BARRETT–JOHNSON INEQUALITIES FOR TOTALLY NONNEGATIVE MATRICES

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Abstract. Given a matrix A, let $A_{I,J}$ denote the submatrix of A determined by rows I and columns J. Fischer's inequalities state that for each $n \times n$ Hermitian positive semidefinite matrix A, and each subset I of $\{1, \ldots, n\}$ and its complement I^c, we have $\det(A) \leq \det(A_{I,I}) \det(A_{I^c,I^c})$. Barrett and Johnson [*Linear Multilinear Algebra* **34**, 1993] extended these to state inequalities for sums of products of principal minors whose orders are given by nonincreasing nonnegative integer sequences $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\mu = (\mu_1, \ldots, \mu_s)$ summing to n. Specifically, for each $n \times n$ real positive semidefinite matrix A we have the inequality

$$\lambda_1! \cdots \lambda_r! \sum_{(I_1,\ldots,I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}) \;\geq\; \mu_1! \cdots \mu_s! \sum_{(J_1,\ldots,J_s)} \det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s}),$$

where the sums are over all sequences of disjoint subsets of $\{1, \ldots, n\}$ satisfying $|I_k| = \lambda_k$, $|J_k| = \mu_k$, if and only if the integer sequences λ, μ satisfy $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i. We show that these inequalities hold for totally nonnegative matrices as well.

1. Introduction

A matrix $A \in \mathrm{Mat}_{n \times n}(\mathbb{C})$ is called *Hermitian* if it satisfies $A^* = A$ where $*$ denotes conjugate transpose. Such a matrix is called *Hermitian positive semi-definite* (HPSD) if we have $x^* A x \geq 0$ for all $x \in \mathbb{C}^n$. For $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$, the Hermitian property reduces to symmetry $A^\top = A$, and the positive semidefinite (PSD) property reduces to $x^\top A x \geq 0$ for all $x \in \mathbb{R}^n$. Hermitian matrices are fundamental to quantum mechanics since they describe operators with real eigenvalues. Positive definite (semi-definite) matrices are one of the matrix analogs of positive (nonnegative) numbers, and correspond to inner products (nonnegative quadratic forms). They appear in many fields such as geometry, numerical analysis, optimization, quantum physics, and statistics. (See, e.g., [2], [16].)

Another matrix analog of nonnegative numbers is the class of *totally nonnegative* (TNN) matrices, consisting of the matrices in $\mathrm{Mat}_{n \times n}(\mathbb{R})$ in which each minor (determinant of a square submatrix) is nonnegative. Initially, total nonnegativity arose in three different areas: it was studied by Gantmacher and Krein in oscillations of vibrating systems, by Schoenberg in applications to the analysis of real roots of polynomials and spline functions, and by Karlin in integral equations and statistics. TNN matrices appear in many other areas of mathematics. (See, e.g., [10].)

Like TNN matrices, HPSD matrices have a well-known characterization in terms of matrix minors. Given an $n \times n$ matrix $A = (a_{i,j})$ and subsets $I, J \subseteq [n] := \{1, \ldots, n\}$, define the submatrix $A_{I,J} := (a_{i,j})_{i \in I, j \in J}$, and define the set $I^c := [n] \smallsetminus I$. While TNN matrices are

Date: May 8, 2026.

characterized by the inequalities

$$\det(A_{I,J}) \geq 0, \qquad \text{for all } I, J \subseteq [n] \text{ with } |I| = |J|, \tag{1.1}$$

HPSD matrices are characterized by the (equalities and) inequalities

$$\begin{aligned} a^*_{j,i} &= a_{i,j}, && \text{for all } i, j \in [n], \\ \det(A_{I,I}) &\geq 0, && \text{for all } I \subseteq [n]. \end{aligned} \tag{1.2}$$

From (1.1) – (1.2), one can deduce many more inequalities satisfied by polynomials in the entries of $n \times n$ HPSD, PSD, and/or TNN matrices.

The following inequalities hold for HPSD matrices and TNN matrices. Hadamard [14] showed that for A HPSD we have

$$\det(A) \leq a_{1,1} \cdots a_{n,n}, \tag{1.3}$$

and Koteljanskii [20, 21] showed that this holds for A TNN as well. Marcus [25] proved a permanental analog

$$\mathrm{per}(A) \geq a_{1,1} \cdots a_{n,n} \tag{1.4}$$

for A HPSD, and this analog clearly holds for A TNN as well. Fischer [12] strengthened (1.3) by showing that for all $I \subseteq [n]$ we have

$$\det(A) \leq \det(A_{I,I}) \det(A_{I^c,I^c}), \tag{1.5}$$

and Ky Fan showed that this holds for A TNN as well (unpublished; see [6]). Lieb [22] proved a permanental analog

$$\mathrm{per}(A) \geq \mathrm{per}(A_{I,I}) \, \mathrm{per}(A_{I^c,I^c}), \tag{1.6}$$

for A HPSD, and this analog holds for A TNN because every term in the expansion of the product on the right-hand side of (1.6) also appears on the left-hand side. Koteljanskii [20, 21] strengthened (1.5) further by proving that for all $I, J \subseteq [n]$ we have

$$\det(A_{I\cup J,I\cup J}) \det(A_{I\cap J,I\cap J}) \leq \det(A_{I,I}) \det(A_{J,J}) \tag{1.7}$$

for A belonging to a class of matrices including HPSD and TNN matrices. Strengthening (1.3) somewhat differently, Schur [32] proved that for each $\mathfrak{S}_n$-character χ, the inequality

$$\det(A) \leq \frac{\mathrm{Imm}_\chi(A)}{\chi(e)} \tag{1.8}$$

holds for A HPSD, where Imm_χ is the immanant corresponding to χ (defined in §3). Stembridge [39, Cor. 3.4] showed that this holds for A TNN as well.

There are other inequalities that hold for TNN matrices, and are not known to hold for HPSD matrices. For instance Charles Johnson showed that the permanental analog of (1.8),

$$\mathrm{per}(A) \geq \frac{\mathrm{Imm}_\chi(A)}{\chi(e)}, \tag{1.9}$$

holds for TNN matrices (unpublished; see [40, p. 1088]). The validity of this inequality for HPSD matrices was conjectured by Lieb [22], is still open, and is known as the *permanental* *dominance conjecture*. Extending the results (1.5) and (1.7), Fallat, Gekhtman, and Johnson [9] and the first author [33] characterized all 8-tuples $(I, J, K, L, I', J', K', L')$ of subsets of $[n]$ for which we have the inequality

$$\det(A_{I,I'})\det(A_{J,J'}) \leq \det(A_{K,K'})\det(A_{L,L'}) \tag{1.10}$$

whenever A is TNN. (See [36] for some progress on products of three or more minors.) Drake, Gerrish, and the first author [8] proved a similar result for products of n matrix entries. In particular, the inequality

$$a_{1,v_1}\cdots a_{n,v_n} \geq a_{1,w_1}\cdots a_{n,w_n} \tag{1.11}$$

holds for all TNN A if and only if $v \leq w$ in the Bruhat order on $\mathfrak{S}_n$. (See, e.g., [4].) Other linear combinations of the monomials in (1.11) which produce valid inequalities for TNN matrices are related to Lusztig's work on canonical bases of quantum groups [24] and are called *Kazhdan–Lusztig immanants* in [29]. Writing these as $\{\mathrm{Imm}_w(A) \,|\, w \in \mathfrak{S}_n\}$, we have

$$\mathrm{Imm}_w(A) \geq 0 \tag{1.12}$$

for all $w \in \mathfrak{S}_n$ and all TNN A. (See [29] for definitions.) We will define and use special cases of these, called *Temperley–Lieb immanants* in [28], in Section 4.

Still other inequalities hold for HPSD matrices, but are not known to hold for TNN matrices. Heyfron [15] extended (1.8) and proved special cases of the permanental dominance conjecture (1.9) by considering the irreducible $\mathfrak{S}_n$-characters that are indexed by "hook" shapes $\lambda = (k, 1, \ldots, 1)$. (See Sections 2 – 3 for definitions.) In particular, we have

$$\mathrm{per}(A) = \frac{\mathrm{Imm}_{\chi^n}(A)}{\chi^n(e)} \geq \frac{\mathrm{Imm}_{\chi^{n-1,1}}(A)}{\chi^{n-1,1}(e)} \geq \frac{\mathrm{Imm}_{\chi^{n-2,1,1}}(A)}{\chi^{n-2,1,1}(e)} \geq \cdots \geq \frac{\mathrm{Imm}_{\chi^{1,\ldots,1}}(A)}{\chi^{1,\ldots,1}(e)} = \det(A) \tag{1.13}$$

for A HPSD. (See [17], [27] for more results of this form.) Borcea and Brändén showed that certain averages of products of pairs of minors appearing in (1.5) satisfy the log-concavity inequalities [5, Cor. 3.1 (b)]

$$\left(\sum_{\substack{I\subseteq[n]\\|I|=k}} \frac{\det(A_{I,I})\det(A_{I^c,I^c})}{\binom{n}{k}}\right)^2 \geq \left(\sum_{\substack{I\subseteq[n]\\|I|=k+1}} \frac{\det(A_{I,I})\det(A_{I^c,I^c})}{\binom{n}{k+1}}\right)\left(\sum_{\substack{I\subseteq[n]\\|I|=k-1}} \frac{\det(A_{I,I})\det(A_{I^c,I^c})}{\binom{n}{k-1}}\right) \tag{1.14}$$

for all A HPSD and $k = 1, \ldots, n-1$. They also showed that these averages satisfy the Maclaurin-like inequalities [5, Cor. 3.1 (c)]

$$\left(\frac{\sum_{|I|=k+1}\det(A_{I,I})\det(A_{I^c,I^c})}{\binom{n}{k+1}\det(A)}\right)^k \geq \left(\frac{\sum_{|I|=k}\det(A_{I,I})\det(A_{I^c,I^c})}{\binom{n}{k}\det(A)}\right)^{k+1} \tag{1.15}$$

for A nonsingular HPSD and for $k = 1, \ldots, n-1$.

