TOTAL NONNEGATIVITY AND INDUCED SIGN CHARACTERS OF THE HECKE ALGEBRA

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ABSTRACT. Let $\mathfrak{S}_{[i,j]}$ be the subgroup of the symmetric group $\mathfrak{S}_n$ generated by adjacent transpositions $(i, i+1), \ldots, (j-1, j)$, assuming $1 \leq i < j \leq n$. We give a combinatorial rule for evaluating induced sign characters of the type A Hecke algebra $H_n(q)$ at all elements of the form $\sum_{w \in \mathfrak{S}_{[i,j]}} T_w$ and at all products of such elements. This includes evaluation at some elements $C'_w(q)$ of the Kazhdan-Lusztig basis.

1. INTRODUCTION

Concepts of total nonnegativity, first explored by Gantmacher and Krein [9] have found their way into many areas of mathematics, including the study of polynomials $p(x_{1,1}, \ldots, x_{n,n})$ satisfying $p(a_{1,1}, \ldots, a_{n,n}) \geq 0$ for every totally nonnegative $n \times n$ matrix $A = (a_{i,j})$. We call these *totally nonnegative (TNN)* polynomials. In particular, work of Lusztig [17] implies that in $\mathbb{Z}[x] := \mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$, certain elements which are related to the dual canonical basis of the quantum group $\mathcal{O}_q(SL_n(\mathbb{C}))$ are TNN polynomials.

In practice, it is sometimes possible to use cluster algebras [8] and a computer to demonstrate that a polynomial is TNN by expressing it as a subtraction-free rational expression in matrix minors. On the other hand, no simple characterization of TNN polynomials is known. To improve our understanding of TNN polynomials, one might begin by investigating the *immanant subspace* $\text{span}_{\mathbb{Z}}\{x_{1,w_1} \cdots x_{n,w_n} \mid w \in \mathfrak{S}_n\}$ of $\mathbb{Z}[x]$, especially the generating functions

$$(1.1) \qquad \text{Imm}_\theta(x) := \sum_{w \in \mathfrak{S}_n} \theta(w) x_{1,w_1} \cdots x_{n,w_n}$$

for class functions $\theta : \mathfrak{S}_n \to \mathbb{Z}$. Or, since some published results relate total nonnegativity to the Hecke algebra $H_n(q)$ and its traces $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, one might investigate these.

In particular, let $\{\widetilde{C}_w(q) \mid w \in \mathfrak{S}_n\}$ be the (modified, signless) *Kazhdan-Lusztig basis* of $H_n(q)$, defined by

$$(1.2) \qquad \widetilde{C}_w(q) := q^{\frac{\ell(w)}{2}} C'_w(q) = \sum_{v \leq w} P_{v,w}(q) T_v,$$

where $\{T_w \mid w \in \mathfrak{S}_n\}$ is the natural basis of $H_n(q)$, $\{P_{v,w}(q) \mid v, w \in \mathfrak{S}_n\}$ are the Kazhdan-Lusztig polynomials [13], and $\leq$ denotes the Bruhat order. (See, e.g., [3, §2].) Specializing at $q^{\frac{1}{2}} = 1$ we have $T_v \mapsto v$ and $H_n(1) \cong \mathbb{Z}[\mathfrak{S}_n]$. For $1 \leq a < b \leq n$, let $s_{[a,b]} \in \mathfrak{S}_n$ be the *reversal* whose one-line notation is $1 \cdots (a-1)b(b-1) \cdots (a+1)a(a+2) \cdots n$. Stembridge [21,

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Thm. 2.1] showed that for any linear function $\theta : \mathbb{Z}[\mathfrak{S}_n] \to \mathbb{Z}$, the immanant $\mathrm{Imm}_\theta(x)$ is TNN if for all sequences $J_1, \ldots, J_m$ of subintervals of $[n] := \{1, \ldots, n\}$, we have

$$(1.3) \qquad \theta(\widetilde{C}_{s_{J_1}}(1) \cdots \widetilde{C}_{s_{J_m}}(1)) \geq 0.$$

Furthermore, by Lindström's Lemma [16] and its converse [4], [6], a combinatorial interpretation of the above expression would immediately yield a combinatorial interpretation of the number $\mathrm{Imm}_\theta(A)$ for A a TNN matrix. Now if a linear function $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ specializes at $q^{\frac{1}{2}} = 1$ to θ, then (1.3) is clearly a consequence of the condition

$$(1.4) \qquad \theta_q(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)) \in \mathbb{N}[q],$$

and a combinatorial interpretation of coefficients of the above polynomial would yield combinatorial interpretations of the earlier expressions. Haiman [10, Appendix] observed that (1.4) in turn follows from the condition that for all $w \in \mathfrak{S}_n$ we have

$$(1.5) \qquad \theta_q(\widetilde{C}_w(q)) \in \mathbb{N}[q],$$

since results in [1], [20] imply that any product of Kazhdan-Lusztig basis elements belongs to $\mathrm{span}_{\mathbb{N}[q]}\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n\}$. On the other hand, it is not clear that the coefficients in the resulting linear combination would lead to a combinatorial interpretation of the expression in (1.4).

Stembridge [21, Cor. 3.3] proved that for each irreducible $\mathfrak{S}_n$-character χ^λ, the evaluation (1.3) belongs to $\mathbb{N}$. Haiman [10, Lem. 1.1] proved that for each corresponding irreducible $H_n(q)$-character χ_q^λ, the evaluation (1.5) belongs to $\mathbb{N}[q]$. They conjectured the same [22, Conj. 2.1], [10, Conj. 2.1] for functions ϕ^λ (ϕ_q^λ), called *monomial traces*, related to irreducible characters by the inverse Kostka numbers. Both sets $\{\chi^\lambda \,|\, \lambda \vdash n\}$, $\{\phi^\lambda \,|\, \lambda \vdash n\}$ ($\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, $\{\phi_q^\lambda \,|\, \lambda \vdash n\}$), where $\lambda \vdash n$ denotes that λ varies over all partitions of n, form bases of the space of $\mathfrak{S}_n$-class functions ($H_n(q)$-trace space). (See, e.g., [3, §A3.9], [22, §1].)

None of these results or conjectures included a combinatorial interpretation. For the purposes of understanding TNN polynomials of the form (1.1), it would be desirable to solve the following problem.

Problem 1.1. *Give combinatorial interpretations of all of the expressions in (1.4) when θ_q varies over all elements of any basis of the $H_n(q)$-trace space.*

So far, only some special cases have such interpretations. In the case that w avoids the patterns 3412 and 4231, the Kazhdan-Lusztig basis element $\widetilde{C}_w(q)$ is closely related to a product of the form appearing in (1.4). Combinatorial interpretations of the corresponding expressions (1.4) and (1.5) were given in [5] for $\theta_q \in \{\chi_q^\lambda \,|\, \lambda \vdash n\}$, and for θ_q belonging to several other bases of the $H_n(q)$-trace space, including the basis $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$ of induced sign characters. Combinatorial interpretations for $\theta_q = \phi_q^\lambda$ were given only when λ has at most two parts [5, Thm. 10.3], or when λ has rectangular shape and $q = 1$ [22, Thm. 2.8].

In the case that all reversals $s_{J_1}, \ldots, s_{J_m}$ in (1.4) are adjacent transpositions, combinatorial interpretations of

$$\epsilon_q^\lambda(\widetilde{C}_{s_{[j_1, j_1+1]}}(q) \cdots \widetilde{C}_{s_{[j_m, j_m+1]}}(q))$$

for all $\lambda \vdash n$ were given in [11, Thm. 5.4], nearly solving Problem 1.1. To complete the solution of Problem 1.1, we generalize [11, Thm. 5.4] to interpret all evaluations of the form

$$(1.6) \qquad \epsilon_q^\lambda(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)),$$

where the intervals $J_1, \ldots, J_m \subseteq [n]$ have arbitrary cardinality. The justification of our solution closely follows the steps used in [11, §3-5], with definitions and propositions generalized as necessary.

In Section 2, we discuss the symmetric group algebra $\mathbb{Z}[\mathfrak{S}_n]$, the Hecke algebra $H_n(q)$, and a noncommutative q-analog $\mathcal{A}(n, q)$ of $\mathbb{Z}[x]$ known as the quantum matrix bialgebra. The q-analogs $\mathrm{Imm}_{\theta_q}(x)$ of immanants (1.1) belong to a q-analog $\mathcal{A}_{[n],[n]} \subset \mathcal{A}$ of the immanant subspace of $\mathbb{Z}[x]$, and $\mathrm{Imm}_{\epsilon_q^\lambda}(x)$ has a natural expression in terms of certain monomials which we write as $\{x^{u,w} \mid u, w \in \mathfrak{S}_n\}$. In Section 3, we consider elements of the form $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q) \in H_n(q)$, and we associate to each a planar network G, a related matrix B, and a linear map $\sigma_B : \mathcal{A}_{[n],[n]} \to \mathbb{Z}[q^{\frac{1}{2}}, \bar{q}^{\frac{1}{2}}]$. In Section 4, we interpret $\sigma_B(x^{u,w}) \in \mathbb{N}[q]$ for each pair $u, w \in \mathfrak{S}_n$ in terms of statistics on families of source-to-sink paths in G and on tableaux filled with such paths. In Section 5, we show that for each $H_n(q)$-trace θ_q and its generating function $\mathrm{Imm}_{\theta_q}(x)$, we have the identity

$$(1.7) \qquad \theta_q(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)) = \sigma_B(\mathrm{Imm}_{\theta_q}(x)).$$

This identity leads to our main result in Section 6, where we use path families, tableaux, and $\mathrm{Imm}_{\epsilon_q^\lambda}(x)$ to combinatorially evaluate the expressions (1.6).

2. Algebraic background

Define the *symmetric group algebra* $\mathbb{Z}[\mathfrak{S}_n]$ and the *(type A Iwahori-) Hecke algebra* $H_n(q)$ to be the algebras with multiplicative identity elements e and T_e, respectively, generated over $\mathbb{Z}$ and $\mathbb{Z}[q^{\frac{1}{2}}, \bar{q}^{\frac{1}{2}}]$ by elements $s_1, \ldots, s_{n-1}$ and $T_{s_1}, \ldots, T_{s_{n-1}}$, subject to the relations

$$(2.1) \quad \begin{array}{lll} s_i^2 = e & T_{s_i}^2 = (q-1)T_{s_i} + qT_e & \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i = s_j s_i s_j & T_{s_i}T_{s_j}T_{s_i} = T_{s_j}T_{s_i}T_{s_j} & \text{for } |i-j| = 1, \\ s_i s_j = s_j s_i & T_{s_i}T_{s_j} = T_{s_j}T_{s_i} & \text{for } |i-j| \geq 2. \end{array}$$

Analogous to the natural basis $\{w \mid w \in \mathfrak{S}_n\}$ of $\mathbb{Z}[\mathfrak{S}_n]$ is the natural basis $\{T_w \mid w \in \mathfrak{S}_n\}$ of $H_n(q)$, where we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$ whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced (short as possible) expression for w in $\mathfrak{S}_n$. We call ℓ the *length* of w and write $\ell = \ell(w)$. We define the *one-line notation* $w_1 \cdots w_n$ of $w \in \mathfrak{S}_n$ by letting any expression for w act on the word $1 \cdots n$, where each generator $s_j = s_{[j,j+1]}$ acts on an n-letter word by swapping the letters in positions j and $j+1$, i.e., $s_j \circ v_1 \cdots v_n = v_1 \cdots v_{j-1} v_{j+1} v_j v_{j+2} \cdots v_n$. $\ell(w)$ is equal to $\mathrm{INV}(w)$, the number of inversions in the one-line notation $w_1 \cdots w_n$ of w. If this one-line notation contains no subword matching the pattern 3412 or 4231 (e.g., if w is a reversal), then the Kazhdan-Lusztig polynomials in (1.2) are identically 1 [15].

For any subinterval $J = [a, b]$ of $[n] := \{1, \ldots, n\}$, define $\mathfrak{S}_J$ to be the subgroup of $\mathfrak{S}_n$ generated by $\{s_a, \ldots, s_{b-1}\}$. The longest element of $\mathfrak{S}_J \cong \mathfrak{S}_{b-a+1}$ is the reversal $s_J := s_{[a,b]}$. Define $\mathfrak{S}_-^J$ to be the set of minimum length representatives of left cosets $w\mathfrak{S}_J$, i.e., the elements $w \in \mathfrak{S}_n$ satisfying $ws_i > w$ (in the Bruhat order) for all $i \in [a, b-1]$. A bijection

of $\mathfrak{S}_n$ with $\mathfrak{S}_-^J \times \mathfrak{S}_J$ is given by the fact that each element $w \in \mathfrak{S}_n$ factors uniquely as

$$(2.2) \qquad\qquad\qquad w = w_- w^*,$$

with $(w_-, w^*) \in \mathfrak{S}_-^J \times \mathfrak{S}_J$ and $\ell(w_-) + \ell(w^*) = \ell(w)$. This factorization satisfies the identity

$$(2.3) \qquad\qquad T_w \widetilde{C}_{s_J}(q) = T_{w_-} T_{w^*} \widetilde{C}_{s_J}(q) = q^{\ell(w^*)} T_{w_-} \widetilde{C}_{s_J}(q).$$

A similar factorization result employs the set $\mathfrak{S}_J^-$ of minimum length representatives of right cosets $\mathfrak{S}_J w$, and defines a bijection from $\mathfrak{S}_n$ to $\mathfrak{S}_J \times \mathfrak{S}_J^-$.

