INEQUALITIES IN PRODUCTS OF MINORS OF TOTALLY NONNEGATIVE MATRICES

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Outline

1. Totally nonnegative (TNN) matrices
2. Path families in planar networks
3. TNN polynomials
4. Inequalities in products of minors
5. Characterization theorems
6. Open questions

Total nonnegativity

Let $\Delta_{I,I'}$ denote the (I, I')-*minor* of A: the determinant of the submatrix corresponding to rows I and columns I'.

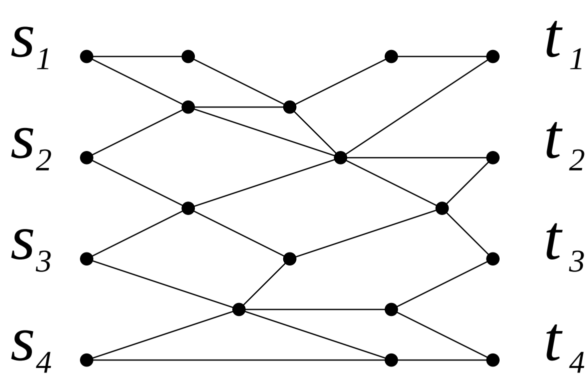
$$A = \begin{bmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix},$$

$$\Delta_{\{1,3\},\{2,3\}} = \det \begin{bmatrix} 6 & 3 \\ 4 & 4 \end{bmatrix} = 12.$$

Define a matrix to be *totally nonnegative* (TNN) if each of its minors is nonnegative.

Planar networks

Define a *planar network of order n* to be a directed acyclic planar graph, in which $2n$ boundary vertices are labeled counterclockwise as $s_1, \ldots, s_n, t_n, \ldots, t_1$ as below.



Define the *path matrix* $A = [a_{ij}]$ by

$$a_{ij} = \# \text{ paths from } s_i \text{ to } t_j.$$

Theorem: (K-McG'59, L'73) The path matrix of a planar network is always TNN.

Proof idea: $\Delta_{I,I'}$ counts families of nonintersecting paths from sources

$$S_I = \{s_i \mid i \in I\}$$

to sinks

$$T_{I'} = \{t_i \mid i \in I'\}.$$

Theorem: (W'52, L'55, C'76, B'95) All TNN matrices are essentially path matrices of planar networks.

TNN polynomials

Call a polynomial f in n^2 variables *TNN* if for any TNN matrix A we have

$$f(a_{1,1}, \ldots, a_{n,n}) \geq 0.$$

Theorem: (Lusztig '94) The elements of Zelevinsky and Berenstein's dual canonical basis for the coordinate ring of GL_n are TNN.

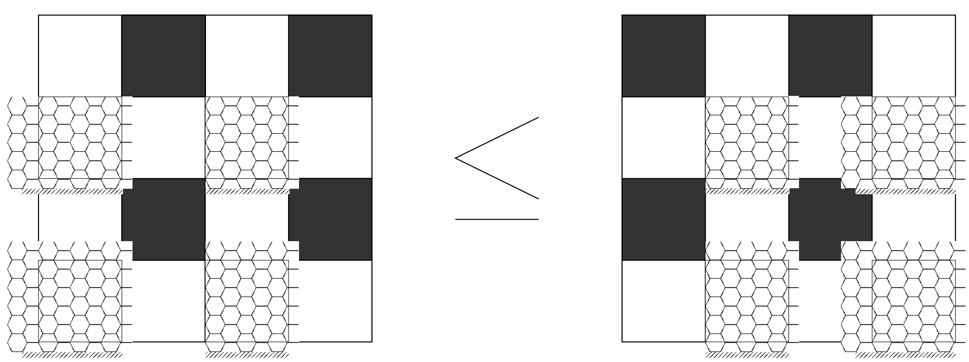
Problem: Find a simple description of the dual canonical basis for GL_n.

Problem: Find a simple description of any family of TNN polynomials.