Nearly belonging to the list of HPSD inequalities above are the Barrett–Johnson inequalities [1] for (real) PSD matrices: for nonincreasing nonnegative integer sequences

$\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1, \dots, \mu_s)$ summing to n, we have

$$(1.16)\quad \lambda_1! \cdots \lambda_r! \sum_{(I_1,\dots,I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}) \;\geq\; \mu_1! \cdots \mu_s! \sum_{(J_1,\dots,J_s)} \det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s}),$$

where the sums are over sequences of disjoint subsets of $[n]$ satisfying $|I_k| = \lambda_k$, $|J_k| = \mu_k$, if and only if the integer sequences λ, μ satisfy $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for $i = 1, \dots, \min(r, s)$. We will show that these inequalities also hold for all TNN matrices.

The remainder of the paper is organized as follows. In Section 2, we review basic facts about the symmetric group, the space of its traces (class functions), and symmetric functions. In Section 3 we apply these ideas and tools to form certain polynomials called *immanants* in the entries of $n \times n$ matrices, and we discuss the evaluation of these at TNN matrices. In Section 4, we discuss the Temperley–Lieb algebra, related immanants, and their relationship to products of matrix minors. In Section 5, we present our main result that the Barrett–Johnson inequalities hold for all $n \times n$ TNN matrices. In Section 6 we summarize some open problems exploring the validity of inequalities which are conspicuously absent from the lists in Section 1.

2. The symmetric group, its traces, and symmetric functions

The *symmetric group algebra* $\mathbb{C}[\mathfrak{S}_n]$ is generated over $\mathbb{C}$ by $s_1, \dots, s_{n-1}$, subject to relations

$$\begin{aligned} s_i^2 &= e && \text{for } i = 1, \dots, n-1, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } |i - j| = 1, \\ s_i s_j &= s_j s_i && \text{for } |i - j| \geq 2. \end{aligned}$$

We define the *one-line notation* $w_1 \cdots w_n$ of $w \in \mathfrak{S}_n$ by letting any expression for w act on the word $1 \cdots n$, where each generator $s_j = s_{[j,j+1]}$ acts on an n-letter word by swapping the letters in positions j and $j+1$, i.e., $s_j \circ v_1 \cdots v_n = v_1 \cdots v_{j-1} v_{j+1} v_j v_{j+2} \cdots v_n$. Whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced (shortest possible) expression for $w \in \mathfrak{S}_n$, we call ℓ the *length* of w and write $\ell = \ell(w)$. It is known that $\ell(w)$ is equal to $\textsc{inv}(w)$, the number of inversions in $w_1 \cdots w_n$.

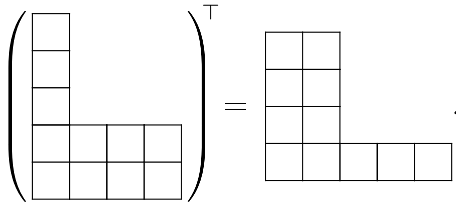
Recall that each conjugacy class in $\mathfrak{S}_n$ consists of all permutations w that have a fixed *cycle type*, denoted by $\mathrm{ctype}(w)$. Cycle types are *integer partitions* of n, weakly decreasing positive integer sequences $\lambda = (\lambda_1, \dots, \lambda_\ell)$ satisfying $\lambda_1 + \cdots + \lambda_\ell = n$. The $\ell = \ell(\lambda)$ components of λ are called its *parts*, and we let $|\lambda| = n$ and $\lambda \vdash n$ denote that λ is a partition of n. Sometimes it is convenient to express a partition in *exponential notation*, omitting parentheses and commas, so that $4^2 1^3 := (4, 4, 1, 1, 1)$. Given $\lambda \vdash n$, we define the *transpose* partition $\lambda^\top = (\lambda^\top_1, \dots, \lambda^\top_{\lambda_1})$ by

$$(2.1)\qquad \lambda^\top_i = \#\{j \mid \lambda_j \geq i\}.$$

The transpose of a partition λ is easy to read from its *Young diagram* of λ_i left-justified boxes in row i, ordered from bottom to top. For instance, the equation

$$(4^2 1^3)^\top = (4, 4, 1, 1, 1)^\top = 5 2^3 = (5, 2, 2, 2)$$

can represented in terms of Young diagrams by reflecting the first diagram in the line of slope 1 passing through its lower left-hand corner:



An important partial order on the set of partitions of n is the *majorization order*, defined by

$$\lambda \preceq \mu \quad \text{iff} \quad \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \quad \text{for all } i. \tag{2.2}$$

It is known that if λ is covered by μ in this partial order, i.e., if no partition ν satisfies $\lambda \prec \nu \prec \mu$, then λ, μ have equal parts except in two positions $i < j$ where we have

$$\mu_i = \lambda_i + 1, \qquad \mu_j = \lambda_j - 1. \tag{2.3}$$

Let Λ be the ring of symmetric functions in $x = (x_1, x_2, \dots)$ having integer coefficients, and let Λ_n be the $\mathbb{Z}$-submodule of the homogeneous functions of degree n. This submodule has rank equal to the number of partitions of n. Five standard bases of Λ_n consist of monomial $\{m_\lambda \,|\, \lambda \vdash n\}$, elementary $\{e_\lambda \,|\, \lambda \vdash n\}$, (complete) homogenous $\{h_\lambda \,|\, \lambda \vdash n\}$, power sum $\{p_\lambda \,|\, \lambda \vdash n\}$, and Schur $\{s_\lambda \,|\, \lambda \vdash n\}$ symmetric functions. (See, e.g., [37, Ch. 7] for definitions.)

The change-of-basis matrix relating $\{e_\lambda \,|\, \lambda \vdash n\}$ to $\{m_\lambda \,|\, \lambda \vdash n\}$ is given by the equations

$$e_\lambda = \sum_{\mu \preceq \lambda^\top} M_{\lambda,\mu} m_\mu, \tag{2.4}$$

where $M_{\lambda,\mu}$ equals the number of *column-strict Young tableaux of shape* $\lambda^\top$ *and content* μ. That is, $M_{\lambda,\mu}$ is the number of histograms having λ_i boxes in column i for all i, filled with μ_1 ones, μ_2 twos, etc., with column entries strictly increasing from bottom to top. For example we have

$$e_{221} = m_{32} + 2m_{311} + 5m_{221} + 12m_{2111} + 30m_{11111},$$

and the coefficients $M_{221,32} = 1$ of m_{32}, $M_{221,311} = 2$ of m_{311}, $M_{221,221} = 5$ of m_{221} count the column-strict tableaux

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\atop\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}, \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}\atop\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array}\atop\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array},$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\atop\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}\atop\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array}\atop\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}\atop\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array}\atop\begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline \end{array}.$$

Let $\mathcal{T}_n$ be the $\mathbb{Z}$-module of $\mathbb{Z}[\mathfrak{S}_n]$-*traces* (equivalently, $\mathfrak{S}_n$-class functions), the linear functionals $\theta : \mathbb{Z}[\mathfrak{S}_n] \to \mathbb{Z}$ satisfying $\theta(gh) = \theta(hg)$ for all $g, h \in \mathbb{Z}[\mathfrak{S}_n]$. Like the $\mathbb{Z}$-module Λ_n, the trace space $\mathcal{T}_n$ has dimension equal to the number of positive integer partitions of n. The Frobenius $\mathbb{Z}$-module isomorphism (2.5)

$$\begin{aligned} \mathrm{Frob} : \mathcal{T}_n &\to \Lambda_n \\ \theta &\mapsto \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \theta(w) p_{\mathrm{ctype}(w)} \end{aligned} \tag{2.5}$$

defines bijections between the standard bases of Λ_n and of $\mathcal{T}_n$. Schur functions correspond to irreducible characters of $\mathfrak{S}_n$,

$$s_\lambda \leftrightarrow \chi^\lambda,$$

while elementary and homogeneous symmetric functions correspond to induced sign and trivial characters,

$$e_\lambda \leftrightarrow \epsilon^\lambda = \mathrm{sgn}{\uparrow}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}, \qquad h_\lambda \leftrightarrow \eta^\lambda = \mathrm{triv}{\uparrow}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n},$$

where $\mathfrak{S}_\lambda$ is the Young subgroup of $\mathfrak{S}_n$ corresponding to λ. On the other hand, the power sum and monomial bases of Λ_n correspond to bases of $\mathcal{T}_n$ which are not characters. We call these the *power sum* $\{\psi^\lambda \,|\, \lambda \vdash n\}$ and *monomial* $\{\phi^\lambda \,|\, \lambda \vdash n\}$ traces, respectively. These are the bases related to the irreducible character bases by the same matrices of character evaluations and inverse Koskta numbers that relate the power sum and monomial symmetric functions to the Schur functions:

$$\begin{aligned} p_\lambda &= \sum_\mu \chi^\mu(\lambda) s_\mu, \qquad & \psi^\lambda &= \sum_\mu \chi^\mu(\lambda)\chi^\mu, \\ m_\lambda &= \sum_\mu K^{-1}_{\lambda,\mu} s_\mu, \qquad & \phi^\lambda &= \sum_\mu K^{-1}_{\lambda,\mu}\chi^\mu, \end{aligned} \tag{2.6}$$

where $\chi^\mu(\lambda) := \chi^\mu(w)$ for any $w \in \mathfrak{S}_n$ having cycle type equal to λ.