For any partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, the subgroup

$$\mathfrak{S}_\lambda := \mathfrak{S}_{[1,\lambda_1]} \times \mathfrak{S}_{[\lambda_1+1,\lambda_1+\lambda_2]} \times \cdots \times \mathfrak{S}_{[n-\lambda_r+1,n]} \cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r}$$

of $\mathfrak{S}_n$ generated by

$$(2.4) \qquad\qquad \{s_1, \ldots, s_{n-1}\} \smallsetminus \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, s_{\lambda_1+\lambda_2+\lambda_3}, \ldots, s_{n-\lambda_r}\}$$

is called the *Young subgroup* indexed by λ. Define $H_\lambda(q)$ to be the subalgebra of $H_n(q)$ generated by the corresponding elements T_{s_i}. Define $\mathfrak{S}_\lambda^-$ to be the set of minimum length representatives of right cosets $\mathfrak{S}_\lambda u$, i.e., the elements $u \in \mathfrak{S}_n$ satisfying $s_i u > u$ (in the Bruhat order) for all s_i in the set (2.4). Equivalently, $\mathfrak{S}_\lambda^-$ consists of all $u \in \mathfrak{S}_n$ whose one-line notations are concatenations of r increasing subwords

$$(2.5) \qquad u_1 \cdots u_{\lambda_1}, \quad u_{\lambda_1+1} \cdots u_{\lambda_1+\lambda_2}, \quad u_{\lambda_1+\lambda_2+1} \cdots u_{\lambda_1+\lambda_2+\lambda_3}, \quad \ldots, \quad u_{n-\lambda_r+1} \cdots u_n.$$

The bijections of $\mathfrak{S}_n$ with $\mathfrak{S}_J \times \mathfrak{S}_J^-$, for $J = [1, \lambda_1], [\lambda_1+1, \lambda_1+\lambda_2], \ldots, [n-\lambda_r+1, n]$, extend to a bijection of $\mathfrak{S}_n$ with $\mathfrak{S}_\lambda \times \mathfrak{S}_\lambda^-$ and a unique factorization of each element $w \in \mathfrak{S}_n$ as

$$(2.6) \qquad\qquad\qquad w = w^\circ w^-$$

with $(w^\circ, w^-) \in \mathfrak{S}_\lambda \times \mathfrak{S}_\lambda^-$ and $\ell(w^\circ) + \ell(w^-) = \ell(w)$.

We call a linear function $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ an $H_n(q)$-*trace* if it satisfies $\theta_q(gh) = \theta_q(hg)$ for all $g, h \in H_n(q)$. Such functions form a free $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module of rank equal to the number of partitions of n, and include the characters of all finite representations of $H_n(q)$. The specialization at $q^{\frac{1}{2}} = 1$ of a trace is an $\mathfrak{S}_n$-class function. Two bases of the module of $H_n(q)$-trace functions are the set of irreducible characters $\{\chi_q^\lambda \mid \lambda \vdash n\}$ and the set of induced sign characters $\{\epsilon_q^\lambda \mid \lambda \vdash n\}$, where

$$\epsilon_q^\lambda = \mathrm{sgn}\!\uparrow_{H_\lambda(q)}^{H_n(q)}, \qquad \mathrm{sgn}(T_{s_i}) = -1.$$

Define the *quantum matrix bialgebra* $\mathcal{A} = \mathcal{A}(n, q)$ to be to be the associative algebra with unit 1 generated over $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ by n^2 variables $x = (x_{1,1}, \ldots, x_{n,n})$, subject to the relations

$$(2.7) \qquad\begin{aligned} x_{i,\ell} x_{i,k} &= q^{\frac{1}{2}} x_{i,k} x_{i,\ell}, & x_{j,k} x_{i,\ell} &= x_{i,\ell} x_{j,k}, \\ x_{j,k} x_{i,k} &= q^{\frac{1}{2}} x_{i,k} x_{j,k}, & x_{j,\ell} x_{i,k} &= x_{i,k} x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x_{i,\ell} x_{j,k}, \end{aligned}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. (See [18].) The counit map $\varepsilon(x_{i,j}) = \delta_{i,j}$, and coproduct map $\Delta(x_{i,j}) = \sum_{k=1}^n x_{i,k} \otimes x_{k,j}$ give $\mathcal{A}$ a bialgebra structure. While not a Hopf algebra, $\mathcal{A}$ is closely related to the quantum group $\mathcal{O}_q(SL_n(\mathbb{C})) \cong \mathbb{C} \otimes \mathcal{A}/(\det_q(x) - 1)$, where

$$(2.8) \qquad\qquad \det_q(x) := \sum_{v \in \mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(v)} x_{1,v_1} \cdots x_{n,v_n}$$

is the $(n \times n)$ *quantum determinant* of the matrix $x = (x_{i,j})$. The antipode map of this Hopf algebra is $\mathcal{S}(x_{i,j}) = (-q^{\frac{1}{2}})^{j-i}\det_q(x_{[n]\smallsetminus\{j\},[n]\smallsetminus\{i\}})$, where $x_{L,M} := (x_{\ell,m})_{\ell\in L, m\in M}$, and $\det_q(x_{L,M})$ is defined analogously to (2.8), assuming $|L| = |M|$. Specializing $\mathcal{A}$ at $q^{\frac{1}{2}} = 1$, we obtain the commutative ring $\mathbb{Z}[x]$.

As a $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module, $\mathcal{A}$ has a natural basis $\{x_{1,1}^{a_{1,1}} \cdots x_{n,n}^{a_{n,n}} \,|\, a_{1,1}, \ldots, a_{n,n} \in \mathbb{N}\}$ of monomials in which variables appear in lexicographic order, and the relations (2.7) provide an algorithm for expressing any other monomial in terms of this basis. The submodule $\mathcal{A}_{[n],[n]}$ spanned by the monomials $\{x^{u,w} := x_{u_1,w_1} \cdots x_{u_n,w_n} \,|\, u, w \in \mathfrak{S}_n\}$ has rank $n!$ and natural basis $\{x^{e,w} \,|\, w \in \mathfrak{S}_n\}$. Alternatively, for any fixed u, the set $\{x^{u,w} \,|\, w \in \mathfrak{S}_n\}$ is a basis as well.

For any linear function $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$, we may define the generating function

$$(2.9) \qquad \mathrm{Imm}_{\theta_q}(x) := \sum_{w\in\mathfrak{S}_n} q^{\frac{-\ell(w)}{2}} \theta_q(T_w) x_{1,w_1} \cdots x_{n,w_n}$$

in $\mathcal{A}_{[n],[n]}$ which is a q-analog of the generating function (1.1). For example, the generating function for the induced sign character ϵ_q^λ is

$$(2.10) \qquad \mathrm{Imm}_{\epsilon_q^\lambda}(x) = \sum_I \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}),$$

where the sum is over all ordered set partitions $I = (I_1, \ldots, I_r)$ of $[n]$ having *type* λ, i.e., satisfying $|I_j| = \lambda_j$ [14, Thm. 5.4]. Fixing an ordered set partition I and expanding the corresponding term on the right-hand side of (2.10), we obtain

$$(2.11) \qquad \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}) = \sum_{y\in\mathfrak{S}_\lambda} (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} x^{u,yu},$$

where $u = u(I)$ is the element of $\mathfrak{S}_\lambda^-$ whose r increasing subwords (2.5) are the increasing rearrangements of the blocks $I_1, \ldots, I_r$ of I. As I varies over all ordered set partitions of type λ, u varies over all elements of $\mathfrak{S}_\lambda^-$ and we have

$$(2.12) \qquad \mathrm{Imm}_{\epsilon_q^\lambda}(x) = \sum_{u\in\mathfrak{S}_\lambda^-} \sum_{y\in\mathfrak{S}_\lambda} (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} x^{u,yu}.$$

A special case of the induced sign character immanant (2.10) is the generating function for the Hecke algebra sign character $\epsilon_q^n = \chi_q^{1^n}$: the quantum determinant (2.8).

3. Star networks and the evaluation map

To each element of the form $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q) \in H_n(q)$, we associate a planar network G, a related matrix B, and a linear map $\sigma_B : \mathcal{A}_{[n],[n]} \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. These three objects play crucial roles in the justification of the identity (1.7) in Theorem 5.2, which in turn is necessary for the justification of our main result in Theorem 6.5.

These three definitions are straightforward generalizations of those given in [11, §3.1–3.2] for elements of the form $\widetilde{C}_{s_{i_1}}(q) \cdots \widetilde{C}_{s_{i_m}}(q) \in H_n(q)$, i.e., where $|J_1| = \cdots = |J_m| = 2$. Other predecessors of the definitions, appearing in [5, §3], [19, §5], pertained to sequences $(J_1, \ldots, J_m)$ of intervals satisfying the strict "zig-zag" conditions stated in [5, §3], [19, §3].

3.1. The network G associated to $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$.

To the Kazhdan-Lusztig basis element $\widetilde{C}_{s_{[a,b]}}(q)$, we associate a (simple, directed, planar) graph $G_{[a,b]}$ on $2n + 1$ vertices. For $1 \le a < b \le n$, define $G_{[a,b]}$ as follows.

(1) On the left is a column of n vertices labeled *source 1* , ..., *source n*, from bottom to top.

(2) On the right is a column of n vertices, labeled *sink 1* , ..., *sink n*, from bottom to top.

(3) An interior vertex is placed between the sources and sinks.

(4) For $i = 1, \ldots, a - 1$ and $i = b + 1, \ldots, n$, a directed edge begins at source i and terminates at sink i.

(5) For $i = a, \ldots, b$, a directed edge begins at source i and terminates at the interior vertex, and another directed edge begins at the interior vertex and terminates at sink i.

For $a = 1, \ldots, n$ we define $G_{[a,a]}$ to be the similar directed planar graph on n sources and n sinks with n edges, each from source i to sink i, for $i = 1, \ldots, n$. Call each of the above graphs a *simple star network*. In figures we will not explicitly draw vertices or show edge orientations (assumed to be from left to right). For $n = 4$, there are seven simple star networks: $G_{[1,4]}$, $G_{[2,4]}$, $G_{[1,3]}$, $G_{[3,4]}$, $G_{[2,3]}$, $G_{[1,2]}$, $G_{[1,1]} = \cdots = G_{[4,4]}$, respectively,

(3.1)

Define a *star network* to be the concatenation of finitely many simple star networks. We write $G \circ H$ for the network in which sink i of G is identified with source i of H, for $i = 1, \ldots, n$. For example, when $n = 4$ we have

(3.2)
$$G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]} = \quad , \qquad G_{[2,4]} \circ G_{[1,3]} = \quad .$$

Let $\pi = (\pi_1, \ldots, \pi_n)$ be a sequence of source-to-sink paths in a wiring diagram G. We call π a *path family* if there exists a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ such that π_i is a path from source i to sink w_i. In this case, we say more specifically that π has *type w*. We say that the path family *covers* G if it contains every edge exactly once.

The simple star network $G_{[a,b]}$ represents the Kazhdan-Lusztig basis element $\widetilde{C}_{s_{[a,b]}}(q)$ in the sense that we have

$$\widetilde{C}_{s_{[a,b]}}(q) = \sum_{\pi} T_{\text{type}(\pi)},$$

where the sum is over all path families which cover $G_{[a,b]}$. The concatenation $G_{J_1} \circ \cdots \circ G_{J_m}$ represents the product $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$ in a sense which we will make more precise in Corollary 5.3.

3.2. The matrix B associated to $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$.

One can enhance any planar network by associating to each edge a *weight* belonging to some ring R, and by defining the *weight of a path* to be the product of its edge weights. If R is noncommutative, then one multiplies weights in the order that the corresponding edges appear in the path. For a *family* $\pi = (\pi_1, \ldots, \pi_n)$ of n paths in a planar network, one defines

$\mathrm{wgt}(\pi) = \mathrm{wgt}(\pi_1)\cdots\mathrm{wgt}(\pi_n)$. The *(weighted) path matrix* $B = B(G) = (b_{i,j})$ of G is defined by letting $b_{i,j}$ be the sum of weights of all paths in G from source i to sink j. Thus the product $b_{1,w_1}\cdots b_{n,w_n}$ is equal to the sum of weights of all path families of type w in G (covering G or not). It is easy to show that path matrices respect concatenation: $B(G_1 \circ G_2) = B(G_1)B(G_2)$. When R is commutative, a result known as *Lindström's Lemma* [12], [16] asserts that for row and column sets I, J with $|I| = |J|$, the minor $\det(B_{I,J})$ is equal to the sum of weights of all nonintersecting path families from sources indexed by I to sinks indexed by J.