Fact: The inequality

$$\Delta_{13,24}\Delta_{24,13} \leq \Delta_{13,13}\Delta_{24,24}$$

holds for all TNN matrices.



Equivalently, the polynomial

$$\Delta_{13,13}\Delta_{24,24} - \Delta_{13,24}\Delta_{24,13}$$

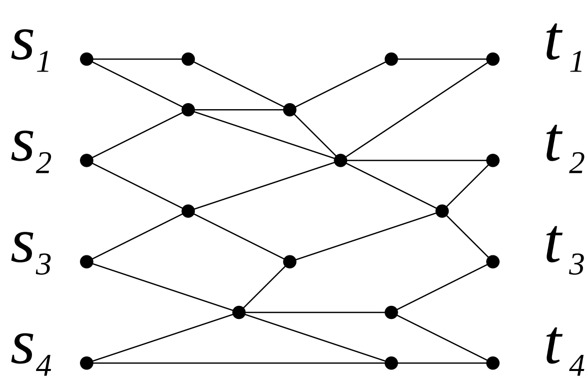
is TNN.

Combinatorial interpretation of the products of minors

$\Delta_{I,I'}\Delta_{\overline{I},\overline{I'}}$ counts families of n paths from S to T such that

1. paths from S_I to $T_{I'}$ don't intersect.
2. paths from $S_{\overline{I}}$ to $T_{\overline{I}'}$ don't intersect.

We will say that such a path family *obeys the* (I, I') *crossing rule*.



Question: When is

$$\Delta_{J,J'}\Delta_{\overline{J},\overline{J'}} - \Delta_{I,I'}\Delta_{\overline{I},\overline{I'}}$$

a TNN polynomial?

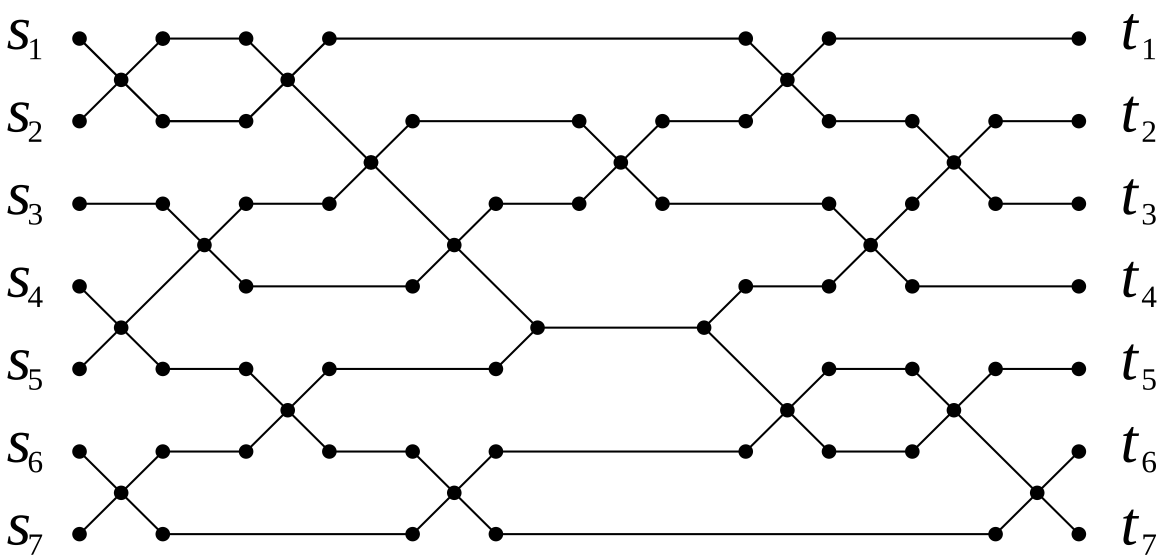
Question: When does

$$\Delta_{I,I'}\Delta_{\overline{I},\overline{I'}} \leq \Delta_{J,J'}\Delta_{\overline{J},\overline{J'}}$$