We remark that for e the identity element of $\mathfrak{S}_n$ and for each character $\chi \in \mathcal{T}_n$, the evaluation $\chi(e)$ equals the dimension of the representation corresponding to χ. Formulas for these dimensions are often quite simple. For example, the hook shape irreducible characters appearing in (1.13) satisfy

$$\chi^{k1^{n-k}}(e) = \binom{n-1}{k-1}; \tag{2.7}$$

induced sign and trivial characters indexed by $\lambda = (\lambda_1 \dots, \lambda_\ell)$ satisfy

$$\epsilon^\lambda(e) = \eta^\lambda(e) = \frac{n!}{\lambda_1! \cdots \lambda_\ell!}. \tag{2.8}$$

(See, e.g., [31, §1.2, Thm. 3.10.2].)

3. Immanants and totally nonnegative polynomials

Each of the inequalities listed in Section 1 may be reformulated in terms of a polynomial in matrix entries. Specifically, let $x = (x_{i,j})_{i,j \in [n]}$ be a matrix of n^2 indeterminates, and for $p(x) \in \mathbb{C}[x] := \mathbb{C}[x_{i,j}]_{i,j\in[n]}$ and $A = (a_{i,j})$ an $n \times n$ matrix, define

$$p(A) := p(a_{1,1}, a_{1,2}, \dots, a_{n,n}).$$

While few of the polynomial inequalities in Section 1 specifically refer to the symmetric group, all of them notably involve polynomials which are linear combinations of monomials of the form

$$\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in \mathfrak{S}_n\}.$$

Following [23, 38], we call such polynomials *immanants*. Specifically, given $f : \mathfrak{S}_n \to \mathbb{C}$ define the *f-immanant* to be the polynomial

$$\mathrm{Imm}_f(x) := \sum_{w \in \mathfrak{S}_n} f(w) x_{1,w_1} \cdots x_{n,w_n} \in \mathbb{C}[x]. \tag{3.1}$$

The sign character ($w \mapsto (-1)^{\ell(w)}$) immanant and trivial character ($w \mapsto 1$) immanant are the determinant and permanent,

$$\det(x) = \sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}, \qquad \mathrm{per}(x) = \sum_{w \in \mathfrak{S}_n} x_{1,w_1} \cdots x_{n,w_n}.$$

Simple formulas for the induced sign and trivial character immanants follow from work of Littlewood [23] and Merris–Watkins [26],

$$\begin{aligned} \mathrm{Imm}_{\epsilon^\lambda}(x) &= \sum_{(I_1,\dots,I_r)} \det(x_{I_1,I_1}) \cdots \det(x_{I_r,I_r}), \\ \mathrm{Imm}_{\eta^\lambda}(x) &= \sum_{(I_1,\dots,I_r)} \mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r}), \end{aligned} \tag{3.2}$$

where $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ and the sums are over all sequences of pairwise disjoint subsets of $[n]$ satisfying $|I_j| = \lambda_j$ for $j = 1, \dots, r$. Call such a sequence an *ordered set partition* of $[n]$ *of type* λ. Combining (3.2) and (2.8), we see that the Barrett–Johnson inequalities (1.16) may be restated as

$$\frac{\mathrm{Imm}_{\epsilon^\lambda}(A)}{\epsilon^\lambda(e)} \geq \frac{\mathrm{Imm}_{\epsilon^\mu}(A)}{\epsilon^\mu(e)}, \qquad \text{whenever } \lambda \preceq \mu. \tag{3.3}$$

The simple form (3.2) of induced sign character immanants makes it easy to construct block-diagonal matrices on which some of the immanants vanish but others do not. Specifically, given $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, define $B(\lambda)$ to be the block-diagonal matrix whose ith diagonal block is a $\lambda_i \times \lambda_i$ matrix of ones, e.g.,

$$B(4,2,1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.4}$$

Proposition 3.1. *For* $\lambda, \mu \vdash n$, *we have* $\mathrm{Imm}_{\epsilon^\lambda}(B(\mu^\top)) > 0$ *if* $\lambda \preceq \mu$; $\mathrm{Imm}_{\epsilon^\lambda}(B(\mu^\top)) = 0$ *otherwise.*

Proof. (See also [1, Lem. 2].) Fix $\mu = (\mu_1, \dots, \mu_s)$ and let $J_1, \dots, J_{\mu_1}$ denote the subintervals of $[n]$ corresponding to the diagonal blocks of $B(\mu^\top)$,

$$J_1 = [1, \mu_1^\top], \quad J_2 = [\mu_1^\top + 1,\ \mu_1^\top + \mu_2^\top], \quad \dots, \quad J_{\mu_1} = [n - \mu_{\mu_1}^\top + 1, n]. \tag{3.5}$$

Now consider $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$. Clearly for each ordered set partition $(I_1, \dots, I_r)$ of $[n]$ of type λ, the term

$$\det(B(\mu^\top)_{I_1,I_1}) \cdots \det(B(\mu^\top)_{I_r,I_r}) \tag{3.6}$$

of $\mathrm{Imm}_{\epsilon^\lambda}(B(\mu^\top))$ is nonnegative. This term is positive if the set partition $(I_1, \dots, I_r)$ has the property that for $k = 1, \dots, r$, the elements of I_k belong to λ_k distinct subintervals (3.5); it is zero otherwise. But this property of $(I_1, \dots, I_r)$ is equivalent to the existence of a 0-1, $r \times \mu_1$ matrix $C = (c_{i,j})$ having row sums λ and column sums $\mu^\top$ (with $c_{i,j} = 1$ if I_i contains an element of J_j). The existence of C is known to be equivalent to the majorization condition $\lambda \preceq \mu$. (See, e.g., [37, pp. 291–292].)

Thus if $\lambda \preceq \mu$, the term (3.6) of $\mathrm{Imm}_{\epsilon^\lambda}(B(\mu^\top))$ is positive, all other terms of $\mathrm{Imm}_{\epsilon^\lambda}(B(\mu^\top))$ are nonnegative, and we have $\mathrm{Imm}_{\epsilon^\lambda}(B(\mu^\top)) > 0$. Otherwise all terms are zero and we have $\mathrm{Imm}_{\epsilon^\lambda}(B(\mu^\top)) = 0$. □

Some current interest in immanants and their connection to TNN matrices was inspired by Lusztig's work with canonical bases of quantum groups. (See, e.g., [24].) In particular, one such quantum group has a special basis whose elements can be described in terms of immanants which evaluate nonnegatively on TNN matrices. Call a polynomial $p(x)$ *totally nonnegative* (TNN) if $p(A) \geq 0$ whenever A is a totally nonnegative matrix. There is no known procedure to decide if a given polynomial is TNN.

The formulas (3.2) make it obvious that $\mathrm{Imm}_{\epsilon^\lambda}(x)$ and $\mathrm{Imm}_{\eta^\lambda}(x)$ are TNN polynomials for each partition $\lambda \vdash n$. A stronger result [39, Cor. 3.3] asserts that irreducible character immanants $\mathrm{Imm}_{\chi^\lambda}(x)$ are TNN as well. It is clear that the $\mathfrak{S}_n$-trace immanants

$$\{\mathrm{Imm}_\theta(x) \,|\, \theta \in \mathcal{T}_n, \mathrm{Imm}_\theta(x) \text{ is TNN}\}$$

form a cone, i.e., they are closed under taking nonnegative real linear combinations. Stembridge has conjectured [40, Conj. 2.1] that the extreme rays of this cone are generated by the monomial trace immanants

$$\{\mathrm{Imm}_{\phi^\lambda}(x) \,|\, \lambda \vdash n\}, \tag{3.7}$$

and has shown [40, Prop. 2.3] that the cone of TNN $\mathfrak{S}_n$-trace immanants lies inside of the cone generated by (3.7).

Proposition 3.2. *Each immanant of the form* $\mathrm{Imm}_\theta(x)$ *with* $\theta \in \mathcal{T}_n$ *is a totally nonnegative polynomial only if it is equal to a nonnegative linear combination of monomial trace immanants.*

Thus it is conjectured that an $\mathfrak{S}_n$-trace immanant is TNN if and only if it is equal to a nonnegative linear combination of monomial trace immanants. Indeed it is known that some monomial trace immanants generate extremal rays of the cone of TNN $\mathfrak{S}_n$-trace immanants [7, Thm. 10.3]. (See Theorem 4.8.)