Assigning weights to the edges of $G = G_{J_1} \circ \cdots \circ G_{J_m}$ can aid in the evaluation of a linear function $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ at $\widetilde{C}_{s_{J_1}}(q)\cdots\widetilde{C}_{s_{J_m}}(q)$ by relating this evaluation to the generating function (2.9). In particular, let $\{z_{h,p,k}\,|\,1 \le p \le r; i_p \le h \le j_p; 1 \le k \le 2\}$ be indeterminate weights satisfying

(3.3) $\qquad z_{h_2,p_2,k_2} z_{h_1,p_1,k_1} = \begin{cases} z_{h_1,p_1,k_1} z_{h_2,p_2,k_2} & \text{if } p_1 \ne p_2, \text{ or } k_1 \ne k_2, \\ q^{\frac{1}{2}} z_{h_1,p_1,k_1} z_{h_2,p_2,k_2} & \text{if } p_1 = p_2, \ k_1 = k_2, \text{ and } h_1 < h_2. \end{cases}$

We assign weights to the edges of $G_{J_p} = G_{[i_p,j_p]}$ as follows.

(1) Assign weight 1 to the $n - j_p + i_p - 1$ edges not incident upon the central vertex.
(2) Assign weights $z_{i_p,p,1}, z_{i_p+1,p,1}, \ldots, z_{j_p,p,1}$, to the $j_p - i_p + 1$ edges entering the central vertex, from bottom to top.
(3) Assign weights $z_{i_p,p,2}, z_{i_p+1,p,2}, \ldots, z_{j_p,p,2}$, to the $j_p - i_p + 1$ edges leaving the central vertex, from bottom to top.

For example, the star network $G = G_{J_1} \circ G_{J_2} = G_{[2,4]} \circ G_{[1,3]}$ is weighted as

(3.4)
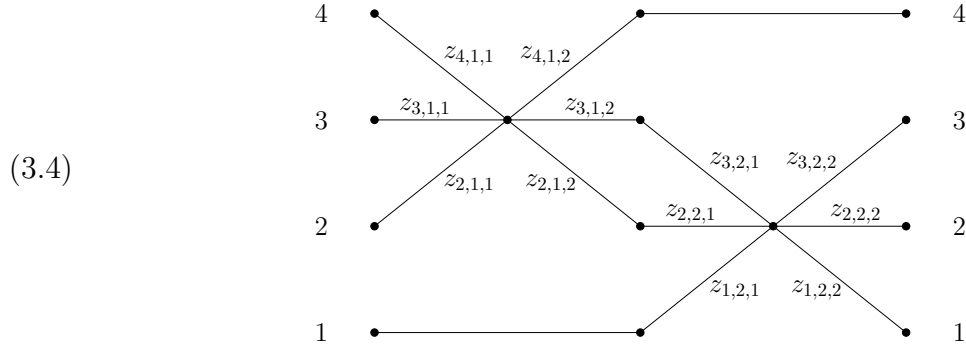
and the weighted path matrix of G is

(3.5)

$$
B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_{2,1,1}z_{2,1,2} & z_{2,1,1}z_{3,1,2} & z_{2,1,1}z_{4,1,2} \\ 0 & z_{3,1,1}z_{2,1,2} & z_{3,1,1}z_{3,1,2} & z_{3,1,1}z_{4,1,2} \\ 0 & z_{2,1,1}z_{2,1,2} & z_{2,1,1}z_{3,1,2} & z_{2,1,1}z_{4,1,2} \end{bmatrix} \begin{bmatrix} z_{1,2,1}z_{1,2,2} & z_{1,2,1}z_{2,2,2} & z_{1,2,1}z_{3,2,2} & 0 \\ z_{2,2,1}z_{1,2,2} & z_{2,2,1}z_{2,2,2} & z_{2,2,1}z_{3,2,2} & 0 \\ z_{3,2,1}z_{1,2,2} & z_{3,2,1}z_{2,2,2} & z_{3,2,1}z_{3,2,2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} z_{1,1,1}z_{1,2,2} & z_{1,2,1}z_{2,2,2} & z_{1,2,1}z_{3,2,2} & 0 \\ z_{2,1,1}(z_D + z_U)z_{1,2,2} & z_{2,1,1}(z_D + z_U)z_{2,2,2} & z_{2,1,1}(z_D + z_U)z_{3,2,2} & z_{2,1,1}z_{4,1,2} \\ z_{3,1,1}(z_D + z_U)z_{1,2,2} & z_{3,1,1}(z_D + z_U)z_{2,2,2} & z_{3,1,1}(z_D + z_U)z_{3,2,2} & z_{3,1,1}z_{4,1,2} \\ z_{4,1,1}(z_D + z_U)z_{1,2,2} & z_{4,1,1}(z_D + z_U)z_{2,2,2} & z_{4,1,1}(z_D + z_U)z_{3,2,2} & z_{4,1,1}z_{4,1,2} \end{bmatrix},
$$

where we have defined $z_D = z_{2,1,2}z_{2,2,1}$, $z_U = z_{3,1,2}z_{3,2,1}$.

3.3. The map σ_B associated to $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$.

The final, algebraic, object which we associate to the product $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$ is a map σ_B which allows us to evaluate a linear functional θ_q at this product by considering the corresponding generating function $\mathrm{Imm}_{\theta_q}(x) \in \mathcal{A}_{[n],[n]}$. This strategy can be quite helpful when one has a simpler expression for $\mathrm{Imm}_{\theta_q}(x)$ than for its coefficients $\theta_q(T_w)$. In our case, an expression for $\mathrm{Imm}_{\epsilon_q^\lambda}(x)$ is given by (2.10), while we have no explicit formula for $\epsilon_q^\lambda(T_w)$.

Let Z_G be the quotient of the noncommutative ring

$$\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]\langle z_{h_p,p,k} \mid p = 1, \ldots, r; h_p = i_p, \ldots, j_p; k = 1, 2\rangle$$

modulo the ideal generated by the relations (3.3), and assume that $q^{\frac{1}{2}}$, $q^{\frac{1}{2}}$ commute with all other indeterminates. Let z_G be the product of all indeterminates $z_{h_p,p,k}$, in lexicographic order, and for $f \in Z_G$, let $[z_G]f$ denote the coefficient of z_G in f. Let B be the weighted path matrix of G. We define σ_B to be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-linear map

$$
\begin{aligned}
(3.6) \qquad \sigma_B : \mathcal{A}_{[n],[n]} &\to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] \\
x_{1,v_1} \cdots x_{n,v_n} &\mapsto [z_G]b_{1,v_1} \cdots b_{n,v_n},
\end{aligned}
$$

where $[z_G]b_{1,v_1} \cdots b_{n,v_n}$ denotes the coefficient of z_G in $b_{1,v_1} \cdots b_{n,v_n}$, taken after $b_{1,v_1} \cdots b_{n,v_n}$ is expanded in the lexicographic basis of Z_G. Note that the "substitution" $x_{i,j} \mapsto b_{i,j}$ is performed only for monomials of the form $x^{e,v}$ in $\mathcal{A}_{[n],[n]}$: we define $\sigma_B(x^{u,w})$ by first expanding $x^{u,w}$ in the basis $\{x^{e,v} \mid v \in \mathfrak{S}_n\}$, and *then* performing the substitution.

To illustrate the utility of the function σ_B, we consider both sides of the identity (1.7) when the product of Kazhdan-Lusztig basis elements is

$$(3.7) \qquad \widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,3]}}(q) = (1 + q) \sum_{w \leq 3421} T_w \in H_4(q),$$

and when the linear function $\theta_q : H_4(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ is defined by

$$
\theta_q(T_w) = \begin{cases} 1 & \text{if } w = 3412, \\ -1 & \text{if } w = 4312, \\ 0 & \text{otherwise.} \end{cases}
$$

To compute the left-hand side of (1.7), we evaluate

$$(3.8) \qquad \theta_q(\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[2,4]}}(q)) = (1 + q)(1) + (0)(-1) = 1 + q,$$

since in (3.7) T_{3412} has coefficient $1+q$ and T_{4312} has coefficient 0. To compute the right-hand side of (1.7), we first write

$$(3.9) \qquad \mathrm{Imm}_{\theta_q}(x) = q^{-2}x_{1,3}x_{2,4}x_{3,1}x_{4,2} - q^{\frac{-5}{2}}x_{1,4}x_{2,3}x_{3,1}x_{4,2},$$

since $\ell(3412) = 4$ and $\ell(4312) = 5$. Now observe that the star network and weighted path matrix corresponding to $\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[2,4]}}(q)$ are $G = G_{[2,4]} \circ G_{[1,3]}$ in (3.4) and B in (3.5). Thus we compute $\sigma_B(\mathrm{Imm}_{\theta_q}(x))$ by substituting $x_{i,j} = b_{i,j}$ in (3.9) to obtain

$$
\begin{aligned}
(3.10) \qquad & q^{-2}b_{1,3}b_{2,4}b_{3,1}b_{4,2} = q^{-2}z_{1,2,1}z_{3,2,2}z_{2,1,1}z_{4,1,2}z_{3,1,1}(z_D + z_U)z_{1,2,2}z_{4,1,1}(z_D + z_U)z_{2,2,2}, \\
& q^{\frac{-5}{2}}b_{1,4}b_{2,3}b_{3,1}b_{4,2} = 0,
\end{aligned}
$$

and extract the coefficient of $z_G = z_{1,2,1} z_{1,2,2} z_{2,1,1} \cdots z_{4,1,2}$ from these. Since z_D^2 and z_U^2 are not square-free, we ignore two monomials in the expansion of (3.10). Now, using the relations (3.3) to express the remaining two monomials in lexicographic order, we introduce multiples of $q^{\frac{1}{2}}$ for pairs $(z_{h_1,p,k}, z_{h_2,p,k})$ of variables sharing second and third indices,

$$z_{3,2,2} z_{4,1,2} z_D = q^{\frac{1}{2}} z_D z_{3,2,2} z_{4,1,2}, \qquad z_{3,2,2} z_{1,2,2} = q^{\frac{1}{2}} z_{1,2,2} z_{3,2,2},$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z_U z_D = q z_D z_U.$$
$$z_{3,2,2} z_{4,1,2} z_U = q^{\frac{1}{2}} z_U z_{3,2,2} z_{4,1,2}, \qquad z_{3,2,2} z_{2,2,2} = q^{\frac{1}{2}} z_{2,2,2} z_{3,2,2},$$

Specifically, we introduce a multiple of q^2 for the monomial in which z_D precedes z_U, and a multiple of q^3 in the other monomial. Thus we have

$$[z_G] q^{-2} (q^2 + q^3) z_{1,2,1} z_{1,2,2} z_{2,1,1} z_D z_{2,2,2} z_{3,1,1} z_U z_{3,2,2} z_{4,1,1} z_{4,1,2} = 1 + q,$$

which matches (3.8), as desired.

The maps σ behave well with respect to concatenation and matrix products. In particular, the result [11, Prop. 3.4] applies verbatim to the networks, matrices, and maps generalized as above from their earlier definitions in [11].

Proposition 3.1. *Let star networks G, H have weighted path matrices B, C, respectively. Then for all $u, w \in \mathfrak{S}_n$ we have*

$$(3.11) \qquad\qquad \sigma_{BC}(x^{u,w}) = \sum_{v \in \mathfrak{S}_n} \sigma_B(x^{u,v}) \sigma_C(x^{v,w}).$$

Proof. Extend the proof of [11, Prop. 3.4] to include elements of the ring Z_G, as defined in Subsection 3.2. $\qquad\square$

4. G-TABLEAUX AND THE COMBINATORICS OF THE EVALUATION MAP σ_B

Justification of the identity (1.7), which allows us to prove Theorem 6.5, depends upon a combinatorial interpretation of evaluations $\sigma_B(x^{u,v})$, which is stated in Proposition 4.5. This result and the preparatory Lemmas 4.1 – 4.4 extend earlier work in [5, §5–6] and [11, §3–4] on special cases of the evaluations.

First we observe two cases in which such an evaluation is quite simple.

Lemma 4.1. *Let star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$ have weighted path matrix B, and fix $u, v \in \mathfrak{S}_n$.*

    (1) If no path family of type $u^{-1}v$ covers G, then $\sigma_B(x^{u,v}) = 0$.
    (2) If exactly one path family of type v covers G, then $\sigma_B(x^{e,v}) = q^{\frac{\ell(v)}{2}}$.