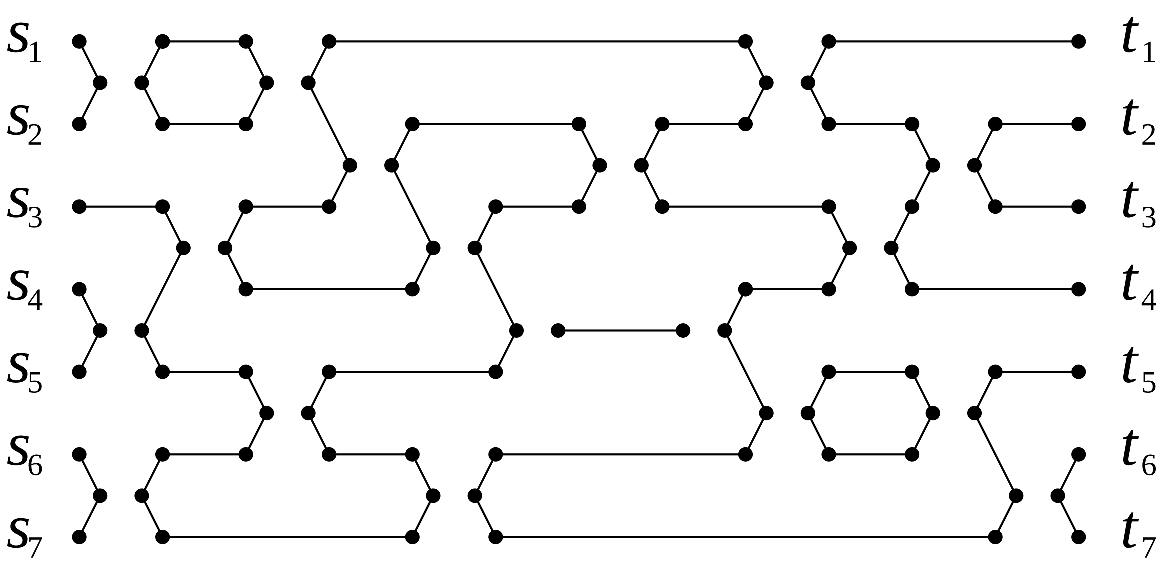
hold for all TNN matrices?

Question: When do path families which obey the (J, J') crossing rule outnumber those which obey the (I, I') crossing rule in every planar network?

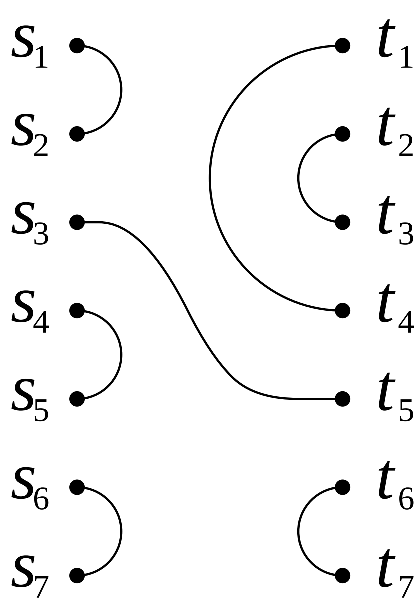
A planar network G.



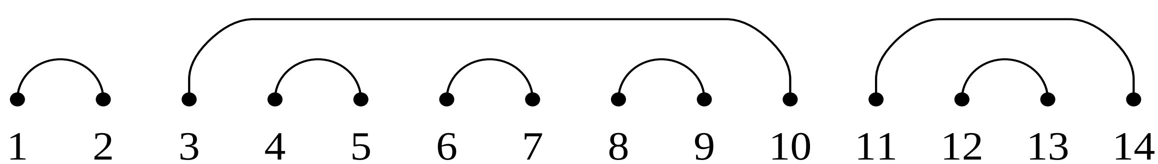
From G, define $\phi(G)$.