4. The Temperley–Lieb algebra and 2-colorings

Given a complex number ξ, we define the *Temperley–Lieb algebra* $T_n(\xi)$ to be the $\mathbb{C}$-algebra generated by elements $t_1, \ldots, t_{n-1}$ subject to the relations

$$\begin{aligned} t_i^2 &= \xi t_i, && \text{for } i = 1, \ldots, n-1, \\ t_i t_j t_i &= t_i, && \text{if } |i-j| = 1, \\ t_i t_j &= t_j t_i, && \text{if } |i-j| \geq 2. \end{aligned}$$

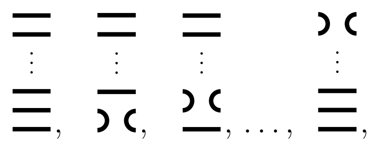
When $\xi = 2$ we have the isomorphism $T_n(2) \cong \mathbb{C}[\mathfrak{S}_n]/(e + s_1 + s_2 + s_1s_2 + s_2s_1 + s_1s_2s_1)$. (See e.g. [11], [13, § 2.1, § 2.11], [41, § 7].) Specifically, this isomorphism is given by

$$\begin{aligned} \sigma : \mathbb{C}[\mathfrak{S}_n] &\to T_n(2), \\ s_i &\mapsto t_i - 1. \end{aligned} \tag{4.1}$$

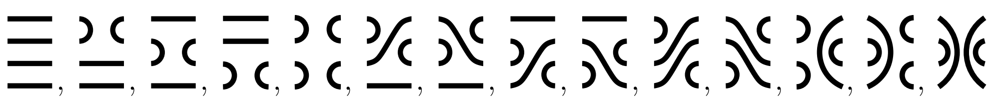
Let $\mathcal{B}_n$ be the multiplicative monoid generated by $t_1, \dots, t_{n-1}$ when $\xi = 1$, which is also called the standard basis of $T_n(\xi)$. It is known that $|\mathcal{B}_n|$ is the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, and that elements of $\mathcal{B}_n$ correspond bijectively to elements w of $\mathfrak{S}_n$ whose one-line notation $w_1 \cdots w_n$ is 321-*avoiding*, i.e., contains no 3-letter decreasing subsequence $w_{j_1} > w_{j_2} > w_{j_3}$ with $j_1 < j_2 < j_3$. Furthermore it is straightforward to show that the bijection between these sets given by $s_{i_1} \cdots s_{i_\ell} \mapsto t_{i_1} \cdots t_{i_\ell}$ maps reduced expressions to reduced expressions.

Diagrams of the basis elements of $T_n(\xi)$, made popular by Kauffman [18, § 4] are (undirected) graphs with $2n$ vertices and n edges. The identity and generators $1, t_1, \dots, t_{n-1}$ are represented by

(4.2)

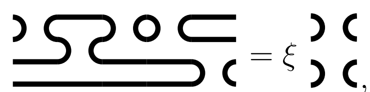


and multiplication of these elements corresponds to concatenation of diagrams, with cycles contributing a factor of ξ. For instance, the fourteen basis elements of $T_4(\xi)$ are



and the equality $t_3t_2t_3t_3t_1 = \xi t_1t_3$ in $T_4(\xi)$ is represented by

(4.3)



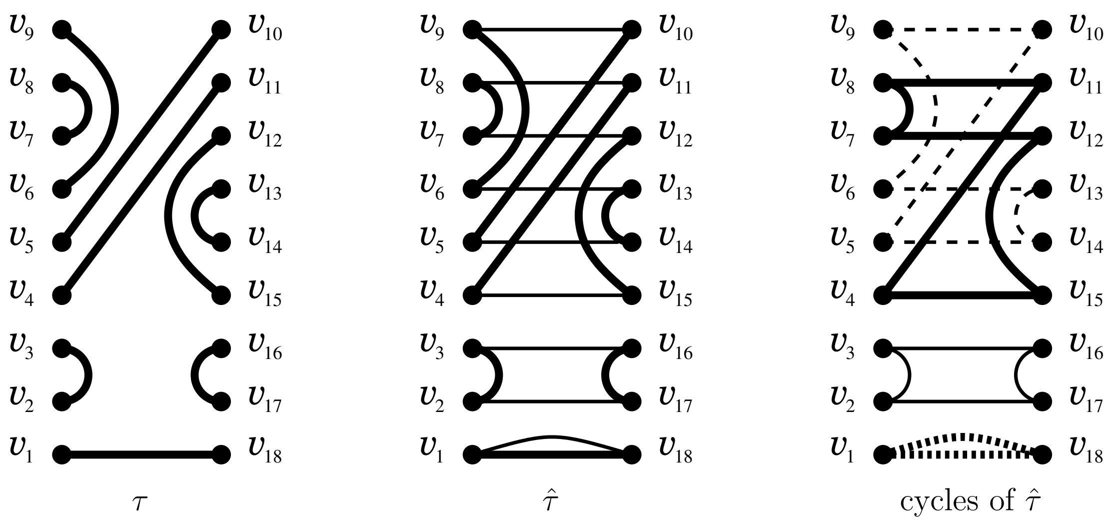
with the "bubble" becoming the scalar multiple ξ. We will identify each element $\tau \in \mathcal{B}_n$ with its Kauffman diagram. Additionally, we will label vertices $v_1, \dots, v_n$, clockwise from the lower left, and will define the *height* of a vertex to be

$$\mathrm{hgt}(v_i) = \begin{cases} i & \text{if } 1 \leq i \leq n, \\ 2n+1-i & \text{if } n+1 \leq i \leq 2n. \end{cases}$$

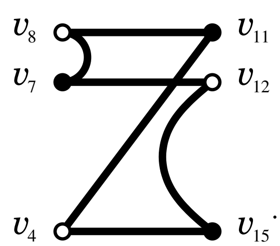
For instance, the first diagram in (4.5) shows the element $t_7t_6t_8t_5t_7t_4t_6t_5t_2 \in \mathcal{B}_9$ with vertex labels $v_1, \dots, v_{18}$; vertex heights are $1, \dots, 9$ from bottom to top. It is easy to see that for all $\tau \in \mathcal{B}_n$, each edge (v_i, v_j) satisfies

$$\mathrm{hgt}(v_i) - \mathrm{hgt}(v_j) = \begin{cases} 1 \pmod 2 & \text{if } i, j \leq n \text{ or } i, j \geq n+1, \\ 0 \pmod 2 & \text{otherwise.} \end{cases} \tag{4.4}$$

Define $\hat{\tau}$ to be the graph obtained from τ by adding edges $(i, 2n+1-i)$ for all i (even if such an edge already exists). Since each vertex in $\hat{\tau}$ has degree 2, it is clear that this graph is a disjoint union of cycles. For example, corresponding to the element τ on the left of (4.5) we have $\hat{\tau}$ to its right, and the decomposition of this graph into four disjoint cycles.

(4.5)


The Temperley–Lieb algebra $T_n(2)$ sometimes arises in the 2-coloring of combinatorial objects, for instance in the following case. Define a *principal coloring* of $\tau \in \mathcal{B}_n$ to be a map κ : vertices$(\tau) \to \{\text{black}, \text{white}\}$ which is a *proper* coloring of $\hat{\tau}$, i.e., $\text{color}(v_i) \neq \text{color}(v_{2n+1-i})$ for $i = 1, \ldots, n$ and $\text{color}(v_i) \neq \text{color}(v_j)$ if v_i and v_j are adjacent in τ. Let (τ, κ) denote the graph τ with its vertices colored by κ. Principal colorings of τ are closely related to cycles in $\hat{\tau}$. Given τ principally colored, it is clear that vertex colors must alternate along each cycle of $\hat{\tau}$. Moreover, it is also true that vertex colors must alternate as one views the vertices of the cycle in clockwise order, ignoring the edges of that cycle. For example, consider a 2-coloring of the cycle $(v_4, v_{11}, v_8, v_7, v_{12}, v_{15})$ of $\hat{\tau}$ in (4.5),



Proposition 4.1. *If $\hat{\tau}$ is a single cycle, then there are two principal colorings of τ. In each of them, the vertices of odd index and of even index have opposite colors.*

Proof. This is clear if $n = 1$. Assume that $n \geq 2$ and suppose we have a principal coloring of τ. It suffices to show that $\text{color}(v_i) \neq \text{color}(v_{i+1})$ for $i = 1, \ldots, n-1$. Consider a subpath of the cycle of $\hat{\tau}$ from v_i to v_{i+1}. Since v_i, v_{i+1} belong to the left column, this path contains an even number of edges having vertices in different columns. By (4.4), for each such edge (v_j, v_k) the difference $\text{hgt}(v_j) - \text{hgt}(v_k)$ is even. Again by (4.4), since $\text{hgt}(v_{i+1}) - \text{hgt}(v_i) = 1$ the path contains an odd number of edges with vertices in a single column. Thus the path consists of an odd number of edges, and $\text{color}(v_i) \neq \text{color}(v_{i+1})$. □

It is easy to see that if κ is a principal coloring of $\tau \in \mathcal{B}_n$ and if $\hat{\tau}$ consists of a single cycle, then we have

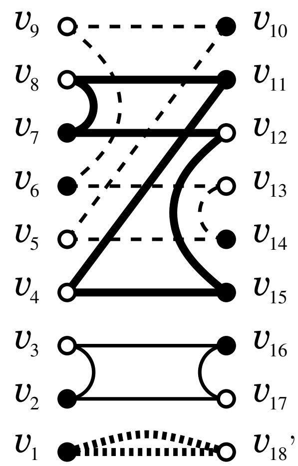
(4.6)
$$|\#(\text{white vertices on left of } (\tau,\kappa)) - \#(\text{white vertices on right of } (\tau,\kappa))| = \begin{cases} 0 & \text{if } n \text{ even}, \\ 1 & \text{if } n \text{ odd}. \end{cases}$$

In this situation we call (τ, κ) *balanced* if n is even, and *unbalanced* otherwise. More specifically, we call (τ, κ) *left-unbalanced* (*right-unbalanced*) if it has more white vertices on the left (right).