Proof. (1) If no path family of type $u^{-1}v$ covers G, then we have $b_{u_1,v_1} \cdots b_{u_n,v_n} = 0$.
(2) Suppose that there is exactly one path family π of type v which covers G. Then for $i < j$, the pair (v_i, v_j) is an inversion in v if and only if the paths π_i and π_j cross. Since π is the unique path family of type v, the paths cannot cross more than once. If π_i, π_j do not intersect, or intersect without crossing, then each variable in $\text{wgt}(\pi_i)$ is lexicographically less than each variable in $\text{wgt}(\pi_j)$. Relative to one another, all of these variables appear in lexicographic order in the expression

$$(4.1) \qquad\qquad \text{wgt}(\pi) = \text{wgt}(\pi_1) \cdots \text{wgt}(\pi_n)$$

and contribute q^0 to the coefficient of z_G. On the other hand, if π_i, π_j do intersect at the central vertex of G_{J_p}, then for some indices $a < b$, the variable $z_{b,p,2}$ appears in $\text{wgt}(\pi_i)$,

earlier in (4.1) than the lexicographically lesser variable $z_{b,p,2}$ which appears in $\mathrm{wgt}(\pi_j)$. By (3.3) this pair of variables contributes $q^{\frac{1}{2}}$ to the coefficient of z_G.   □

In other cases, the evaluation $\sigma_B(x^{u,v})$ is less simple, and we will interpret such an evaluation by covering a star network

$$(4.2) \qquad\qquad G = G_{J_1} \circ \cdots \circ G_{J_m}$$

with a path family $\pi = (\pi_1, \ldots, \pi_n)$ and arranging the component paths into a (French) Young diagram whose shape is some partition λ of n. Following [5], we call the resulting structure a *G-tableau*, or more specifically a *π-tableau* of *shape λ*. If $\mathrm{type}(\pi) = w$, we say that the tableau has type w. Since a π tableau contains each path of π exactly once, and since there are no constraints on where paths can be placed, there are always $n!$ π-tableaux of a fixed shape λ.

Given a π-tableau U of shape λ, we define two Young tableaux $L(U)$ and $R(U)$ of shape λ by replacing each path in π by its source and sink index, respectively. A fixed path family π of type v and a permutation $u \in \mathfrak{S}_n$ determine a path tableau $U(\pi, u, uv) = \pi_{u_1} \cdots \pi_{u_n}$ which satisfies $L(U(\pi, u, uv)) = u_1 \cdots u_n$ and $R(U(\pi, u, uv)) = (uv)_1 \cdots (uv)_n$. The inclusion of uv in our notation $U(\pi, u, uv)$ is superfluous but makes clear the ordering of sinks as they appear in the tableau.

We define statistics on path families and on the tableaux containing them. Suppose that two paths π_i, π_j pass through the central vertex of some star network G_{J_p} in the factorization (4.2). We call the triple $(\pi_i, \pi_j, p) = (\pi_j, \pi_i, p)$ a *crossing* of π if the two paths cross there, and a *noncrossing* otherwise. Define

$$(4.3) \qquad\qquad \mathrm{CR}(\pi) = \#\{(\pi_i, \pi_j, p) \mid (\pi_i, \pi_j, p) \text{ is a crossing}\}.$$

Suppose that the triple (π_i, π_j, p) is a noncrossing, with π_i entering and leaving the central vertex of G_{J_p} below π_j. If U is any π-tableau, then we call the triple an *inverted noncrossing of U* if π_j appears in an earlier column of U than does π_i. Define

$$(4.4) \qquad \mathrm{INVNC}(U) = \#\{(\pi_i, \pi_j, p) \mid (\pi_i, \pi_j, p) \text{ is an inverted noncrossing in } U\}.$$

Now using the statistics CR and INVNC, we can combinatorially evaluate $\sigma_C(x^{u,v})$ in the case that G is a simple star network with weighted path matrix C.

Lemma 4.2. *Let simple star network G_J have weighted path matrix C. For $u, w \in \mathfrak{S}_n$ we have*

$$(4.5) \qquad\qquad \sigma_C(x^{u,w}) = \begin{cases} q^{\frac{\mathrm{CR}(\pi)}{2}} q^{\mathrm{INVNC}(U(\pi,u,w))} & \text{if } u^{-1}w \in \mathfrak{S}_J, \\ 0 & \text{otherwise}, \end{cases}$$

where π is the unique path family of type $u^{-1}w$ covering G_J.

Proof. If $u^{-1}w \notin \mathfrak{S}_J$, then there is no path family of type $u^{-1}w$ which covers G_J and the right-hand side of (4.5) is 0. By Lemma 4.1, the left-hand side is 0 as well. Assume therefore that we have $u^{-1}w \in \mathfrak{S}_J$.

Suppose $u = e$. Then the left-hand side of (4.5) is $q^{\frac{\ell(w)}{2}}$ by Lemma 4.1. The right-hand side is also $q^{\frac{\mathrm{CR}(\pi)}{2}} q^{\mathrm{INVNC}(U(\pi,u,w))} = q^{\frac{\ell(w)}{2}}$, since the left tableau $u_1 \cdots u_n = 1 \cdots n$ guarantees that $U(\pi, u, w)$ has no inverted noncrossings.

Now suppose the claim to be true when $\ell(u) = 0, \ldots, k$ and consider u of length $k + 1$. Factor u as $u = s_i u'$ with $\ell(u') = k$. By [11, Eq. (2.5)], we have

$$\sigma_C(x^{u,w}) = \sigma_C(x^{s_i u', w}) = \begin{cases} \sigma_C(x^{u', s_i w}) & \text{if } s_i w > w, \\ \sigma_C(x^{u', s_i w}) + (q^{\frac{1}{2}} - \bar{q}^{\frac{1}{2}})\sigma_C(x^{u', w}) & \text{if } s_i w < w. \end{cases}$$

By induction, and since $(u')^{-1} s_i w = u^{-1} w$, this is

$$(4.6) \qquad \begin{cases} q^{\frac{\text{CR}(\pi)}{2}} q^{\text{INVNC}(U(\pi, u', s_i w))} & \text{if } s_i w > w, \\ q^{\frac{\text{CR}(\pi)}{2}} q^{\text{INVNC}(U(\pi, u', s_i w))} + (q^{\frac{1}{2}} - \bar{q}^{\frac{1}{2}})(q^{\frac{\text{CR}(\pi')}{2}} q^{\text{INVNC}(U(\pi', u', w))}) & \text{if } s_i w < w, \end{cases}$$

where π, as in the statement of the lemma, is the unique path family of type $(u')^{-1} s_i w = u^{-1} w$ covering G_J, and where π' is the unique path family of type $(u')^{-1} w = u^{-1} s_i w$ covering G_J.

Let us consider the two G_J-tableaux appearing in (4.6), and relate their crossings and inverted noncrossings to those of $U(\pi, u, w)$.

The first tableau,

$$U(\pi, u', s_i w) = \pi_{u_1} \cdots \pi_{u_{i-1}} \pi_{u_{i+1}} \pi_{u_i} \pi_{u_{i+2}} \cdots \pi_{u_n},$$

differs from $U(\pi, u, w)$ only by the transposition of paths π_{u_i} and $\pi_{u_{i+1}}$. Thus if $s_i w > w$, then we have $u_i > u_{i+1}$ and $w_i < w_{i+1}$, i.e., the paths π_{u_i} and $\pi_{u_{i+1}}$ cross. It follows that we have $\text{INVNC}(U(\pi, u', s_i w)) = \text{INVNC}(U(\pi, u, w))$, and the claim is true. On the other hand, if $s_i w < w$, then we have $u_i > u_{i+1}$ and $w_i > w_{i+1}$, i.e., the paths π_{u_i} and $\pi_{u_{i+1}}$ form a (noninverted) noncrossing in $U(\pi, u', s_i w)$ and an inverted noncrossing in $U(\pi, u, w)$. It follows that we have

$$(4.7) \qquad \text{INVNC}(U(\pi, u', s_i w)) = \text{INVNC}(U(\pi, u, w)) - 1.$$

The second tableau,

$$\begin{aligned} U(\pi', u', w) &= \pi'_{u_1} \cdots \pi'_{u_{i-1}} \pi'_{u_{i+1}} \pi'_{u_i} \pi'_{u_{i+2}} \cdots \pi'_{u_n} \\ &= \pi_{u_i} \cdots \pi_{u_{i-1}} \pi'_{u_{i+1}} \pi'_{u_i} \pi_{u_{i+2}} \cdots \pi_{u_n}, \end{aligned}$$

differs from the tableau $U(\pi, u, w)$ only in the pair $(\pi'_{u_{i+1}}, \pi'_{u_i})$ from sources (u_{i+1}, u_i) to sinks (w_i, w_{i+1}), respectively. This pair forms a crossing in $U(\pi', u', w)$ in place of the inverted noncrossing of $(\pi_{u_i}, \pi_{u_{i+1}})$ in $U(\pi, u, w)$. It is straightforward but tedious to check that all other contributions to the differences $\text{CR}(\pi) - \text{CR}(\pi')$ and $\text{INVNC}(U(\pi, u, w)) - \text{INVNC}(U(\pi', u', w))$ are due to paths π_{u_j} from sources u_j to sinks w_j with indices j belonging to

$$Q_i := \{j \mid u_{i+1} < u_j < u_i; \ w_{i+1} < w_j < w_i\}.$$

Each such path π_{u_j} crosses neither π_{u_i} nor $\pi_{u_{i+1}}$ in π, but crosses both π'_{u_i} and $\pi'_{u_{i+1}}$ in π'. Similarly, for each such path π_{u_j}, exactly one of the pairs (π_{u_j}, π_{u_i}), $(\pi_{u_j}, \pi_{u_{i+1}})$ forms an inverted noncrossing in π while neither of the pairs (π_{u_j}, π'_{u_i}), $(\pi_{u_j}, \pi'_{u_{i+1}})$ forms an inverted noncrossing in π'. Thus we have

$$(4.8) \qquad \begin{aligned} \text{CR}(\pi') &= \text{CR}(\pi) + 1 + 2|Q_i|, \\ \text{INVNC}(U(\pi', u', w)) &= \text{INVNC}(U(\pi, u, w)) - 1 - |Q_i|. \end{aligned}$$

In summary, when $s_i w < w$, we combine (4.7) and (4.8) to see that the expression (4.6) is equal to

$$q^{\frac{\text{CR}(\pi)}{2}} q^{\text{INVNC}(U(\pi,u,w))-1} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) q^{\frac{\text{CR}(\pi')+1+2|Q_i|}{2}} q^{\text{INVNC}(U(\pi,u,w))-1-|Q_i|}$$

$$= q^{\frac{\text{CR}(\pi)}{2}} q^{\text{INVNC}(U(\pi,u,w))} (q^{-1} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) q^{\frac{1}{2}} q^{\frac{2|Q_i|}{2}} q^{-1} q^{-|Q_i|})$$

$$= q^{\frac{\text{CR}(\pi)}{2}} q^{\text{INVNC}(U(\pi,u,w))} (q^{-1} + 1 - q^{-1}),$$

and again the claim is true.   □

Alternatively, when $G = G_J$ is a simple star network with weighted path matrix C, we can algebraically evaluate $\sigma_C(x^{u,w})$ in terms of the factorizations in (2.2).

Lemma 4.3. *Let simple star network G_J have weighted path matrix C, and let $u, w \in \mathfrak{S}_n$ have the unique $\mathfrak{S}_-^J \times \mathfrak{S}_J$ factorizations $u = u_- u^*$, $w = w_- w^*$. Then we have*

$$\sigma_C(x^{u,w}) = \begin{cases} q^{\frac{\ell(u^*)}{2}} q^{\frac{\ell(w^*)}{2}} & \text{if } u^{-1}w \in \mathfrak{S}_J, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 4.2 it is sufficient to show that $q^{\frac{\ell(u^*)}{2}} q^{\frac{\ell(w^*)}{2}} = q^{\frac{\text{CR}(\pi)}{2}} q^{\text{INVNC}(U(\pi,u,w))}$, equivalently

(4.9) $\text{INV}(u^*) + \text{INV}(w^*) = \text{CR}(\pi) + 2\,\text{INVNC}(U(\pi, u, w)),$

whenever $u^{-1}w \in \mathfrak{S}_J$.

Consider a pair (i,j) with $i < j$ and the contributions of this pair and the triple $(\pi_i, \pi_j, 1)$ to both sides of (4.9). If the paths do not intersect, then i or j does not belong to J and we have a contribution of 0 to both sides of (4.9). Assume therefore that the paths intersect.

If the triple $(\pi_i, \pi_j, 1)$ is a crossing, then (i,j) is an inversion in exactly one of u^* and w^*. Thus we have a contribution of 1 to both sides of (4.9). On the other hand, if $(\pi_i, \pi_j, 1)$ is a noncrossing, then π_j intersects π_i from above. Thus $(\pi_i, \pi_j, 1)$ is an inverted noncrossing if and only if π_j precedes π_i in $U(\pi, u, w)$, i.e., if and only if j precedes i in u. But this condition is equivalent to j preceding i in w since π_i and π_j do not cross. Hence we have a contribution of 2 to both sides of (4.9) or a contribution of 0 to both sides.   □

To combinatorially evaluate expressions of the form $\sigma_B(x^{u,w})$ where B is the weighted path matrix of an arbitrary star network G, we will use the fact that G is a concatenation of smaller star networks. When a star network G can be decomposed as a concatenation $G' \circ H$, it is natural to similarly decompose any path family $\pi = (\pi_1, \ldots, \pi_n)$ that covers G as $\pi = \pi^{G'} \circ \pi^H$, where $\pi^{G'} = (\pi_1^{G'}, \ldots, \pi_n^{G'})$, $\pi^H = (\pi_1^H, \ldots, \pi_n^H)$, and $\text{type}(\pi) = \text{type}(\pi^{G'})\text{type}(\pi^H)$. One can also decompose a single path π_i as $(\pi_i)^{G'} \circ (\pi_i)^H$ but one must be careful however: if $\text{type}(\pi^{G'}) = y$ then we have $(\pi_i)^{G'} = (\pi^{G'})_i = \pi_i^{G'}$, but $(\pi_i)^H = (\pi^H)_{y_i}$, which is not in general equal to $(\pi^H)_i = \pi_i^H$, the component path of π^H which starts at source i of H.