$\phi(G)$ induces a perfect matching of $S \cup T$

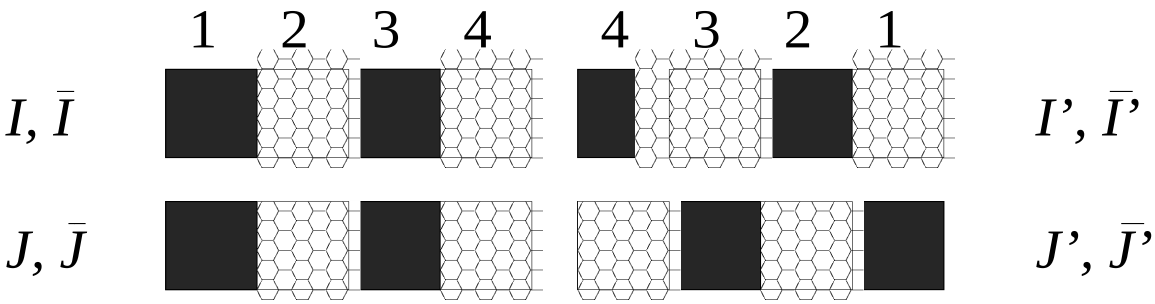


whose edges define subintervals of $[2n]$.

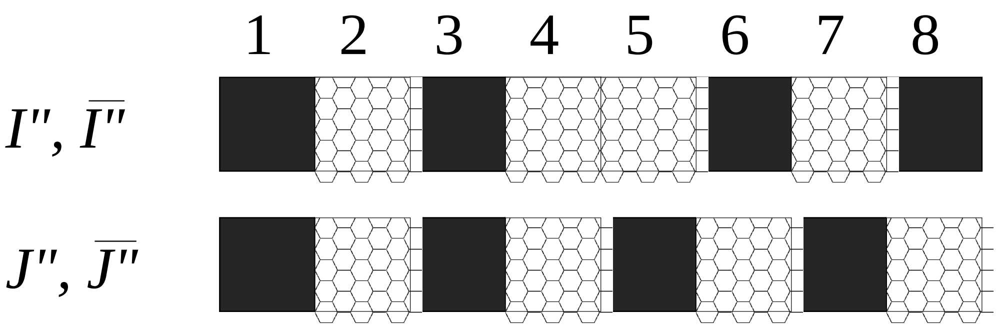


Q: When is $\Delta_{J,J'}\Delta_{\overline{J},\overline{J'}} - \Delta_{I,I'}\Delta_{\overline{I},\overline{I'}}$ TNN?

Write $(I', \overline{I'})$ and $(J', \overline{J'})$ backwards.



Swap $(I', \overline{I'})$, swap $(J', \overline{J'})$, renumber.



A: (**Theorem**) When the set partition $(J'', \overline{J''})$ of $[2n]$ is at least as sparse as $(I'', \overline{I''})$.

Corollary of main theorem

Corollary: (MS '01) If the polynomial

$$\Delta_{J,J'}\Delta_{\overline{J},\overline{J'}} - \Delta_{I,I'}\Delta_{\overline{I},\overline{I'}}$$

is TNN, it counts path families which

1. obey the (J, J') crossing rule
2. can not be covered by any path family which obeys the (I, I') crossing rule.

Theorem: (MS '01) The inequality

$$\Delta_{I,I'}\Delta_{K,K'} \leq \Delta_{J,J'}\Delta_{L,L'}$$

holds for all TNN matrices if and only if we can delete repeated indices and reduce to the previous theorem.

Corollary: All of the inequalities of the above form are consequences of inequalities of the form

$$\Delta_{I,I}\Delta_{\overline{I},\overline{I}} \leq \Delta_{J,J}\Delta_{\overline{J},\overline{J}},$$

where I, J are n-subsets of $[2n]$. (Studied by FGJ'01.)

Open questions

1. Which TNN polynomials can be written as subtraction-free rational expressions (or Laurent polynomials) in matrix minors?

2. How can one characterize inequalities that hold between products of k minors of TNN matrices, for $k > 2$?

3. Which TNN polynomials when applied to Jacobi-Trudi matrices evaluate to Schur-positive symmetric functions?