DUMONT'S STATISTIC ON WORDS

Mark Skandera

(Massachusetts Institute of Technology)

Outline

1. Definitions
2. The bijection
3. A related order complex

Eulerian Statistics

Permutation statistics whose distributions on S_n are given by the nth Eulerian polynomial,

$$A_n(x) = \sum_{k=1}^{n} A(n, k-1)x^k,$$

are known as *Eulerian statistics*.

Two important examples are “des” (descents) and “exc” (excedances).

$$\mathrm{des}(\pi) = \#\{i|\pi_i > \pi_{i+1}; i = 1, \ldots, n-1\}.$$
$$\mathrm{exc}(\pi) = \#\{i|\pi_i > i; i = 1, \ldots, n-1\}.$$

A third Eulerian statistic defined by Dumont, counts the number of distinct non-zero letters in the code of a permutation.

Example.

$$\begin{aligned} \pi &= 2\ 8\ 4\ 3\ 6\ 7\ 9\ 5\ 1 \\ \mathrm{code}(\pi) &= 1\ 6\ 2\ 1\ 2\ 2\ 2\ 1\ 0 \end{aligned}$$

The non-zero letters in $\mathrm{code}(\pi)$ are $LC(\pi) = \{1, 2, 6\}$. Thus, $\mathrm{dmc}(\pi) = 3$.

Generalizing permutations on n letters are *words* $w = w_1 \cdots w_m$ on n letters, where letters may be repeated in w.

Given a word w, we define $R(w)$ to be the set of all rearrangements of w.

Example.

$$\begin{aligned} w &= 3\ 2\ 3\ 3\ 1\ 1\ 2\ 1 \\ u &= 1\ 3\ 2\ 1\ 2\ 3\ 1\ 3 \in R(w) \\ v &= 1\ 1\ 1\ 2\ 2\ 3\ 3\ 3 \in R(w) \end{aligned}$$

Dumont's statistic is easy to define on words.

Example.

$$\begin{array}{lcl} w & = & 3\ 2\ 3\ 3\ 1\ 1\ 2\ 1 \\ \mathrm{code}(w) & = & 5\ 3\ 4\ 4\ 0\ 0\ 1\ 0 \end{array}$$

$$LC(w) = \{1, 3, 4, 5\}, \mathrm{dmc}(w) = 4.$$

We define the *non-decreasing rearrangement* of w to be the unique rearrangement $\bar{w}$ satisfying $\bar{w}_1 \leq \cdots \leq \bar{w}_m$.

To each word w we will associate the *biword*

$$\tilde{w} = \begin{pmatrix} \bar{w} \\ w \end{pmatrix}$$

Example. Let $w = 31231121$. Then,

$$\tilde{w} = \begin{pmatrix} \bar{w} \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 1 & 2 & 1 \end{pmatrix}$$

Excedances

We will call position i of w an *excedance* if $w_i \geq \bar{w}_i$. We will refer to w_i as the *value* of this excedance.

For example, if

$$\tilde{w} = \begin{pmatrix} \bar{w} \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 1 & 2 & 1 \end{pmatrix},$$

then the excedance set of w is $E(w) = \{1, 3, 4\}$, and these excedances have values $3, 2,$ and 3.

MacMahon showed that the word statistics des and exc are equally distributed on the rearrangement class of any word w.

That is, for any word w,

$$\begin{aligned}&\#\{v \in R(w)|\mathrm{exc}(v) = k\}\\ &= \#\{v \in R(w)|\mathrm{des}(v) = k\}.\end{aligned}$$

Theorem. For any word w, the distribution on $R(w)$ of the word statistic dmc is the same as that of des and exc.

$$\begin{aligned}
&\#\{y \in R(w) | \mathrm{dmc}(y) = k\} \\
&= \#\{y \in R(w) | \mathrm{des}(y) = k\} \\
&= \#\{y \in R(w) | \mathrm{exc}(y) = k\}.
\end{aligned}$$

Proof. We will define a bijection $\lambda : R(w) \to R(w)$, satisfying

$$E(w) = LC(\lambda(w)).$$

Clearly, this guarantees that

$$\mathrm{exc}(w) = \mathrm{dmc}(\lambda(w)).$$

□

We define the bijection in three steps.

Let $\begin{pmatrix} w \\ \mathrm{code}(w) \end{pmatrix} = \begin{pmatrix} w_1 & \cdots & w_m \\ c_1 & \cdots & c_m \end{pmatrix}$.

1. Define the word $d = d_1 \cdots d_m$ by

$$d_i = \begin{cases} i & \text{if } i \text{ is an excedance in } w, \\ 0 & \text{if } c_i = 0, \\ \beta(i) & \text{otherwise,} \end{cases}$$

where $\beta(i)$ is the c_ith excedance of w with value at least w_i.