The above decompositions of path families and path tableaux satisfy some simple identities.

Lemma 4.4. *Let path family π cover star network $G = G' \circ H$, and let $\pi^{G'}, \pi^H$ be its restrictions to G', H. Then we have*

$$\text{CR}(\pi) = \text{CR}(\pi^{G'}) + \text{CR}(\pi^H).$$

Now fix $u \in \mathfrak{S}_n$ and define $w = u \cdot \text{type}(\pi)$, $v = u \cdot \text{type}(\pi^{G'})$. Then we have

$$\text{INVNC}(U(\pi, u, w)) = \text{INVNC}(U(\pi^{G'}, u, v)) + \text{INVNC}(U(\pi^H, v, w)).$$

Proof. Write $G' = G_{J_1} \circ \cdots \circ G_{J_k}$, $H = G_{J_{k+1}} \circ \cdots \circ G_{J_m}$ and let paths π_i, π_j intersect at the central vertex of some factor G_{J_p}. Suppose that this point of intersection is a crossing. Then it is a crossing of $\pi^{G'}$ if $p \leq k$ and a crossing of π^H if $p > k$. Thus it contributes 1 to $\text{CR}(\pi)$ and 1 to either $\text{CR}(\pi^{G'})$ (if $p \leq k$) or $\text{CR}(\pi^H)$ (if $p > k$).

Suppose on the other hand that the point of intersection is a noncrossing. Then it is a noncrossing of $\pi^{G'}$ if $p \leq k$ and a noncrossing of π^H if $p > k$. Moreover, π_i enters and departs from the central vertex of G_{J_p} above π_j if and only if the restrictions of these paths to G' or H do the same. Now consider the positions of these paths in their respective tableaux. Let a, b, be the positions of π_i and π_j in $U(\pi, u, w)$: $\pi_i = \pi_{u_a}$, $\pi_j = \pi_{u_b}$, and assume without loss of generality that $a < b$. Then the restrictions of the two paths to G' and H also appear in positions a and b, for we have

tableau	path in position a	path in position b
$U(\pi, u, w)$	π_{u_a} from source u_a of G to sink w_a	π_{u_b} from source u_b of G to sink w_b
$U(\pi^{G'}, u, v)$	$\pi_{u_a}^{G'}$ from source u_a of G' to sink v_a	$\pi_{u_b}^{G'}$ from source u_b of G' to sink v_b
$U(\pi^H, v, w)$	$\pi_{v_a}^H$ from source v_a of H to sink w_a	$\pi_{u_b}^H$ from source v_b of H to sink w_b

with $\pi_{u_a} = \pi_{u_a}^{G'} \circ \pi_{v_a}^H$, $\pi_{u_b} = \pi_{u_b}^{G'} \circ \pi_{v_b}^H$. Thus the triple $(\pi_{u_a}, \pi_{u_b}, p)$ contributes 1 to $\text{INVNC}(U(\pi, u, w))$ if and only if $(\pi_{u_a}^{G'}, \pi_{u_b}^{G'}, p)$ contributes 1 to $\text{INVNC}(U(\pi^{G'}, u, v))$ $(p \leq k)$, or $(\pi_{u_a}^H, \pi_{u_b}^H, p)$ contributes 1 to $\text{INVNC}(U(\pi^H, v, w))$ $(p > k)$. $\qquad\square$

Combining the simple star network evaluation $\sigma_C(x^{u,w})$ from Lemma 4.2 with the identities in Lemma 4.4, we can now inductively prove a combinatorial evaluation of $\sigma_B(x^{u,w})$ where B is the weighted path matrix of an arbitrary star network.

Proposition 4.5. *Let star network G have weighted path matrix B. For $u, w \in \mathfrak{S}_n$ we have*

$$(4.10) \qquad \sigma_B(x^{u,w}) = \sum_\pi q^{\frac{\text{CR}(\pi)}{2}} q^{\text{INVNC}(U)},$$

where the sum is over path families π of type $u^{-1}w$ covering G, and $U = U(\pi, u, w)$ is the unique π-tableaux of shape (n) satisfying $L(U) = u_1 \cdots u_n$, $R(U) = w_1 \cdots w_n$.

Proof. By Lemma 4.2, the claim is true when G is the simple star network $G_{[i_1, j_1]}$.

Now suppose the claim is true for G a concatenation of $1, \ldots, m$ simple star networks and consider $G = G' \circ H$ where $G' = G_{J_1} \circ \cdots \circ G_{J_m}$ and $H = G_{J_{m+1}}$. Let B', C be the path matrices of G', H, respectively so that $B = B'C$ is the path matrix of G.

By Proposition 3.1 and induction we have

$$(4.11) \qquad \sigma_{B'C}(x^{u,w}) = \sum_{t \in \mathfrak{S}_{J_{m+1}}} \sum_{\pi^{G'}} q^{\frac{\text{CR}(\pi^{G'})}{2}} q^{\text{INVNC}(U(\pi^{G'}, u, wt^{-1}))} q^{\frac{\text{CR}(\pi^H)}{2}} q^{\text{INVNC}(U(\pi^H, wt^{-1}, w))},$$

where $\pi^{G'}$ varies over all path families of type $u^{-1}wt^{-1}$ which cover G'. Since pairs of path families of types $u^{-1}wt^{-1}$ and t covering G' and H, respectively, correspond to path families of type $u^{-1}w$ covering G, we may apply Lemma 4.4 to the right-hand side above to obtain

$$\sum_{\pi} q^{\frac{\text{CR}(\pi)}{2}} q^{\text{INVNC}(U(\pi,u,w))},$$

where the sum is over all path families π of type $u^{-1}w$ which cover G. $\qquad\qquad\square$

5. Products $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$ and the q-immanant evaluation theorem

We complete our justification of the crucial identity (1.7) in Theorem 5.2, first using Proposition 5.1 to relate evaluations of the form $\sigma_B(x^{e,w})$ to coefficients in the natural expansion of $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$. As a corollary, we obtain a second interpretation of the above coefficients which generalizes a result of Deodhar. The first two results generalize [11, Prop. 3.6 – Thm. 3.7].

Proposition 5.1. *Let star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$ have weighted path matrix B, and fix $w \in \mathfrak{S}_n$. Then the coefficient of T_w in $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$ is equal to $q^{\frac{-\ell(w)}{2}} \sigma_B(x^{e,w})$.*

Proof. Consider the simple star network $G = G_{J_1}$ and its weighted path matrix B. By Lemma 4.3 we have

$$\sigma_B(x^{e,w}) = \begin{cases} q^{\frac{\ell(w)}{2}} & \text{if } w \in \mathfrak{S}_{J_1}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\widetilde{C}_{s_{J_1}}(q) = \sum_{w \in \mathfrak{S}_{J_1}} T_w$, the result is true for any simple star network.

Now assume that the result holds for concatenations of $1, \ldots, m-1$ simple star networks, and define $\{a_w \mid w \in \mathfrak{S}_n\} \subseteq \mathbb{Z}[q]$ by

$$\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_{m-1}}}(q) = \sum_{w \in \mathfrak{S}_n} a_w T_w.$$

Consider the star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$ with weighted path matrix B and decompose G as $G' \circ H$, where $G' = G_{J_1} \circ \cdots \circ G_{J_{m-1}}$ has weighted path matrix B' and $H = G_{J_m}$ has weighted path matrix C. By Proposition 3.1 we have

$$\sigma_B(x^{e,w}) = \sum_{u \in \mathfrak{S}_{J_m}} \sigma_{B'}(x^{e,wu^{-1}}) \sigma_C(x^{wu^{-1},w}),$$

where the sum is taken over $u \in \mathfrak{S}_{J_m}$ by Lemma 4.3. Now factor w as $w_- w^*$ with $w_- \in \mathfrak{S}_-^{J_m}$, $w^* \in \mathfrak{S}_{J_m}$, and define $t = w^* u^{-1}$ so that we have $wu^{-1} = w_- w^*(w^*)^{-1} t = w_- t$. Making this substitution in the sum and observing that as u varies over $\mathfrak{S}_{J_m}$, so does t, we apply Lemma 4.3 to the second factor in each summand to obtain

$$\sum_{t \in \mathfrak{S}_{J_m}} \sigma_{B'}(x^{e,w_-t}) \sigma_C(x^{w_-t,w}) = \sum_{t \in \mathfrak{S}_{J_m}} \sigma_{B'}(x^{e,w_-t}) q^{\frac{\ell((w_-t)^*)}{2}} q^{\frac{\ell(w^*)}{2}} = q^{\frac{\ell(w^*)}{2}} \sum_{t \in \mathfrak{S}_{J_m}} \sigma_{B'}(x^{e,w_-t}) q^{\frac{\ell(t)}{2}}.$$

By induction, this is

$$q^{\frac{\ell(w^*)}{2}} \sum_{t \in \mathfrak{S}_{J_m}} q^{\frac{\ell(w_-t)}{2}} a_{w_-t} q^{\frac{\ell(t)}{2}} = q^{\frac{\ell(w^*)}{2}} \sum_{t \in \mathfrak{S}_{J_m}} q^{\frac{\ell(w_-)}{2}} q^{\frac{\ell(t)}{2}} a_{w_-t} q^{\frac{\ell(t)}{2}} = q^{\frac{\ell(w)}{2}} \sum_{t \in \mathfrak{S}_{J_m}} a_{w_-t} q^{\ell(t)}.$$

On the other hand, consider the element

$$(5.1) \qquad \widetilde{C}_{s_{J_m}}(q) \cdots \widetilde{C}_{s_{J_m}}(q) = \Big(\sum_{w \in \mathfrak{S}_n} a_w T_w \Big) \widetilde{C}_{s_{J_m}}(q) = \sum_{w \in \mathfrak{S}_n} a_w T_{w_-} T_{w^*} \widetilde{C}_{s_{J_m}}(q).$$

By (1.2), (2.3) and the fact that reversals avoid the patterns 3412 and 4231, this is

$$\sum_{y \in \mathfrak{S}_-^{J_m}} T_y \sum_{t \in \mathfrak{S}_{J_m}} a_{yt} T_t \widetilde{C}_{s_{J_m}}(q) = \sum_{y \in \mathfrak{S}_-^{J_m}} T_y \widetilde{C}_{s_{J_m}}(q) \sum_{t \in \mathfrak{S}_{J_m}} a_{yt} q^{\ell(t)} = \sum_{y \in \mathfrak{S}_-^{J_m}} T_y \sum_{v \in \mathfrak{S}_{J_m}} T_v \sum_{t \in \mathfrak{S}_{J_m}} a_{yt} q^{\ell(t)}.$$

Now $q^{\frac{\ell(w)}{2}}$ times the coefficient of T_w in this expression is

$$q^{\frac{\ell(w)}{2}} \sum_{t \in \mathfrak{S}_{J_m}} a_{w_- t} q^{\ell(t)}. \qquad \square$$

Now we have a quick proof of the q-immanant evaluation theorem for star networks.

Theorem 5.2. *Let $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, \bar{q}^{\frac{1}{2}}]$ be linear, and let star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$ have weighted path matrix B. Then we have*

$$(5.2) \qquad \theta_q(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)) = \sigma_B(\mathrm{Imm}_{\theta_q}(x)).$$

Proof. Write $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q) = \sum_{w \in \mathfrak{S}_n} a_w T_w$. Then the right-hand side of (5.2) is

$$\sigma_B \Big(\sum_{w \in \mathfrak{S}_n} \theta_q(T_w) q^{\frac{-\ell(w)}{2}} x^{e,w} \Big) = \sum_{w \in \mathfrak{S}_n} \theta_q(T_w) q^{\frac{-\ell(w)}{2}} \sigma_B(x^{e,w}) = \sum_{w \in \mathfrak{S}_n} \theta_q(T_w) a_w = \theta_q \Big(\sum_{w \in \mathfrak{S}_n} a_w T_w \Big),$$

where the second equality follows from Proposition 5.1. But this is the left-hand side of (5.2). $\square$

Proposition 5.1 and the special case $u = e$ of Proposition 4.5 yield a proof of a generalization of Deodhar's defect formula [7, Prop. 3.5] for coefficients in the expansion of $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$, which we will use in Section 6. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a path family covering a star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$. If two paths π_i, π_j intersect at the central vertex of G_{J_p}, call the triple (π_i, π_j, p) *defective* or *a defect* if the paths have previously crossed an odd number of times (i.e., in $G_{J_1}, \ldots, G_{J_{i_{p-1}}}$). Let $\mathrm{D}(\pi)$ denote the number of defects of π,

$$(5.3) \qquad \mathrm{D}(\pi) = \#\{(\pi_i, \pi_j, p) \mid (\pi_i, \pi_j, p) \text{ defective }\}.$$

The original definition of $\mathrm{D}(\pi)$ given in [7, Defn. 2.2] (as interpreted in [2, Rmk. 6]) and used throughout [11] is simply the number of *indices p* appearing in defective triples (p, π_i, π_j), since each simple star $G_{J_p} = G_{[j_p, j_p+1]}$ considered in those papers completely determines the two paths to intersect at its central vertex. The proof of the following result is nearly identical to that of [11, Cor. 4.2]. We include it here, both for the convenience of the reader, and because it relies upon Propositions 4.5 and 5.1 in this paper, which differ from the corresponding (weaker) results [11, Prop. 4.1], [11, Prop. 3.6] in that paper.