Now consider $\tau \in \mathcal{B}_n$ with $\hat{\tau}$ a disjoint union of cycles $C_1, \dots, C_d$, and κ a principal coloring of τ. Define

$$\begin{aligned} \alpha = \alpha(\tau,\kappa) &:= \#\{i \,|\, (\tau_{C_i}, \kappa) \text{ is right-unbalanced}\}, \\ \beta = \beta(\tau,\kappa) &:= \#\{i \,|\, (\tau_{C_i}, \kappa) \text{ is left-unbalanced}\}. \end{aligned} \tag{4.7}$$

For example, recall τ and $\hat{\tau}$ from (4.5). The proper 2-coloring κ of $\hat{\tau}$,



corresponds to a principal coloring of τ which satisfies $\alpha(\tau, \kappa) = 1$, $\beta(\tau, \kappa) = 2$, and $d = 4$. Also note that there is one balanced cycle.

It is easy to characterize the colorings κ of a given Temperley–Lieb basis element τ for which the numbers α, β in (4.7) are constant.

Lemma 4.2. *Let (τ, κ), (τ, κ') be principal colorings having j white vertices on the left, for some $j \leq \lfloor \frac{n}{2} \rfloor$. Then we have*

$$\alpha(\tau,\kappa) = \alpha(\tau,\kappa'), \qquad \beta(\tau,\kappa) = \beta(\tau,\kappa').$$

Proof. It is easy to see that the number of cycles of $\hat{\tau}$ of cardinality 2 (mod 4) is

$$\alpha(\tau,\kappa) + \beta(\tau,\kappa) = \alpha(\tau,\kappa') + \beta(\tau,\kappa'). \tag{4.8}$$

On the other hand, we have by our assumption that the number of white vertices on the right of (τ, κ) (or of (τ, κ')) minus the number of white vertices on the left is

$$(n - j) - j = n - 2j. \tag{4.9}$$

Since each balanced subgraph (τ_{C_i}, κ) (or (τ_{C_i}, κ')) contributes the same number of white vertices to both sides and each unbalanced subgraph contributes one more white vertex to one side than to the other (4.6), we see that (4.9) equals

$$\alpha(\tau,\kappa) - \beta(\tau,\kappa) = \alpha(\tau,\kappa') - \beta(\tau,\kappa'). \tag{4.10}$$

Combining (4.8) and (4.10), we have the desired equalities. □

This lemma allows us to define

$$\alpha(\tau,j) := \alpha(\tau,\kappa) \quad \text{and} \quad \beta(\tau,j) := \beta(\tau,\kappa) \tag{4.11}$$

if there exists a principal coloring of τ in which exactly j vertices on the left are white.

Just as $T_n(2)$ is related to 2-coloring, it is related to total nonnegativity of polynomials in the subspace of immanants of the form

$$\mathrm{span}_{\mathbb{R}}\{\det(x_{I,I})\det(x_{I^c,I^c}) \,|\, I \subseteq [n]\}. \tag{4.12}$$

To state this relationship explicitly, we define an immanant $\mathrm{Imm}_\tau(x)$ for each $\tau \in \mathcal{B}_n$ via the function [28, §3]

$$\begin{aligned} f_\tau : \mathbb{C}[\mathfrak{S}_n] &\to \mathbb{C} \\ w &\mapsto \text{ coefficient of } \tau \text{ in } \sigma(w), \end{aligned} \tag{4.13}$$

(extended linearly). To economize notation, we will write Imm_τ instead of Imm_{f_τ},

$$\mathrm{Imm}_\tau(x) = \sum_{w \in \mathfrak{S}_n} f_\tau(w) x_{1,w_1} \cdots x_{n,w_n}. \tag{4.14}$$

For example, consider the case $n = 3$ and $\tau = t_1 \in \mathcal{B}_3$. Extracting the coefficients of t_1 in the expressions

$$\begin{gathered} \sigma(e) = 1, \qquad \sigma(s_1) = t_1 - 1, \qquad \sigma(s_2) = t_2 - 1, \\ \sigma(s_1s_2) = (t_1-1)(t_2-1) = t_1t_2 - t_1 - t_2 + 1, \\ \sigma(s_2s_1) = (t_2-1)(t_1-1) = t_2t_1 - t_1 - t_2 + 1, \\ \sigma(s_1s_2s_1) = (t_1-1)(t_2-1)(t_1-1) = t_1 + t_2 - t_1t_2 - t_2t_1 - 1, \end{gathered}$$

we have $f_{t_1}(e) = 0$, $f_{t_1}(s_1) = 1$, $f_{t_1}(s_2) = 0$, $f_{t_1}(s_1s_2) = f_{t_1}(s_2s_1) = -1$, $f_{t_1}(s_1s_2s_1) = 1$, and

$$\mathrm{Imm}_{t_1}(x) = x_{1,2}x_{2,1}x_{3,3} - x_{1,3}x_{2,1}x_{3,2} - x_{1,2}x_{2,3}x_{3,1} + x_{1,3}x_{2,2}x_{3,1}.$$

Note that in the special case $\tau = 1$, the function f_τ maps a permutation w to $(-1)^{\mathrm{INV}(w)}$. Thus the determinant is a Temperley–Lieb immanant:

$$\det(x) = \mathrm{Imm}_1(x).$$

It was shown in [28] that Temperley–Lieb immanants form a basis of the space (4.12), and moreover that they are TNN. Furthermore, they are the extreme rays of the cone of TNN immanants in this space [28, Thm. 10.3].

Proposition 4.3. *Each immanant of the form*

$$\mathrm{Imm}_f(x) = \sum_{\substack{I,J \subseteq [n] \\ |I|=|J|}} c_{I,J} \det(x_{I,J}) \det(x_{\overline{I},\overline{J}}) \tag{4.15}$$

is a totally nonnegative polynomial if and only if it is equal to a nonnegative linear combination of Temperley–Lieb immanants.

In fact each product of complementary minors is a 0-1 linear combination of Temperley–Lieb immanants [28, Prop. 4.4].

Theorem 4.4. *For $I \subseteq [n]$ we have*

$$\det(x_{I,I})\det(x_{I^c,I^c}) = \sum_{\tau \in \mathcal{B}_n} b_\tau \mathrm{Imm}_\tau(x), \tag{4.16}$$

where

(4.17)

$$b_\tau = \begin{cases} 1 & \textit{if there is a principal coloring of } \tau \textit{ with } \{v_i \,|\, i \in I\} \textit{ white, } \{v_i \,|\, i \in [n] \smallsetminus I\} \textit{ black,} \\ 0 & \textit{otherwise.} \end{cases}$$

By (3.2) we have that for each two-part partition $\lambda = (n-j, j)$ of n, the corresponding induced sign character immanant belongs to (4.12). Furthermore, we have the following explicit expansion of these immanants in terms of the Temperley–Lieb immanant basis.

Theorem 4.5. *For $j = 0, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, we have*

$$\mathrm{Imm}_{\epsilon^{n-j,j}}(x) = \sum_{\tau \in \mathcal{B}_n} d_{j,\tau} \mathrm{Imm}_\tau(x),$$

where $d_{j,\tau}$ is the number of principal colorings of τ having j white vertices on the left. Explicitly, if such a coloring exists, then $d_{j,\tau} = 2^{d-\alpha-\beta}\binom{\alpha+\beta}{\alpha}$, where d is the number of cycles of $\hat{\tau}$, and $\alpha = \alpha(\tau, j)$, $\beta = \beta(\tau, j)$ are defined as in (4.7), (4.11).

Proof. The combinatorial description follows immediately from Theorem 4.4, and it remains to show the desired explicit formula for $d_{j,\tau}$. Now suppose $\hat{\tau}$ consists of cycles $C_1, \ldots, C_d$ and consider the proper 2-colorings of these cycles that combine to form a principal coloring of τ having j white vertices on the left. For each of the $d - \alpha - \beta$ balanced induced subgraphs τ_{C_i}, both of the two possible colorings contribute $\frac{|C_i|}{2}$ white vertices to the left column of τ. There are $2^{d-\alpha-\beta}$ colorings of the corresponding vertices. Besides these, each unbalanced subgraph τ_{C_i} must be colored so that it contributes more white vertices either to the left or to the right. We choose α of these $\alpha + \beta$ unbalanced subgraphs to be right-unbalanced in $\binom{\alpha+\beta}{\alpha}$ ways. □

More generally, for any function $f : \mathbb{C}[\mathfrak{S}_n] \to \mathbb{C}$, with $\mathrm{Imm}_f(x)$ belonging to the space (4.12), we can determine the coefficients in

$$\mathrm{Imm}_f(x) = \sum_{\tau \in \mathcal{B}_n} c_\tau \mathrm{Imm}_\tau(x) \tag{4.18}$$

by applying f to appropriate elements of $\mathbb{C}[\mathfrak{S}_n]$.