Example. Let w be the word 431431421, and let c be its code.

$$\begin{pmatrix} w' \\ w \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 \\ \bar{4} & \bar{3} & 1 & \bar{4} & 3 & 1 & 4 & 2 & 1 \\ 6 & 4 & 0 & 4 & 3 & 0 & 2 & 1 & 0 \end{pmatrix}$$

Since w has excedances at positions $1, 2$, and 4, we write these values into d.

$$\begin{pmatrix} w \\ d \end{pmatrix} = \begin{pmatrix} \bar{4} & \bar{3} & 1 & \bar{4} & 3 & 1 & 4 & 2 & 1 \\ 1 & 2 & & 4 & & & & & \end{pmatrix}.$$

For non-excedances i, we set d_i equal to the c_ith excedance having value at least w_i,

$$\begin{pmatrix} w \\ d \end{pmatrix} = \begin{pmatrix} \bar{4} & \bar{3} & 1 & \bar{4} & 3 & 1 & 4 & 2 & 1 \\ 1 & 2 & & 4 & 4 & & 4 & 1 & \end{pmatrix}.$$

We place zeros elsewhere.

$$d = \begin{pmatrix} 1 & 2 & 0 & 4 & 4 & 0 & 4 & 1 & 0 \end{pmatrix}$$

2. Define the biword $y = \begin{pmatrix} w \\ d \end{pmatrix}$, and let $y' = \begin{pmatrix} w' \\ d' \end{pmatrix}$, be the unique rearrangement of y satisfying $\mathrm{code}(w') = d'$.

3. Set $\lambda(w) = w'$.

Example. Rearranging

$$y = \begin{pmatrix} w \\ d \end{pmatrix} = \begin{pmatrix} 4 & 3 & 1 & 4 & 3 & 1 & 4 & 2 & 1 \\ 1 & 2 & 0 & 4 & 4 & 0 & 4 & 1 & 0 \end{pmatrix},$$

we have

$$y' = \begin{pmatrix} w' \\ d' \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 & 4 & 1 & 3 & 2 & 4 & 1 \\ 4 & 0 & 4 & 4 & 0 & 2 & 1 & 1 & 0 \end{pmatrix}.$$

We therefore set

$$\lambda(431431421) = 314413241.$$

Proposition. Given the word w, construct d as in the bijection. Then we may rearrange the biword $y = \begin{pmatrix} w \\ d \end{pmatrix}$, as $y' = \begin{pmatrix} w' \\ d' \end{pmatrix}$, so that $\mathrm{code}(w') = d'$.

Proof. The necessary and sufficient condition on $y_i = \begin{pmatrix} w_i \\ d_i \end{pmatrix}$ for it to appear in the biword

$$\begin{pmatrix} w' \\ \text{code}(w') \end{pmatrix}$$

is that d_i be no greater than the number of letters (with multiplicities counted) in w which are less than w_i.

We compare w_i to $\tilde{w}_i$ and consider three cases.

Case 1: $(w_i > \tilde{w}_i)$. Position i is an excedance in w, so $d_i = i$, and w_i is greater than the i letters $\tilde{w}_i \geq \cdots \geq \tilde{w}_1$.

Case 2: $(w_i \leq \tilde{w}_i;\ c_i = 0)$. $d_i = 0$.

Case 3: $(w_i \leq \tilde{w}_i;\ c_i > 0)$. Let k be the number of letters in w which are strictly less than w_i. Since c_i of these letters appear to the right of position i, then at least c_i of the positions $1, \ldots, k$ are excedances.
d_i, being of one of these, is at most k.

f-vectors and h-vectors

The f vector of a $(d-1)$-dimensional simplicial complex is

$$f = (f_0, f_1, \ldots, f_{d-1}),$$

where f_i counts the number of faces of dimension i.

The h-vector may be defined as

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i},$$

where $f_{-1} = 1$, by convention.

Knowing the h-vector of a simplicial complex is equivalent to knowing the f-vector.

Sometimes the h-vector of one simplicial complex is the f-vector of another.

Theorem. If Σ is a Cohen Macauley complex, then its h-vector is the f-vector of some multicomplex.

Theorem. If Σ is a *balanced* Cohen Macauley complex, then its h-vector is the f-vector of some *simplicial* complex.

Fact. If $J(P)$ is a *distributive lattice*, then its h-vector is the f-vector of some simplicial complex.

Conjecture. If $J(P)$ is a distributive lattice, then its h-vector is the f-vector of some *poset*.

Theorem. If the distributive lattice $J(P)$ is a *product of chains*, then its h-vector is the f-vector of some poset.

The rearrangement class $R(w)$ of any word corresponds to linear extensions of a product of chains $J(P)$.

Leting $C(w)$ be the set of codes of $R(w)$, we see that the h-vector of $J(P)$ counts codes in $C(w)$ by dmc.

Fact. We may define a poset Q on the one-letter codes of $C(w)$ so that k-element chains correspond to k-letter codes of $C(w)$.

Thus, Q satisfies $h_{J(P)} = f_Q$.

Let c and d be codes whose nonzero letters are j and k, respectively. Define the poset Q by setting $c <_Q d$ if

1. $j < k$.
2. The multiplicity of j in c is strictly *greater* than that of k in d.
3. For each position i such that $d_i = k$, we have $c_{i+k-j} = j$.

Example. Consider the codes $c <_Q d$.

$$\begin{aligned} d &= 4\ 4\ 0\ 0\ 4\ 0\ 0\ 0\ 0 \\ c &= 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0 \end{aligned}$$

We see the comparison most easily by sliding the letters of d to the right and decreasing them.

$$\begin{aligned} d &= 4\ 4\ 0\ 0\ 4\ 0\ 0\ 0\ 0 \\ &\ 0\ 3\ 3\ 0\ 0\ 3\ 0\ 0\ 0 \\ &\ 0\ 0\ 2\ 2\ 0\ 0\ 2\ 0\ 0 \\ &\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0 \end{aligned}$$

$$c = 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0$$

Example. The code 4501400200 corresponds to the following chain.

0500000000

|

4040400000

|

0020202200

|

0101010110