Corollary 5.3. *The coefficients in the expansion $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q) = \sum_w a_w T_w$ are given by*

$$a_w = \sum_{\pi} q^{\mathrm{D}(\pi)},$$

where the sum is over all path families of type w which cover the star network $G_{J_1} \circ \cdots \circ G_{J_m}$.

Proof. Substitute $u = e$ in Proposition 4.5. The right-hand side of (4.10) is a sum over path families π of type w, and each tableau $U = U(\pi, e, w)$ is simply the sequence $\pi = (\pi_1, \ldots, \pi_n)$. Thus an inverted noncrossing of U is simply a noncrossing in which the upper path has index less than that of the lower path, i.e., a defective noncrossing. To relate these to all defects, let us temporarily define

$$\text{DNC}(\pi) = \text{ number of defective noncrossings of } \pi,$$

$$\text{DC}(\pi) = \text{ number of defective crossings of } \pi,$$

so that $\text{D}(\pi) = \text{DNC}(\pi) + \text{DC}(\pi)$. Now observe that for any path familiy of type w we have

$$\text{CR}(\pi) = \text{INV}(w) + 2\text{DC}(\pi)$$

because if paths π_a, π_b cross k times, then at most one of those crossings contributes to $\text{INV}(w)$, while exactly half of the remaining crossings are defective. Thus the right-hand side of (4.10) becomes

$$(5.4) \qquad \sum_\pi q^{\frac{\text{INV}(w)+2\text{DC}(\pi)}{2}} q^{\text{DNC}(\pi)} = \sum_\pi q^{\frac{\ell(w)}{2}} q^{\text{DC}(\pi)+\text{DNC}(\pi)} = q^{\frac{\ell(w)}{2}} \sum_\pi q^{\text{D}(\pi)},$$

where the sum is over all path families of type w which cover the star network $G_{J_1} \circ \cdots \circ G_{J_m}$.

By Proposition 5.1, the left-hand side of (4.10) is $q^{\frac{\ell(w)}{2}} a_w$. Combining this with (5.4), we have the desired result. $\qquad \square$

6. EVALUATION OF INDUCED SIGN CHARACTERS

We now complete the solution of Problem 1.1 by providing a combinatorial interpretation in Theorem 6.5 for all expressions of the form

$$(6.1) \qquad \epsilon_q^\lambda(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)),$$

with $\lambda \vdash n$ and $(J_1, \ldots, J_m)$ a sequence of subintervals of $[n]$. Generalizing [11, Thm. 5.4], Theorem 6.5 relies upon several lemmas and definitions which are straightforward generalizations of those given in [11, §5]. On the other hand, the fact that not all reversals appearing as subscripts in (6.1) are adjacent transpositions complicates matters significantly, requiring a new involution (6.6) and four new identities collected in Lemma 6.3.

By Theorem 5.2 and Equation (2.12), each expression (6.1) is equal to

$$(6.2) \qquad \sigma_B(\text{Imm}_{\epsilon_q^\lambda}(x)) = \sum_{u \in \mathfrak{S}_\lambda^-} \sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} \sigma_B(x^{u,yu}),$$

where B is the weighted path matrix of the star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$. Thus it suffices to combinatorially interpret the right-hand side of (6.2). To do this, we will apply Proposition 4.5 to each expression $\sigma_B(x^{u,yu})$ and will compute statistics for tableaux belonging to the set

$$\mathcal{U}_I = \mathcal{U}_I(G) := \{U(\pi, u, yu) \,|\, \pi \text{ covers } G, u = u(I), y \in \mathfrak{S}_\lambda\},$$

where $I \mapsto u(I)$ is the bijection defined after (2.11). Note that our restriction on y forces the sink indices of paths in components

$$(6.3) \qquad (\lambda_1 + \cdots + \lambda_{k-1} + 1), \ldots, (\lambda_1 + \cdots + \lambda_k)$$

of $U(\pi, u, yu)$ to be a permutation of the source indices of the same paths.

Following the outline of [11, §5], we will apply a sign-reversing involution to the tableaux in each set $\mathcal{U}_I$ by defining an involution on a second set $\mathcal{T}_I$ of tableaux, in obvious bijection with $\mathcal{U}_I$. For a wiring diagram G let $\mathcal{T}_I = \mathcal{T}_I(G)$ be the set of all column-closed, left column-strict G-tableaux W of shape $\lambda^\top$ such that $L(W^\top)_k = I_k$ (as sets) for $k = 1, \ldots, r$. Define the bijection

(6.4)
$$\begin{aligned} \delta = \delta_I \colon \mathcal{U}_I &\to \mathcal{T}_I \\ U &\mapsto W, \end{aligned}$$

as in [11, §5], by letting W be the left column-strict G-tableau of shape $\lambda^\top$ whose kth column consists of entries (6.3) of U.

Since $U \in \mathcal{U}_I$ and $\delta(U) \in \mathcal{T}_I$ contain the same path family, it is easy to see that δ does not affect the statistic CR. On the other hand, it changes the statistic INVNC in a very simple way, as in [11, Lem. 5.1]. Define $\text{CDNC}(U)$ to be the number of defective noncrossings of pairs of paths appearing in the same column of U.

Lemma 6.1. *Let I be an ordered set partition. For $U \in \mathcal{U}_I$ we have*

(6.5)
$$\text{INVNC}(U) = \text{INVNC}(\delta(U)) + \text{CDNC}(\delta(U)).$$

Proof. Follow the proof of [11, Lem. 5.1], merely replacing each noncrossing *index* as defined in [11]) with a noncrossing *triple* as defined preceding (4.3). $\square$

Now we define the involution

(6.6)
$$\zeta = \zeta_I \colon \mathcal{T}_I \to \mathcal{T}_I$$

(different from ζ in [11]) as follows. Let $G = G_{J_1} \circ \cdots \circ G_{J_m}$, $u = u(I)$, and $\lambda = \text{type}(I)$. Fix $y \in \mathfrak{S}_\lambda$ and let $W = \delta(U(\pi, u, yu))$.

(1) If W is column-strict, then define $\zeta(W) = W$.
(2) Otherwise,
    (a) Let t be the greatest index such that column t of W is not column-strict.
    (b) Let p be the greatest index such that (at least) two paths in column t of W pass through the interior vertex of G_{J_p}.
    (c) Let j, j' be the two positions in the interval $[\lambda_1 + \cdots + \lambda_{t-1} + 1, \lambda_1 + \cdots \lambda_t]$ for which paths π_{u_j}, $\pi_{u_{j'}}$ have the right-to-left lexicographically greatest pair $((yu)_j, (yu)_{j'})$ of sink indices.
    (d) Let $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_n)$ be the path family obtained from π by swapping the terminal subpaths of π_{u_j} and $\pi_{u_{j'}}$, beginning at the interior vertex of G_{J_p}. ($\hat{\pi}_i = \pi_i$ for $i \notin \{u_j, u_{j'}\}$.)
    (e) Define $\zeta(W)$ to be the tableau obtained from W by replacing π_i by $\hat{\pi}_i$, for $i = 1, \ldots, n$.

If $W \in \mathcal{T}_I$ is not a fixed point of ζ, then the tableaux $U, \widehat{U} \in \mathcal{U}_I$ satisfying $\delta(U) = W$, $\delta(\widehat{U}) = \zeta(W)$ are closely related.

Lemma 6.2. *Let $I = (I_1, \ldots I_r)$ be an ordered set partition of type λ, and define $u = u(I)$. Let G-tableaux $W \in \mathcal{T}_I$ and $U, \widehat{U} \in \mathcal{U}_I$ satisfy $W = \delta(U) \neq \zeta(W) = \delta(\widehat{U})$, and define path families π, $\hat{\pi}$ as above. Then for some generator $s \in \mathfrak{S}_\lambda$ and some permutations $y, \hat{y} = ys \in \mathfrak{S}_\lambda$ we have $U = U(\pi, u, yu)$, $\widehat{U} = U(\hat{\pi}, u, \hat{y}u)$.*

Proof. The tableaux U, $\widehat{U}$ contain the same path families as W and $\zeta(W)$, respectively, and these path families are π, $\hat{\pi}$, as defined in the definition of ζ.

Listing the source indices of paths in column t of W, from bottom to top, we obtain the increasing sequence

$$(6.7) \qquad (u_{\lambda_1+\cdots+\lambda_{t-1}+1}, u_{\lambda_1+\cdots+\lambda_{t-1}+2}, \ldots, u_{\lambda_1+\cdots+\lambda_t}).$$

Since elements of $\mathcal{T}_I$ are column-closed, the sink indices of the paths form a permutation of this sequence,

$$((yu)_{\lambda_1+\cdots+\lambda_{t-1}+1}, (yu)_{\lambda_1+\cdots+\lambda_{t-1}+2}, \ldots, (yu)_{\lambda_1+\cdots+\lambda_t}),$$

for some $y \in \mathfrak{S}_\lambda$. In particular, the sink indices $(yu)_j$ and $(yu)_{j'}$ of π_{u_j}, $\pi_{u_{j'}}$ are simply two components u_i and $u_{i'}$ of (6.7).

By our choice of p, no two paths in column t of W intersect to the right of G_{J_p}. By our choice of j and j', no path in this column has a sink index strictly between u_i and $u_{i'}$. It follows that $|i - i'| = 1$. Thus the sink indices of the path family $\hat{\pi}$ are given by $\hat{y}u = ysu$, where s is the adjacent transposition $s_{\min\{i,i'\}} \in \mathfrak{S}_\lambda$. $\qquad\square$

Furthermore, when $W \in \mathcal{T}_I$ is not a fixed point of ζ, the values of the statistics CR, INVNC and CDNC on W and $\zeta(W)$ are closely related, as are $\ell(y)$ and $\ell(\hat{y})$, where y, $\hat{y}$ are defined as in Lemma 6.2.

Lemma 6.3. *Let $W = U(\pi, u, yu) \neq \zeta(W) = U(\hat{\pi}, u, \hat{y}u)$ be as above with $\hat{\pi}$ being defined by the intersection of π_{u_j} and $\pi_{u_{j'}}$ in G_{J_p}. Let b be the number of paths in π (equivalently, in $\hat{\pi}$) which enter G_{J_p} between the paths π_{u_j} and $\pi_{u_{j'}}$, and which leave G_{J_p} between the same two paths. Then we have*

$$(6.8) \qquad \ell(\hat{y}) = \begin{cases} \ell(y) - 1 & \text{if } (\pi_{u_j}, \pi_{u_{j'}}, p) \text{ is a proper crossing or defective noncrossing,} \\ \ell(y) + 1 & \text{if } (\pi_{u_j}, \pi_{u_{j'}}, p) \text{ is a proper noncrossing or defective crossing;} \end{cases}$$

$$(6.9) \qquad \mathrm{CR}(\hat{\pi}) = \begin{cases} \mathrm{CR}(\pi) - 1 - 2b & \text{if } (\pi_{u_j}, \pi_{u_{j'}}, p) \text{ is a crossing,} \\ \mathrm{CR}(\pi) + 1 + 2b & \text{otherwise;} \end{cases}$$

$$(6.10) \qquad \mathrm{INVNC}(\zeta(W)) = \begin{cases} \mathrm{INVNC}(W) + b & \text{if } (\pi_{u_j}, \pi_{u_{j'}}, p) \text{ is a crossing,} \\ \mathrm{INVNC}(W) - b & \text{otherwise;} \end{cases}$$

$$(6.11) \qquad \mathrm{CDNC}(\hat{\pi}) = \begin{cases} \mathrm{CDNC}(\pi) & \text{if } (\pi_{u_j}, \pi_{u_{j'}}, p) \text{ is proper,} \\ \mathrm{CDNC}(\pi) - 1 & \text{if } (\pi_{u_j}, \pi_{u_{j'}}, p) \text{ is a defective noncrossing,} \\ \mathrm{CDNC}(\pi) + 1 & \text{if } (\pi_{u_j}, \pi_{u_{j'}}, p) \text{ is a defective crossing.} \end{cases}$$

*Proof.* (Identity (6.8)) Recall that we have

$$\mathrm{source}(\pi_{u_j}) = \mathrm{source}(\hat{\pi}_{u_j}) = u_j, \qquad \mathrm{sink}(\pi_{u_j}) = \mathrm{sink}(\hat{\pi}_{u_{j'}}) = (yu)_j,$$
$$\mathrm{source}(\pi_{u_{j'}}) = \mathrm{source}(\hat{\pi}_{u_{j'}}) = u_{j'}, \qquad \mathrm{sink}(\pi_{u_{j'}}) = \mathrm{sink}(\hat{\pi}_{u_j}) = (yu)_{j'} > (yu)_j.$$

By Lemma 6.2, we have $y, \hat{y} = ys \in \mathfrak{S}_\lambda$ and therefore $\ell(\hat{y}) = \ell(y) \pm 1$. Since $u \in \mathfrak{S}_\lambda^-$, we also have

$$(6.12) \qquad \ell(yu) = \ell(y) + \ell(u), \qquad \ell(\hat{y}u) = \ell(\hat{y}) + \ell(u).$$

If $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is a proper crossing or a defective noncrossing, then the relative orders of $(u_j, u_{j'})$ and $((yu)_j, (yu)_{j'})$ are different, i.e., $u_{j'} < u_j$. By the definition of j, j' and (6.7) we also have $j' < j$. Thus the pair of letters $((yu)_j, (yu)_{j'})$ is inverted in the word yu and not in the word $\hat{y}u$. It follows that $\ell(yu) > \ell(\hat{y}u)$ and by (6.12) that $\ell(y) > \ell(\hat{y}) = \ell(y) - 1$. The other case of (6.8) is similar.