Proposition 4.6. *Suppose that $f : \mathbb{C}[\mathfrak{S}_n] \to \mathbb{C}$ satisfies (4.18), and that $\tau \in \mathcal{B}_n$ has a reduced expression $t_{i_1} \cdots t_{i_\ell}$. Then the coefficient c_τ in (4.18) is given by*

$$c_\tau = f((e + s_{i_1}) \cdots (e + s_{i_\ell})). \tag{4.19}$$

Proof. For each element $\tau \in \mathcal{B}_n$ with reduced expression $t_{i_1} \cdots t_{i_\ell}$, define the 321-avoiding permutation $w(\tau)$ to be the product $s_{i_1} \cdots s_{i_\ell}$ of generators in $\mathfrak{S}_n$, as in the discussion following (4.1). By [29, Prop. 3.2], [30, Eqn. (3.8)], the coefficient c_τ equals $f(C'_{w(\tau)})$, where

$\{C'_v \,|\, v \in \mathfrak{S}_n\}$ is the (signless) Kazhdan–Lusztig basis of $\mathbb{C}[\mathfrak{S}_n]$, the $(q = 1)$-specialization of the Hecke algebra basis defined in [19].

If $C'_{w(\tau)} = (e + s_{i_1}) \cdots (e + s_{i_\ell})$, then we have (4.19). (See [3, Thm. 1] for a characterization of permutations $w(\tau)$ satisfying this equality.) Otherwise, define the two-sided ideal

$$K := (e + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1) \tag{4.20}$$

of $\mathbb{C}[\mathfrak{S}_n]$, which satisfies

$$K = \mathrm{span}_{\mathbb{C}}\{C'_v \,|\, v \text{ does not avoid the pattern } 321\}. \tag{4.21}$$

(See, e.g., [11, Prop. 3.1.1].) By [11, Thm. 3.8.2], we have that $(e + s_{i_1}) \cdots (e + s_{i_\ell})$ belongs to the coset $C'_{w(\tau)} + K$. Now use (4.14) and (4.18) to express f in terms of the functions $\{f_\tau \,|\, \tau \in \mathcal{B}_n\}$:

$$f = \sum_{\tau \in \mathcal{B}_n} c_\tau f_\tau. \tag{4.22}$$

Each function f_τ satisfies

$$f_\tau(C'_v) = \begin{cases} 1 & \text{if } v = w(\tau), \\ 0 & \text{otherwise.} \end{cases} \tag{4.23}$$

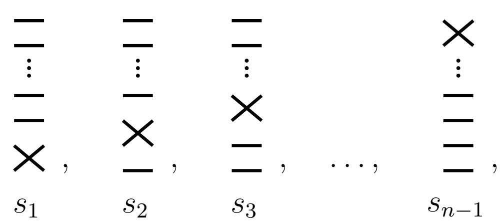
(See [29, Eqn. (2.3), Prop. 3.2].) In particular, whenever v does not avoid the pattern 321, we have $f_\tau(C'_v) = 0$ for all $\tau \in \mathcal{B}_n$ and $f(C'_v) = 0$ as well. It follows that we have

$$f((e + s_{i_1}) \cdots (e + s_{i_\ell})) = f(C'_{w(\tau)}), \tag{4.24}$$

as desired. □

Combining (2.4) (inverting the matrix $M_{\lambda,\mu}$) and (3.2), we see that monomial immanants $\mathrm{Imm}_{\phi^\mu}(x)$ indexed by partitions of the form $\mu = 2^c 1^d \vdash n$ are examples of immanants belonging to the space (4.12). We may therefore expand these in the Temperley–Lieb immanant basis of (4.12) by computing coefficients as in (4.19). In [7], these coefficients were computed combinatorially in terms of *wiring diagrams*, planar networks formed by concatenating a sequence of basic planar networks

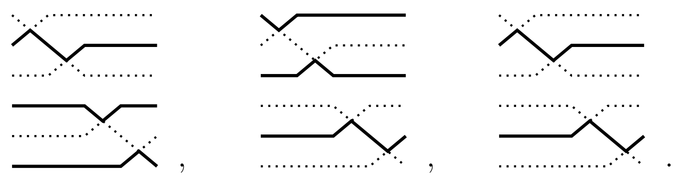
(4.25)



in the order dictated by a fixed (but not necessarily reduced) expression $s_{i_1} \cdots s_{i_k}$. As with Kauffman diagrams, we label the $2n$ boundary vertices of wiring diagrams by $v_1, \ldots, v_{2n}$ clockwise from the lower left, as in (4.5), and often do not draw or label these vertices, as in (4.2) – (4.3).

Define a *path family (of type e) covering* G to be a sequence $\pi = (\pi_1, \ldots, \pi_n)$, with π_i a path in G from v_i to v_{2n+1-i}, and every edge of G belonging to exactly one path. Define a 2-*coloring* of the path family to be an assignment $\pi_i \mapsto \{\text{black, white}\}$ of colors to paths so that equally colored paths do not intersect. Define $\Pi_{p_1,p_2}(G)$ to be the set of all 2-colorings of path families covering G with p_1 white paths and p_2 black paths. For example, when G is the wiring diagram of $s_5s_4s_2s_1$ in $\mathfrak{S}_5$, there are two elements of $\Pi_{3,3}(G)$ and one element of $\Pi_{4,2}(G)$:

(4.26)



The sets $\Pi_{p_1,p_2}(G)$ with p_1 as large as possible appear in the formula for $\phi^\mu((e+s_{i_1})\cdots(e+s_{i_\ell}))$ when $\mu_1 \leq 2$ [7, Thm. 10.3].

Proposition 4.7. *Let $\tau \in \mathcal{B}_n$ have reduced expression $t_{i_1}\cdots t_{i_\ell}$, let G be the wiring diagram of the $\mathfrak{S}_n$-expression $s_{i_1}\cdots s_{i_\ell}$, and fix $2^c1^d \vdash n$. Then we have*

$$\phi^{2^c1^d}((e+s_{i_1})\cdots(e+s_{i_\ell})) = \begin{cases} |\Pi_{c+d,c}(G)| & \text{if } \Pi_{c+d+1,c-1}(G) = \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{4.27}$$

We can simplify this formula somewhat by considering 2-colorings of Kauffman diagrams instead of 2-colorings of path families in wiring diagrams. For each partition $2^c1^d \vdash n$ define $P(2^c1^d)$ to be the set of all basis elements $\tau \in \mathcal{B}_n$ satisfying

(1) there exists a principal coloring of τ with $c+d$ white vertices on the left
(2) there exists no principal coloring of τ with $c+d+1$ white vertices on the left.

Theorem 4.8. *For $\mu = 2^c1^d \vdash n$, the polynomial $\mathrm{Imm}_{\phi^\mu}(x)$ is TNN. In particular we have*

$$\mathrm{Imm}_{\phi^\mu}(x) = \sum_{\tau \in P(\mu)} b_{\mu,\tau}\mathrm{Imm}_\tau(x),$$

where $b_{\mu,\tau} = 2^{\#\text{cycles of } \hat{\tau} \text{ of cardinality } 0 \pmod 4}$.

Proof. Let $t_{i_1}\cdots t_{i_\ell}$ be a minimum-length expression for τ, and let G be the wiring diagram of $s_{i_1}\cdots s_{i_\ell}$. By Proposition 4.6, the coefficient $b_{\mu,\tau}$ equals $\phi^\mu((e+s_{i_1})\cdots(e+s_{i_\ell}))$, which by Proposition 4.7 is the cardinality of the set $\Pi_{c+d,c}(G)$ of path families, assuming c, d satisfy the conditions in (4.27). It is straightforward to show [33, §2] that such path families correspond bijectively to principal colorings (τ,κ) of τ with $c+d$ white and c black vertices on the left, assuming that no principal coloring of τ has $c+d+1$ white vertices on the left.

Theorem 4.5 gives the number of such colorings in terms of the numbers $\alpha = \alpha(\tau,\kappa)$ of right-unbalanced and $\beta = \beta(\tau,\kappa)$ left-unbalanced cycles of (τ,κ), respectively:

$$2^{\#\text{cycles of } \hat{\tau} - \alpha - \beta}\binom{\alpha+\beta}{\beta} = 2^{\#\text{ balanced cycles of } \hat{\tau}}\binom{\alpha+\beta}{\beta}. \tag{4.28}$$

It is clear that each principal coloring (τ,κ) above has no right-unbalanced cycle, for if it did, then we could switch the color of each vertex in this cycle to produce a principal coloring (τ,κ') having $c+d+1$ white verties on the left. Thus we have $\alpha = 0$. Furthermore, a cycle of $\hat{\tau}$ is balanced if and only if its number of vertices is a multiple of 4. Thus (4.28) equals the desired expression. □

5. Main results

Fischer's inequalities (1.5) naturally lead to the questions of how the products

$$\det(A_{I,I})\det(A_{I^c,I^c}) \tag{5.1}$$

of pairs of complementary minors compare to one another, and of whether a greater cardinality difference $|I^c| - |I|$ tends to make the product (5.1) greater or smaller. This second question led Barrett and Johnson [1] to consider the average value of such products when cardinalities are fixed,

$$\frac{1}{\binom{n}{k}} \sum_{\substack{I\subseteq[n]\\|I|=k}} \det(A_{I,I})\det(A_{I^c,I^c}). \tag{5.2}$$

They found that for PSD matrices, a smaller cardinality difference makes the average product of minors greater [1, Thm. 1]. We give two proofs that the same is true for TNN matrices.

