024). Preprint [math.CO/2410.12758](#) on ArXiv.
- [14] G. JAMES. Immanants. *Linear Multilinear Algebra*, **32** (1992) pp. 197–210.
- [15] R. KALISZEWSKI. Hook coefficients of chromatic functions. *J. Comb.*, **6**, 3 (2015) pp. 327–337.
- [16] S. KARLIN AND G. MCGREGOR. Coincidence probabilities. *Pacific J. Math.*, **9** (1959) pp. 1141–1164.
- [17] D. KAZHDAN AND G. LUSZTIG. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, **53** (1979) pp. 165–184.
- [18] E. LIEB. Proofs of some conjectures on permanents. *J. Math. Mech.*, **16** (1966) pp. 127–134.
- [19] B. LINDSTRÖM. On the vector representations of induced matroids. *Bull. London Math. Soc.*, **5** (1973) pp. 85–90.
- [20] D. E. LITTLEWOOD. *The Theory of Group Characters and Matrix Representations of Groups*. Oxford University Press, New York (1940).
- [21] C. LOEWNER. On totally positive matrices. *Math. Z.*, **63** (1955) pp. 338–340.
- [22] G. LUSZTIG. Total positivity in reductive groups. In *Lie Theory and Geometry: in Honor of Bertram Kostant*, vol. 123 of *Progress in Mathematics*. Birkhäuser, Boston (1994), pp. 531–568.
- [23] R. MERRIS. Single-hook characters and Hamiltonian circuits. *Linear Multilinear Algebra*, **14** (1983) pp. 21–35.
- [24] R. MERRIS AND W. WATKINS. Inequalities and identities for generalized matrix functions. *Linear Algebra Appl.*, **64** (1985) pp. 223–242.
- [25] T. H. PATE. Partitions, irreducible characters, and inequalities for generalized matrix functions. *Trans. Amer. Math. Soc.*, **325**, 2 (1991) pp. 875–894.
- [26] B. RHOADES AND M. SKANDERA. Kazhdan–Lusztig immanants and products of matrix minors. *J. Algebra*, **304**, 2 (2006) pp. 793–811.
- [27] B. SAGAN. *The Symmetric Group*. Springer, New York (2001).
- [28] I. SCHUR. Über endliche Gruppen und Hermitesche Formen. *Math. Z.*, **1**, 2-3 (1918) pp. 184–207.
- [29] J. SHARESHIAN AND M. WACHS. Chromatic quasisymmetric functions. *Adv. Math.*, **295** (2016) pp. 497–551.
- [30] M. SKANDERA. On the dual canonical and Kazhdan–Lusztig bases and 3412, 4231-avoiding permutations. *J. Pure Appl. Algebra*, **212** (2008).
- [31] M. SKANDERA. Characters and chromatic symmetric functions. *Electron. J. Combin.*, **28**, 2 (2021). Research Paper P2.19, 39 pages.
- [32] M. SKANDERA AND D. SOSKIN. Barrett–Johnson inequalities for totally nonnegative matrices. *Linear Multilinear Algebra*, **73**, 10 (2025) pp. 2320–2343.
- [33] R. STANLEY. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, **111** (1995) pp. 166–194.
- [34] R. STANLEY. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge (1999).
- [35] R. STANLEY AND J. STEMBRIDGE. On immanants of Jacobi–Trudi matrices and permutations with restricted positions. *J. Combin. Theory Ser. A*, **62** (1993) pp. 261–279.
- [36] J. STEMBRIDGE. Immanants of totally positive matrices are nonnegative. *Bull. London Math. Soc.*, **23** (1991) pp. 422–428.
- [37] J. STEMBRIDGE. Some conjectures for immanants. *Canad. J. Math.*, **44**, 5 (1992) pp. 1079–1099.
- [38] W. T. TROTTER. *Combinatorics and Partially Ordered Sets: Dimension Theory*. Johns Hopkins University Press, Baltimore (1992).
- [39] I. M. WANLESS. Lieb’s permanental dominance conjecture. In *The physics and mathematics of Elliott Lieb—the 90th anniversary. Vol. II*. EMS Press, Berlin (2022), pp. 501–516.
- [40] M. WATSON. Bruhat order on fixed-point-free involutions in the symmetric group. *Electron. J. Combin.*, **21**, 2 (2014). Paper 2.20, 25.
- [41] A. WHITNEY. A reduction theorem for totally positive matrices. *J. Anal. Math.*, **2** (1952) pp. 88–92.