To justify the identities (6.9) – (6.11), we first look closely at paths $\pi_i = \hat{\pi}_i$, $i \notin \{j, j'\}$, which pass through the interior vertex of G_{J_p}. We partition such paths into nine equivalence classes according to whether they enter G_{J_p} below the both of the paths π_{u_j}, $\pi_{u_{j'}}$ (equivalently, below both $\hat{\pi}_{u_j}$, $\hat{\pi}_{u_{j'}}$) between the two paths, or above both paths, and whether they leave G_{J_p} below, between, or above the two paths. We name the nine classes $AD, AE, AF, BD, BE, BF, CD, CE, CF$, according to the (magnified) diagrams of paths intersecting in G_{J_p}

(6.13)
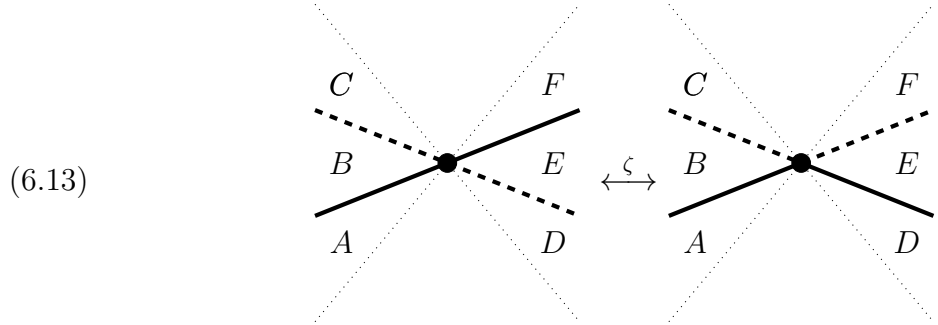
The solid and dashed lines represent edges from π_j, $\pi_{j'}$, $\hat{\pi}_j$ and $\hat{\pi}_{j'}$, and the dotted lines the edges with highest and lowest sink or source indices among edges incident with the central vertex of G_{J_p}. Every other edge meeting this vertex lies in one of the six labeled regions. We may have π_{u_j}, $\pi_{u_{j'}}$ on the left, with $\hat{\pi}_{u_j}$, $\hat{\pi}_{u_{j'}}$ on the right, or vice versa. We may have the paths indexed by u_j entering G_{J_p} above the others in both diagrams, or below the others in both diagrams. All four combinations are possible. Thus, paths in class AE enter G_{J_p} below the two bold paths and leave G_{J_p} between them, while the class BE has cardinality b as defined in the statement of the lemma.

(Identity (6.9)) Consider the contributions of all points of intersection $(\pi_i, \pi_{i'}, k)$ and $(\hat{\pi}_i, \hat{\pi}_{i'}, k)$ to $\mathrm{CR}(\pi)$ and $\mathrm{CR}(\hat{\pi})$, respectively. For $k < p$ and all i, i', we have that $(\pi_i, \pi_{i'}, k)$ is a crossing if and only if $(\hat{\pi}_i, \hat{\pi}_{i'}, k)$ is a crossing. The same is true for $k \geq p$, provided that $i, i' \notin \{u_j, u_{j'}\}$. For $k > p$ and $i \notin \{u_j, u_{j'}\}$, crossings (π_i, π_{u_j}, k) and $(\hat{\pi}_i, \hat{\pi}_{u_{j'}}, k)$ correspond bijectively, as do crossings $(\pi_i, \pi_{u_{j'}}, k)$ and $(\hat{\pi}_i, \hat{\pi}_{u_j}, k)$. By the definition of j, j', there are no crossings of the form $(\pi_{u_j}, \pi_{u_{j'}}, k)$ or $(\hat{\pi}_{u_j}, \hat{\pi}_{u_{j'}}, k)$ for $k > p$. So far, contributions to $\mathrm{CR}(\pi)$ and $\mathrm{CR}(\hat{\pi})$ are equal.

Now consider points of intersection of the form

(6.14) $(\pi_i, \pi_{u_j}, p), \qquad (\pi_i, \pi_{u_{j'}}, p),$

for $i \notin \{u_j, u_{j'}\}$, and their images under ζ,

(6.15) $(\hat{\pi}_i, \hat{\pi}_{u_j}, p), \qquad (\hat{\pi}_i, \hat{\pi}_{u_{j'}}, p).$

If $\pi_i \in AF \cup CD$, then all four triples are crossings; if $\pi_i \in AD \cup CF$, then none is. If $\pi_i \in AE \cup BD \cup BF \cup CE$, then exactly one of the triples (6.14) is a crossing, as is exactly one of the triples (6.15). On the other hand, if π_i is one of the b paths in BE, then either

both triples (6.14) are crossings while both triples (6.15) are noncrossings (if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is a crossing), or both triples (6.14) are noncrossings while both triples (6.15) are crossings (if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is a noncrossing). Finally by the definition of ζ, exactly one of the two triples

$$(6.16) \qquad\qquad (\pi_{u_j}, \pi_{u_{j'}}, p), \qquad (\hat\pi_{u_j}, \hat\pi_{u_{j'}}, p)$$

is a crossing and the other is a noncrossing. Thus the points of intersection of the forms (6.14), (6.15), (6.16) contribute a surplus of $2b+1$ to $\mathrm{CR}(\pi)$ if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is a crossing, and to $\mathrm{CR}(\hat\pi)$ otherwise.

(Identity (6.10)) For $k < p$ and all i, i', we have that $(\pi_i, \pi_{i'}, k)$ is an inverted noncrossing if and only if $(\hat\pi_i, \hat\pi_{i'}, k)$ is an inverted noncrossing. The same is true for $k \geq p$, provided that $i, i' \notin \{u_j, u_{j'}\}$. For $k > p$ and $i \notin \{u_j, u_{j'}\}$, noncrossings (π_i, π_{u_j}, k) and $(\hat\pi_i, \hat\pi_{u_{j'}}, k)$ correspond bijectively, as do noncrossings $(\pi_i, \pi_{u_{j'}}, k)$ and $(\hat\pi_i, \hat\pi_{u_j}, k)$. Moreover, since paths indexed u_j, $u_{j'}$ all appear in the same column t, this correspondence preserves inversion of noncrossings. By the definition of j, j', there are no noncrossings of the form $(\pi_{u_j}, \pi_{u_{j'}}, k)$ or $(\hat\pi_{u_j}, \hat\pi_{u_{j'}}, k)$ for $k > p$. So far, contributions to $\mathrm{INVNC}(\pi)$ and $\mathrm{INVNC}(\hat\pi)$ are equal.

Now consider points of intersection of the forms (6.14), (6.15) for $i \notin \{u_j, u_{j'}\}$. If $\pi_i \in AF \cup CD$, then all four triples are crossings; if $\pi_i \in AD \cup CF$, then all four are noncrossing. Moreover, if we have $\pi_i \in AD$ appearing to the left of column t or $\pi_i \in CF$ appearing to the right of column t, then all four noncrossings are inverted. Otherwise none of the four is inverted. If $\pi_i \in AE \cup BD \cup BF \cup CE$, then exactly one of the triples (6.14) is a noncrossing, as is exactly one of the triples (6.15). These two noncrossings are inverted if we have $\pi_i \in AE \cup BD$ appearing to the right of column t, or if we have $\pi_i \in BF \cup CE$ appearing to the left of column t. Otherwise, the two noncrossings are not inverted. If $\pi_i \in BE$, then either both triples (6.14) are crossings while both triples (6.15) are not (if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is a crossing), or vice versa (if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is a noncrossing). In both cases, exactly one of the two noncrossings is inverted. Finally, observe that neither triple in (6.16) can be an inverted noncrossing, since all four paths appear in column t. Thus the points of intersection of the forms (6.14), (6.15), (6.16) contribute a surplus of b to $\mathrm{INVNC}(\pi)$ if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is a crossing, and to $\mathrm{CR}(\hat\pi)$ otherwise.

(Identity (6.11)) For $k < p$ and all i, i', we have that $(\pi_i, \pi_{i'}, k)$ is a column defective noncrossing if and only if $(\hat\pi_i, \hat\pi_{i'}, k)$ is a column defective noncrossing. The same is true for $k \geq p$, provided that $i, i' \notin \{u_j, u_{j'}\}$. For $k > p$ and $i \notin \{u_j, u_{j'}\}$, no triple (π_i, π_{u_j}, k), $(\pi_i, \pi_{u_{j'}}, k)$, $(\hat\pi_i, \hat\pi_{u_j}, k)$, $(\hat\pi_i, \hat\pi_{u_{j'}}, k)$ can be a column defective noncrossing, since the definition of j, j' guarantees that no path π_i can belong to column t and intersect paths indexed by u_j, $u_{j'}$ to the right of G_{J_p}. So far, contributions to $\mathrm{CDNC}(\pi)$ and $\mathrm{CDNC}(\hat\pi)$ are equal.

Now consider points of intersection of the forms (6.14), (6.15) for $i \notin \{u_j, u_{j'}\}$. If $\pi_i \in AF \cup BF \cup CF \cup AE \cup BE \cup CE$, then it cannot appear in column t by our choice of j, j'. If $\pi_i \in CD$, then all four triples are crossings. If $\pi_i \in AD \cup BD$, then (π_i, π_{u_j}, p) is a column defective noncrossing if and only if $(\hat\pi_i, \hat\pi_{u_j}, p)$ is, and $(\pi_i, \pi_{u_{j'}}, p)$ is a column defective noncrossing if and only if $(\hat\pi_i, \hat\pi_{u_{j'}}, p)$ is. Finally, observe that neither triple in (6.16) can be a column defective noncrossing if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is proper, while exactly one of the two a column defective noncrossing if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is defective. Thus the points of intersection of the forms (6.16) contribute a surplus of 1 to $\mathrm{CDNC}(\pi)$ if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is a defective noncrossing, 1 to $\mathrm{CDNC}(\hat\pi)$ if $(\pi_{u_j}, \pi_{u_{j'}}, p)$ is a defective crossing, and 0 to both otherwise. $\qquad\square$

As a consequence, we have that the map ζ preserves a certain linear combination of the above statistics.

Corollary 6.4. *For $W = \delta(U(\pi, u, yu))$ and $\zeta(W) = \delta(U(\hat{\pi}, u, \hat{y}u))$ in $\mathcal{T}_I$, we have*

$$(6.17) \quad \frac{\mathrm{CR}(\hat{\pi})}{2} + \mathrm{CDNC}(\zeta(W)) + \mathrm{INVNC}(\zeta(W)) - \frac{\ell(\hat{y})}{2} = \frac{\mathrm{CR}(\pi)}{2} + \mathrm{CDNC}(W) + \mathrm{INVNC}(W) - \frac{\ell(y)}{2}.$$

Proof. If W is a fixed point of ζ, then the result is clear. Suppose therefore that $\zeta(W) \neq W$, and consider the triple $(\pi_{u_j}, \pi_{u_{j'}}, p)$ appearing in the definition of ζ. If this triple is a proper noncrossing, then by Lemma 6.3 the left-hand side of (6.17) is

$$\frac{\mathrm{CR}(\pi) + 1 + 2b}{2} + \mathrm{CDNC}(W) + \mathrm{INVNC}(W) - b - \frac{\ell(y) + 1}{2}$$

$$= \frac{\mathrm{CR}(\pi)}{2} + \mathrm{CDNC}(W) + \mathrm{INVNC}(W) - \frac{\ell(y)}{2} + \frac{1 + 2b}{2} - b - \frac{1}{2}.$$

One shows similarly that the result holds when the triple is a defective noncrossing or any crossing. $\qquad\qquad\square$

Finally we can state and justify a subtraction-free formula for $\epsilon_q^\lambda(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q))$. The proof of the following result is nearly identical to that of [11, Thm. 5.4]. We include it here, both for the convenience of the reader, and because it relies upon Theorem 5.2, Proposition 4.5, Lemmas 6.1 – 6.2 in this paper which differ from the corresponding (weaker) results [11, Thm. 3.7], [11, Prop. 4.1], [11, Lem. 5.1 – 5.2], in that paper, and upon Corollary 6.4.