Theorem 5.1. *For all TNN matrices A and for $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, we have*

$$\frac{\sum_{|I|=k} \det(A_{I,I})\det(A_{I^c,I^c})}{\binom{n}{k}} \leq \frac{\sum_{|I|=k+1} \det(A_{I,I})\det(A_{I^c,I^c})}{\binom{n}{k+1}}. \tag{5.3}$$

First proof. By (3.2) it is equivalent to show that the polynomial

$$\frac{\mathrm{Imm}_{\epsilon^{n-k-1,k+1}}(x)}{\binom{n}{k+1}} - \frac{\mathrm{Imm}_{\epsilon^{n-k,k}}(x)}{\binom{n}{k}} \tag{5.4}$$

is totally nonnegative. By Theorem 4.5, this difference belongs to the span of the Temperley–Lieb immanants. In particular, we may multiply the difference by $n!/(k!(n-k-1)!)$ to obtain

$$(k+1)\mathrm{Imm}_{\epsilon^{n-k-1,k+1}}(x) - (n-k)\mathrm{Imm}_{\epsilon^{n-k,k}}(x) = \sum_{\tau\in\mathcal{B}_n} c_\tau \mathrm{Imm}_\tau(x), \tag{5.5}$$

where

$$c_\tau = (k+1)d_{k+1,\tau} - (n-k)d_{k,\tau}. \tag{5.6}$$

and $d_{k+1,\tau}$, $d_{k,\tau}$ are defined in terms of proper colorings of τ as in Theorem 4.5.

We claim that $c_\tau \geq 0$. Suppose we have a proper coloring of τ in which k vertices on the left are white. Let $\alpha = \alpha(\tau, k)$ and $\beta = \beta(\tau, k)$ be the numbers of right-unbalanced and left-unbalanced subgraphs of τ, respectively, as in (4.11).

Let d be the number of cycles in $\hat{\tau}$ as in Theorem 4.5. Then by Theorem 4.5 we have that (5.6) equals

$$\begin{aligned}(k+1)d_{k+1,\tau} - (n-k)d_{k,\tau} &= (k+1)2^{d-\alpha-\beta}\binom{\alpha+\beta}{\alpha-1} - (n-k)2^{d-\alpha-\beta}\binom{\alpha+\beta}{\alpha} \\ &= 2^{d-\alpha-\beta}\left((k+1)\binom{\alpha+\beta}{\alpha-1} - (n-k)\binom{\alpha+\beta}{\alpha}\right) \\ &= 2^{d-\alpha-\beta}\frac{(\alpha+\beta)!}{(\alpha-1)!\beta!}\left(\frac{k+1}{\beta+1} - \frac{n-k}{\alpha}\right).\end{aligned} \tag{5.7}$$

Then using (4.9) and (4.10) to substitute $\alpha = n - 2k + \beta$, we have that the final difference of fractions in (5.7) equals

$$\frac{(k+1)(n-2k+\beta)-(n-k)(\beta+1)}{(\beta+1)(n-2k+\beta)} = \frac{(n-2k-1)(k-\beta)}{(\beta+1)(n-2k+\beta)}.$$

The denominator of this expression is clearly positive. Furthermore, the numerator is nonnegative because of the bounds on k and the definition of β. □

Second proof. Again we show that the polynomial (5.4) is TNN. Multiplying the polynomial by $\frac{n!}{k!(n-k-1)!}$ and expanding it in the monomial immanant basis of the trace immanant space, we have

$$(k+1)\mathrm{Imm}_{\epsilon^{(n-k-1,k+1)}}(x) - (n-k)\mathrm{Imm}_{\epsilon^{(n-k,k)}}(x) = \sum_{\mu \vdash n} c_\mu \mathrm{Imm}_{\phi^\mu}(x), \tag{5.8}$$

where by (2.5) – (2.6) the integers $\{c_\mu \,|\, \mu \vdash n\}$ satisfy

$$(k+1)e_{(n-k-1,k+1)} - (n-k)e_{(n-k,k)} = \sum_{\mu \vdash n} c_\mu m_\mu. \tag{5.9}$$

We claim that $c_\mu \geq 0$ for all relevant partitions μ. To see this, consider the special case of (2.4) for two-part partitions:

$$e_{(n-k,k)} = \sum_{a=0}^{k} M_{(n-k,k),2^a1^{n-2a}} m_{2^a1^{n-2a}}.$$

This formula implies that each partition μ indexing a nonzero coefficient in (5.8) has the form $\mu = 2^a1^{n-2a}$. Recall that this coefficient counts column-strict Young tableaux having two columns, of sizes $n-k$ and $k \leq n-k$, containing the letters $1, 1, 2, 2, \ldots, a, a, a+1, \ldots, n-a$. Column-strictness forces row i of such a tableau to contain only the letter i, for $i = 1, \ldots, a$, and thus the tableau is completely determined by the subset of the letters $\{a+1, \ldots, n-a\}$ appearing in rows $a+1, \ldots, k$ of column 2. Thus there are $\binom{n-2a}{k-a}$ such tableaux, and we have

$$M_{(n-k,k),2^a1^{n-2a}} = \binom{n-2a}{k-a}, \quad \text{for } a = 0, \ldots, k.$$

Similarly, the coefficient of $m_{2^a1^{n-2a}}$ in the monomial expansion of $e_{(n-k-1,k+1)}$ is

$$M_{(n-k-1,k+1),2^a1^{n-2a}} = \binom{n-2a}{k+1-a}, \quad \text{for } a = 0, \ldots, k+1.$$

It follows that in the right-hand side of (5.9), the coefficient of $m_{2^a1^{n-2a}}$ is

$$c_{2^a1^{n-2a}} = \begin{cases} (k+1)\binom{n-2a}{k+1-a} & \text{if } a = k+1, \\ (k+1)\binom{n-2a}{k+1-a} - (n-k)\binom{n-2a}{k-a} & \text{if } 0 \leq a \leq k. \end{cases}$$

If $a = k+1$, this is clearly positive; otherwise, the expression equals

$$\frac{(n-2a)!}{(k-a)!(n-k-a-1)!}\left(\frac{k+1}{k-a+1} - \frac{n-k}{n-k-a}\right)$$
$$= \frac{(n-2a)!}{(k-a)!(n-k-a-1)!}\left(\frac{a(n-2k-1)}{(k-a+1)(n-k-a)}\right).$$

But this also is positive, since $a \le k < \lfloor \frac{n}{2} \rfloor < n-k$. Now by Theorem 4.8 we have that (5.8) is a totally nonnegative polynomial, and that (5.4) is as well. □

Consideration of the average value of products of two complentary minors (5.2) naturally led Barrett and Johnson to consider average values of products of arbitrarily many minors having any fixed cardinality sequences $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$,

$$\frac{1}{\binom{n}{\lambda_1,\dots,\lambda_r}} \sum_{(I_1,\dots,I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}), \tag{5.10}$$

where the sum is over all ordered set partitions of $[n]$ of type λ and the multinomial coefficient $\binom{n}{\lambda_1,\dots,\lambda_r} = \frac{n!}{\lambda_1!\cdots\lambda_r!}$ is the number of such ordered set partitions. They found that for PSD matrices, partitions which appear lower in the majorization order lead to greater average products [1, Thm. 2]. We show that the same is true for TNN matrices.

Theorem 5.2. *Fix partitions $\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1 \dots, \mu_s)$ of n. We have*

$$\lambda_1! \cdots \lambda_r! \sum_{(I_1,\dots,I_r)} \det(A_{I_1,I_1}) \cdots \det(A_{I_r,I_r}) \;\ge\; \mu_1! \cdots \mu_s! \sum_{(J_1,\dots,J_s)} \det(A_{J_1,J_1}) \cdots \det(A_{J_s,J_s}) \tag{5.11}$$

for all TNN matrices if and only if $\lambda \preceq \mu$.

Proof. First suppose that we have $\lambda \preceq \mu$. By (2.3) it suffices to consider λ, μ having equal parts except

$$\mu_i = \lambda_i + 1, \qquad \mu_j = \lambda_j - 1 \tag{5.12}$$

for some $i < j$. (Thus we may assume $s = r$ and will allow $\mu_s = 0$.) Let $\nu = (\nu_1, \dots, \nu_{r-2})$ be the partition of $n - \lambda_i - \lambda_j$ consisting of all other parts.