Theorem 6.5. *Let $G = G_{J_1} \circ \cdots \circ G_{J_m}$. Then for $\lambda \vdash n$ we have*

$$(6.18) \quad \epsilon_q^\lambda(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)) = \sum_\pi q^{\frac{\mathrm{CR}(\pi)}{2}} \sum_W q^{\mathrm{INVNC}(W)},$$

where the sums are over path families π of type e which cover G, and column-strict π-tableaux W of shape $\lambda^\top$.

Proof. Let B be the path matrix of G. By Theorem 5.2 and Equations (2.10) – (2.11), the left-hand side of (6.18) is

$$(6.19) \quad \begin{aligned} \sigma_B(\mathrm{Imm}_{\epsilon_q^\lambda}(x)) &= \sum_I \sigma_B(\det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r})) \\ &= \sum_I \sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} \sigma_B(x^{u(I), yu(I)}), \end{aligned}$$

where the first two sums are over ordered set partitions $I = (I_1, \ldots, I_r)$ of $[n]$ of type λ. Fixing one such partition I and writing $u = u(I)$, we may use Proposition 4.5 and Lemma 6.1 to express the sum over $y \in \mathfrak{S}_\lambda$ as

$$(6.20)$$
$$\sum_{y \in \mathfrak{S}_\lambda} \sum_\pi (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} q^{\frac{\mathrm{CR}(\pi)}{2}} q^{\mathrm{INVNC}(U(\pi, u, yu))} = \sum_{y \in \mathfrak{S}_\lambda} \sum_\pi (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} q^{\frac{\mathrm{CR}(\pi)}{2}} q^{\mathrm{INVNC}(W) + \mathrm{CDNC}(W)},$$

where the inner sums are over path families π of type $u^{-1}yu$ which cover G, and where $W = \delta_I(U(\pi, u, yu))$. As y and π vary in the above sums, $U(\pi, u, yu)$ varies over all tableaux in $\mathcal{U}_I$, and W varies over all tableaux in $\mathcal{T}_I$.

Now consider a tableau $W \in \mathcal{T}_I$ which satisfies $\zeta(W) \neq W$. Let tableaux W and $\zeta(W)$ contain path families π of type $u^{-1}yu$ and $\hat{\pi}$ of type $u^{-1}\hat{y}u$, respectively. Then the two terms on the right-hand side of (6.20) corresponding to $\zeta(W)$ and W sum to

$$(-1)^{\ell(\hat{y})}q^{\frac{\mathrm{CR}(\hat{\pi})}{2}+\mathrm{INVNC}(\zeta(W))+\mathrm{CDNC}(\zeta(W))-\frac{\ell(\hat{y})}{2}} + (-1)^{\ell(y)}q^{\frac{\mathrm{CR}(\pi)}{2}+\mathrm{INVNC}(W)+\mathrm{CDNC}(W)-\frac{\ell(y)}{2}}.$$

By Lemma 6.2 and Corollary 6.4 this is 0. Thus it suffices to sum the right-hand side of (6.20) over only the pairs (y, π) corresponding to tableaux W satisfying $W = \zeta(W)$. By the definition of ζ, each such tableau W is column-strict and therefore satisfies $\mathrm{CDNC}(W) = 0$. Since all tableaux in $\mathcal{T}_I$ are also column-closed, each such tableau W must have type e. Thus we have we have $u^{-1}yu = e$, i.e., $y = e$. It follows that the right-hand side of (6.20) and the third sum in (6.19) are equal to

$$q^{\frac{\mathrm{CR}(\pi)}{2}} \sum_W q^{\mathrm{INVNC}(W)},$$

where the sum is over all tableau W in $\mathcal{T}_I$ which are column-strict of type e. Since all tableaux in $\mathcal{T}_I$ have shape $\lambda^\top$, the three expressions in (6.19) are equal to the right-hand side of (6.18). $\square$

To illustrate the theorem, we compute $\epsilon_q^{211}(\widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,2]}}(q))$ using the wiring diagram

$$(6.21) \qquad\qquad G = G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]} = \ \text{\includegraphics[width=2cm]{wiring}} \ .$$

There are two path families of type e which cover G, and four column-strict G-tableaux of shape $211^\top = 31$ for each:

$$U_\pi^{(1)} = \begin{array}{|c|}\hline \pi_3 \\ \hline \end{array}\!\!\begin{array}{|c|c|c|}\hline \pi_1 & \pi_2 & \pi_4 \\ \hline\end{array}, \ U_\pi^{(2)} = \begin{array}{|c|}\hline \pi_3 \\ \hline\end{array}\!\!\begin{array}{|c|c|c|}\hline \pi_1 & \pi_4 & \pi_2 \\ \hline\end{array}, \ U_\pi^{(3)} = \begin{array}{|c|}\hline \pi_4 \\ \hline\end{array}\!\!\begin{array}{|c|c|c|}\hline \pi_1 & \pi_2 & \pi_3 \\ \hline\end{array}, \ U_\pi^{(4)} = \begin{array}{|c|}\hline \pi_4 \\ \hline\end{array}\!\!\begin{array}{|c|c|c|}\hline \pi_1 & \pi_3 & \pi_2 \\ \hline\end{array};$$

$$U_\rho^{(1)} = \begin{array}{|c|}\hline \rho_3 \\ \hline\end{array}\!\!\begin{array}{|c|c|c|}\hline \rho_2 & \rho_1 & \rho_4 \\ \hline\end{array}, \ U_\rho^{(2)} = \begin{array}{|c|}\hline \rho_3 \\ \hline\end{array}\!\!\begin{array}{|c|c|c|}\hline \rho_2 & \rho_4 & \rho_1 \\ \hline\end{array}, \ U_\rho^{(3)} = \begin{array}{|c|}\hline \rho_4 \\ \hline\end{array}\!\!\begin{array}{|c|c|c|}\hline \rho_2 & \rho_1 & \rho_3 \\ \hline\end{array}, \ U_\rho^{(4)} = \begin{array}{|c|}\hline \rho_4 \\ \hline\end{array}\!\!\begin{array}{|c|c|c|}\hline \rho_2 & \rho_3 & \rho_1 \\ \hline\end{array}.$$

Since the path family π has no crossings, we have $\mathrm{C}(U_\pi^{(i)}) = 0$ for all i, and each tableau $U_\pi^{(i)}$ therefore contributes $q^{\mathrm{INVNC}(U_\pi^{(i)})}$. We have one noncrossing for each of the pairs (π_2, π_3), (π_2, π_4) and (π_3, π_4) and two for the pair (π_1, π_2). Counting the noncrossings only for pairs where the path which intersects the other from above appears in a column left of the other, for instance π_2 and π_3 in $U_\pi^{(1)}$, we find the contributions from $U_\pi^{(1)}, \ldots, U_\pi^{(4)}$ are q, q^2, q^2, q^3, respectively. Since the path family ρ has two crossings, the tableaux for the path family ρ each have two crossings, and one noncrossing for each of the pairs (ρ_1, ρ_3), (ρ_1, ρ_4) and (ρ_3, ρ_4). Adding the contributions together we find the contributions for $U_\rho^{(1)}, \ldots, U_\rho^{(4)}$ are $q^1 q^{2/2} = q^2$, $q^2 q^{2/2} = q^3$, $q^2 q^{2/2} = q^3$ and $q^3 q^{2/2} = q^4$ respectively. Hence we have $\epsilon_q^{211}(\widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,2]}}(q)) = q + 3q^2 + 3q^3 + q^4$.

Theorem 6.5 allows one to combinatorially interpret evaluations of ϵ_q^λ at (multiples of) certain elements $\widetilde{C}_w(q)$ of the Kazhdan-Lusztig basis of $H_n(q)$. In particular, for some elements $\widetilde{C}_w(q)$ there exists a polynomial $g(q)$ such that we have

$$(6.22) \qquad g(q)\widetilde{C}_w(q) = \widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$$

for some sequence $s_{J_1}, \ldots, s_{J_m}$ of reversals. Such permutations include all 3412-avoiding, 4231-avoiding permutations, all of $\mathfrak{S}_4$ (even 4231 and 3412), all of $\mathfrak{S}_5$ except 45312, and all 321-*hexagon-avoiding* permutations. (See [2].)

Corollary 6.6. *Suppose that $\widetilde{C}_w(q)$ satisfies a factorization of the form (6.22) and define $G = G_{J_1} \circ \cdots \circ G_{J_m}$. Then we have*

$$(6.23) \qquad \epsilon_q^\lambda(\widetilde{C}_w(q)) = \frac{1}{g(q)} \sum_U q^{\mathrm{INVNC}(U) + \mathrm{CR}(U)/2},$$

where the sum is over all column-strict G-tableaux of type e and shape $\lambda^\top$.

It would be interesting to know for which Kazhdan-Lusztig basis elements we have the above factorization ([19, Quest. 4.5]).

Recall from Equations (1.4) – (1.5) that Haiman's result [10, Lem. 1.1] that we have

$$\chi_q^\lambda(\widetilde{C}_w(q)) \in \mathbb{N}[q]$$

for all irreducible characters χ_q^λ and all $w \in \mathfrak{S}_n$ implies that we also have

$$(6.24) \qquad \chi_q^\lambda(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)) \in \mathbb{N}[q]$$

for all irreducible characters χ_q^λ and all sequences $(s_{J_1}, \ldots, s_{J_m})$ of reversals. It would therefore be interesting to extend Theorem 6.5 to irreducible characters.

Problem 6.7. *Find a combinatorial interpretation of the polynomial (6.24) which holds for all irreducible charcters χ_q^λ and all sequences $(s_{J_1}, \ldots, s_{J_m})$ of reversals.*

REFERENCES

[1] A. A. BEĬLINSON, J. BERNSTEIN, AND P. DELIGNE. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, vol. 100 of *Astérisque*. Soc. Math. France, Paris (1982), pp. 5–171.

[2] S. C. BILLEY AND G. WARRINGTON. Kazhdan-Lusztig polynomials for 321-hexagon-avoiding permutations. *J. Algebraic Combin.*, **13**, 2 (2001) pp. 111–136.

[3] A. BJÖRNER AND F. BRENTI. *Combinatorics of Coxeter groups*, vol. 231 of *Graduate Texts in Mathmatics*. Springer, New York (2005).

[4] F. BRENTI. Combinatorics and total positivity. *J. Combin. Theory Ser. A*, **71**, 2 (1995) pp. 175–218.

[5] S. CLEARMAN, M. HYATT, B. SHELTON, AND M. SKANDERA. Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements. *Electron. J. Combin.*, **23**, 2 (2016) pp. Paper 2.7, 56.

[6] C. W. CRYER. Some properties of totally positive matrices. *Linear Algebra Appl.*, **15** (1976) pp. 1–25.

[7] V. DEODHAR. A combinatorial setting for questions in Kazhdan-Lusztig theory. *Geom. Dedicata*, **36**, 1 (1990) pp. 95–119.

[8] S. FOMIN AND A. ZELEVINSKY. Total positivity: Tests and parametrizations. *Math. Intelligencer*, **22**, 1 (2000) pp. 23–33.

[9] F. R. GANTMACHER AND M. G. KREIN. *Oscillation matrices and kernels and small vibrations of mechanical systems*. AMS Chelsea Publishing, Providence (2002). Edited by A. Eremenko. Translation based on the 1941 Russian original.

[10] M. Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, **6**, 3 (1993) pp. 569–595.

[11] R. Kaliszewski, J. Lambright, and M. Skandera. Bases of the quantum matrix bialgebra and induced sign characters of the Hecke algebra. *J. Algebraic Combin.*, **49**, 4 (2019) pp. 475–505.

[12] S. Karlin and G. McGregor. Coincidence probabilities. *Pacific J. Math.*, **9** (1959) pp. 1141–1164.

[13] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, **53** (1979) pp. 165–184.

[14] M. Konvalinka and M. Skandera. Generating functions for Hecke algebra characters. *Canad. J. Math.*, **63**, 2 (2011) pp. 413–435.

[15] V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in $SL(n)/B$. *Proc. Indian Acad. Sci. (Math Sci.)*, **100**, 1 (1990) pp. 45–52.

[16] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, **5** (1973) pp. 85–90.

[17] G. Lusztig. Total positivity in reductive groups. In *Lie Theory and Geometry: in Honor of Bertram Kostant*, vol. 123 of *Progress in Mathematics*. Birkhäuser, Boston (1994), pp. 531–568.

[18] Y. I. Manin. *Quantum groups and noncommutative geometry*. Université de Montréal Centre de Recherches Mathématiques, Montreal, QC (1988).

[19] M. Skandera. On the dual canonical and Kazhdan-Lusztig bases and 3412, 4231-avoiding permutations. *J. Pure Appl. Algebra*, **212** (2008).

[20] T. A. Springer. Quelques aplications de la cohomologie d'intersection. In *Séminaire Bourbaki, Vol. 1981/1982*, vol. 92 of *Astérisque*. Soc. Math. France, Paris (1982), pp. 249–273.

[21] J. Stembridge. Immanants of totally positive matrices are nonnegative. *Bull. London Math. Soc.*, **23** (1991) pp. 422–428.

[22] J. Stembridge. Some conjectures for immanants. *Canad. J. Math.*, **44**, 5 (1992) pp. 1079–1099.