As in the proofs of Theorem 5.1, we observe that each sum of products of minors is an induced sign character immanant (3.2). Thus the inequality (5.11) is equivalent to the total nonnegativity of the polynomial

$$\lambda_1! \cdots \lambda_r! \,\mathrm{Imm}_{\epsilon^\lambda}(x) - \mu_1! \cdots \mu_r! \,\mathrm{Imm}_{\epsilon^\mu}(x), \tag{5.13}$$

which by (5.12) and our definition of ν equals

$$\nu_1! \cdots \nu_r! \,(\lambda_j - 1)! \,\lambda_i! \,(\lambda_j \mathrm{Imm}_{\epsilon^\lambda}(x) - (\lambda_i + 1)\mathrm{Imm}_{\epsilon^\mu}(x)). \tag{5.14}$$

Now observe that (3.2) allows us to write

$$\begin{aligned} \mathrm{Imm}_{\epsilon^\lambda}(x) &= \sum_{\substack{J\subseteq[n] \\ |J|=|\nu|}} \mathrm{Imm}_{\epsilon^\nu}(x_{J,J}) \mathrm{Imm}_{\epsilon^{(\lambda_i,\lambda_j)}}(x_{J^c,J^c}), \\ \mathrm{Imm}_{\epsilon^\mu}(x) &= \sum_{\substack{J\subseteq[n] \\ |J|=|\nu|}} \mathrm{Imm}_{\epsilon^\nu}(x_{J,J}) \mathrm{Imm}_{\epsilon^{(\lambda_i+1,\lambda_j-1)}}(x_{J^c,J^c}). \end{aligned} \tag{5.15}$$

It follows that (5.14) equals

$$\nu_1! \cdots \nu_r! \,(\lambda_j - 1)! \,\lambda_i! \sum_{\substack{J\subseteq[n] \\ |J|=|\nu|}} \mathrm{Imm}_{\epsilon^\nu}(x_{J,J}) \big(\lambda_j \mathrm{Imm}_{\epsilon^{(\lambda_i,\lambda_j)}}(x_{J^c,J^c}) - (\lambda_i + 1)\mathrm{Imm}_{\epsilon^{(\lambda_i+1,\lambda_j-1)}}(x_{J^c,J^c})\big).$$

By Theorem 5.1, or more precisely (5.5), this polynomial is TNN.

Now suppose that we have $\lambda \not\preceq \mu$. Setting $A = B(\mu^\top)$ as defined before (3.4), we have by Proposition 3.1 that the left-hand side of (5.11) is 0, while the right-hand side is positive. □

6. Open problems

A casual glance at the results in Section 1 exposes several holes in our understanding of polynomial inequalities satisfied by HPSD and/or TNN matrices. Perhaps the first obvious omission after Equations (1.3) – (1.7) is the natural permanental analog of (1.7),

$$\text{per}(A_{I\cup J,I\cup J})\,\text{per}(A_{I\cap J,I\cap J}) \geq \text{per}(A_{I,I})\,\text{per}(A_{J,J}) \tag{6.1}$$

for all $I, J \subseteq [n]$. Somewhat surprisingly, this inequality holds neither for HPSD matrices nor for TNN matrices: indeed, consider the HPSD, TNN matrix

$$A = \begin{bmatrix} 8 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 8 \end{bmatrix},$$

subsets $\{1, 2\}$, $\{2, 3\}$, and computations

$$\text{per}(A_{123,123})\text{per}(A_{2,2}) = 137 < \text{per}(A_{12,12})\text{per}(A_{23,23}) = 144.$$

Apparently we cannot simply reverse inequalities when we replace determinants with permanents. This makes the problem of extending (1.10) even more intriguing.

Problem 6.1. *Characterize the* 8*-tuples* $(I, J, K, L, I', J', K', L')$ *of subsets which satisfy any of the following.*

(1) $\text{per}(A_{I,I'})\text{per}(A_{J,J'}) \leq \text{per}(A_{K,K'})\text{per}(A_{L,L'})$ *for all PSD matrices A,*
(2) $\text{per}(A_{I,I'})\text{per}(A_{J,J'}) \leq \text{per}(A_{K,K'})\text{per}(A_{L,L'})$ *for all TNN matrices A,*
(3) $\det(A_{I,I'})\det(A_{J,J'}) \leq \det(A_{K,K'})\det(A_{L,L'})$ *for all PSD matrices A.*

(See [35] for progress on Problem 6.1 (2).) One could also extend (1.10) to include products of more than two minors.

Problem 6.2. *Characterize the inequalities satisfied by products of three or more minors of TNN matrices.*

Then there are at least two natural extensions of (1.11).

Problem 6.3. *Characterize pairs* (v, w) *of permutations in* $\mathfrak{S}_n$ *for which we have*

$$a_{1,v_1} \cdots a_{n,v_n} \leq a_{1,w_1} \cdots a_{n,w_n}$$

(1) for all $n \times n$ *PSD matrices A,*
(2) for all $n \times n$ *HPSD matrices A, assuming v, w are involutions.*

Consideration of averages of the products appearing in Problem 6.1 leads to possible extensions of (1.14) – (1.15). To express these we use (3.2) and the fact that $\epsilon^{a,b} = \epsilon^{b,a}$ and $\eta^{a,b} = \eta^{b,a}$.

Problem 6.4. *Decide if*

(1) for all $n \times n$ *TNN matrices A and* $k = 1, \ldots, n$ *we have*

$$\left(\frac{\text{Imm}_{\epsilon^{n-k,k}}(A)}{\binom{n}{k}}\right)^2 \geq \left(\frac{\text{Imm}_{\epsilon^{n-k-1,k+1}}(A)}{\binom{n}{k+1}}\right)\left(\frac{\text{Imm}_{\epsilon^{n-k+1,k-1}}(A)}{\binom{n}{k-1}}\right),$$

(2) for all $n \times n$ HPSD matrices A and $k = 1, \ldots, n$ we have

$$\left(\frac{\mathrm{Imm}_{\eta^{n-k,k}}(A)}{\binom{n}{k}} \right)^2 \leq \left(\frac{\mathrm{Imm}_{\eta^{n-k-1,k+1}}(A)}{\binom{n}{k+1}} \right) \left(\frac{\mathrm{Imm}_{\eta^{n-k+1,k-1}}(A)}{\binom{n}{k-1}} \right), \tag{6.2}$$

(3) for all $n \times n$ TNN matrices A and $k = 1, \ldots, n$ we have (6.2).

Problem 6.5. *Decide if*

(1) for all $n \times n$ TNN matrices A and $k = 1, \ldots, n$ we have

$$\left(\frac{\mathrm{Imm}_{\epsilon^{n-k-1,k+1}}(A)}{\binom{n}{k+1} \det(A)} \right)^k \leq \left(\frac{\mathrm{Imm}_{\epsilon^{n-k,k}}(A)}{\binom{n}{k} \det(A)} \right)^{k+1},$$

(2) for all $n \times n$ HPSD matrices A and $k = 1, \ldots, n$ we have

$$\left(\frac{\mathrm{Imm}_{\eta^{n-k-1,k+1}}(A)}{\binom{n}{k+1} \operatorname{per}(A)} \right)^k \geq \left(\frac{\mathrm{Imm}_{\eta^{n-k,k}}(A)}{\binom{n}{k} \operatorname{per}(A)} \right)^{k+1}, \tag{6.3}$$

(3) for all $n \times n$ TNN matrices A and $k = 1, \ldots, n$ we have (6.3).

Further consideration of averages suggests extending the Barrett–Johnson inequalities (1.16) to all HPSD matrices, and stating a permanental analog as originally suggested in [1]. Again using (3.2) we may state these problems as follows.

Problem 6.6. *Characterize the pairs (λ, μ) of partitions of n for which*

(1) we have $\lambda_1! \cdots \lambda_r! \, \mathrm{Imm}_{\epsilon^\lambda}(A) \leq \mu_1! \cdots \mu_r! \, \mathrm{Imm}_{\epsilon^\mu}(A)$, i.e.,

$$\frac{\mathrm{Imm}_{\epsilon^\lambda}(A)}{\epsilon^\lambda(e)} \leq \frac{\mathrm{Imm}_{\epsilon^\mu}(A)}{\epsilon^\mu(e)},$$

for all $n \times n$ HPSD matrices A,

(2) we have $\lambda_1! \cdots \lambda_r! \, \mathrm{Imm}_{\eta^\lambda}(A) \geq \mu_1! \cdots \mu_r! \, \mathrm{Imm}_{\eta^\mu}(A)$, i.e.,

$$\frac{\mathrm{Imm}_{\eta^\lambda}(A)}{\eta^\lambda(e)} \geq \frac{\mathrm{Imm}_{\eta^\mu}(A)}{\eta^\mu(e)}, \tag{6.4}$$

for all $n \times n$ HPSD matrices A,

(3) we have (6.4) for all $n \times n$ TNN matrices A.

Finally, we have the problem of stating inequalities satisfied by irreducible character immanants. This is likely to be more difficult than Problem 6.6, since irreducible characters are less understood than induced one-dimensional characters.

Problem 6.7. *Characterize the pairs (λ, μ) of partitions of n for which we have*

$$\frac{\mathrm{Imm}_{\chi^\lambda}(A)}{\chi^\lambda(e)} \leq \frac{\mathrm{Imm}_{\chi^\mu}(A)}{\chi^\mu(e)} \tag{6.5}$$

(1) for all $n \times n$ HPSD matrices A,
(2) for all $n \times n$ TNN matrices A.

(See [17, Appendix], [40, §3] for progress.) Perhaps a tractable special case of Problem 6.7 (2) would be the restriction of (6.5) to hook shapes, i.e., a TNN analog of (1.13). In this case, each evaluation $\mathrm{Imm}_{\chi^\lambda}(A)$ has the combinatorial interpretation stated in [34, Thm. 4.11].

7. Acknowledgements

The authors are grateful to Shaun Fallat and Misha Gekhtman for helpful conversations, and to anonymous referees for suggesting many improvements to this article.

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