ON THE DÉSARMÉNIEN-KUNG-ROTA BASIS FOR $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$

Brendon Rhoades and Mark Skandera

University of Minnesota and Lehigh University

Outline

(1) Immanants and $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$.
(2) Bideterminants and the Désarménien-Kung-Rota basis
(3) The dual canonical basis
(4) The iterated dominance posets
(5) The transition matrix between the two bases

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Immanants and $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$

Let $x = (x_{i,j})_{1\leq i,j\leq n}$ be a matrix of variables.

$$\mathcal{O}(SL(n,\mathbb{C})) \cong \mathbb{C}[x_{1,1},\ldots,x_{n,n}]/(\det(x)-1).$$

$$\mathbb{C}[x] = \bigoplus_{r\geq 0} \bigoplus_{(M,M')} \mathcal{A}_{M,M'}(x),$$

Call elements of $\mathcal{A}_{[n],[n]}(x)$ *immanants.*

$$\mathrm{Imm}_f(x) = \sum_{w\in S_n} f(w) x_{1,w(1)} \cdots x_{n,w(n)},$$

$$\det(x) = \sum_{w\in S_n} (-1)^{\ell(w)} x_{1,w(1)} \cdots x_{n,w(n)}.$$

The Désarménien-Kung-Rota basis of $\mathcal{A}_{[n],[n]}(x)$

Define the I, J *submatrix* and I, J *minor* of x by

$$x_{I,J} = (x_{i,j})_{i \in I, j \in J},$$
$$\Delta_{I,J}(x) = \det(x_{I,J}).$$

Given Young tableaux (T, U) of shape $\lambda \vdash n$, define the *bideterminant* $[T : U](x)$ as in the example

$$\left[\begin{matrix} 1 & 3 & 6 \\ 2 & 4 & 7 \\ 5 & & \end{matrix} \; : \; \begin{matrix} 1 & 2 & 5 \\ 3 & 4 & 6 \\ 7 & & \end{matrix} \right](x) = \Delta_{125,137}(x)\Delta_{34,24}(x)\Delta_{67,56}(x).$$

Thm: (D-K-R, '78) A basis for $\mathcal{A}_{[n],[n]}(x)$ is given by

$$\mathbf{D} \underset{\text{def}}{=} \bigcup_{\lambda \vdash n} \{[T : U](x) \,|\, T, U \text{ standard of shape } \lambda\}.$$

The Désarménien-Kung-Rota basis of $\mathcal{A}_{[3],[3]}(x_{1,1},\ldots,x_{3,3})$

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}.$$

$$\begin{aligned}
\left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} : \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right](x) &= \det(x), \\
\left[\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} : \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}\right](x) &= \Delta_{13,13}(x)x_{2,2}, \\
\left[\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} : \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}\right](x) &= \Delta_{13,12}(x)x_{2,3}, \\
\left[\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} : \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}\right](x) &= \Delta_{12,13}(x)x_{3,2}, \\
\left[\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} : \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}\right](x) &= \Delta_{12,12}(x)x_{3,3}, \\
\left[\begin{smallmatrix} 1 & 2 & 3 \end{smallmatrix} : \begin{smallmatrix} 1 & 2 & 3 \end{smallmatrix}\right](x) &= x_{1,1}x_{2,2}x_{3,3}.
\end{aligned}$$

The dual canonical basis of $\mathcal{A}_{[n],[n]}(x)$

Immanants in the dual canonical basis have the form

$$\mathrm{Imm}_v(x) \underset{\mathrm{def}}{=} \sum_{w \geq v} (-1)^{\ell(w)-\ell(v)} Q_{v,w}(1) x_{1,w(1)} \cdots x_{n,w(n)},$$

where $Q_{v,w}(q) = P_{w_0 w, w_0 v}(q)$ are inverse Kazhdan-Lusztig polynomials.

$$\mathrm{Imm}_e(x) = \det(x),$$
$$\mathrm{Imm}_{w_0}(x) = x_{1,n} x_{2,n-1} \cdots x_{n,1}.$$

Thm: (D, '92) A basis for $\mathcal{A}_{[n],[n]}(x)$ is given by

$$\mathbf{L} \underset{\mathrm{def}}{=} \{\mathrm{Imm}_w(x) \,|\, w \in S_n\}.$$

The dual canonical basis of $\mathcal{A}_{[3],[3]}(x_{1,1},\ldots,x_{3,3})$

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}.$$

$$\begin{aligned}
\mathrm{Imm}_{123}(x) &= \det(x), \\
\mathrm{Imm}_{213}(x) &= x_{1,2}x_{2,1}x_{3,3} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2} \\
&\quad + x_{1,3}x_{2,2}x_{3,1}, \\
\mathrm{Imm}_{132}(x) &= x_{1,1}x_{2,3}x_{3,2} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2} \\
&\quad + x_{1,3}x_{2,2}x_{3,1}, \\
\mathrm{Imm}_{231}(x) &= x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,2}x_{3,1}, \\
\mathrm{Imm}_{312}(x) &= x_{1,3}x_{2,1}x_{3,2} - x_{1,3}x_{2,2}x_{3,1}, \\
\mathrm{Imm}_{321}(x) &= x_{1,3}x_{2,2}x_{3,1}.
\end{aligned}$$

The iterated dominance order on SYT

For a SYT T, define $T_i =$ subtableau containing $1, \ldots, i$.

Given SYTx T, U, define $T \leq_I U$ if

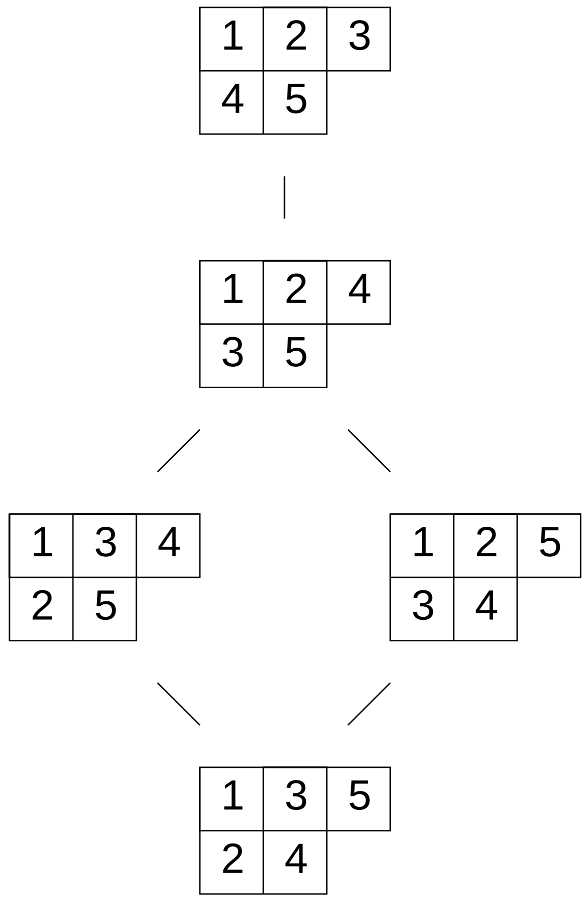
$$\text{shape}(T_i) \leq \text{shape}(U_i), \qquad i = 1, \ldots, n$$

in ordinary dominance.

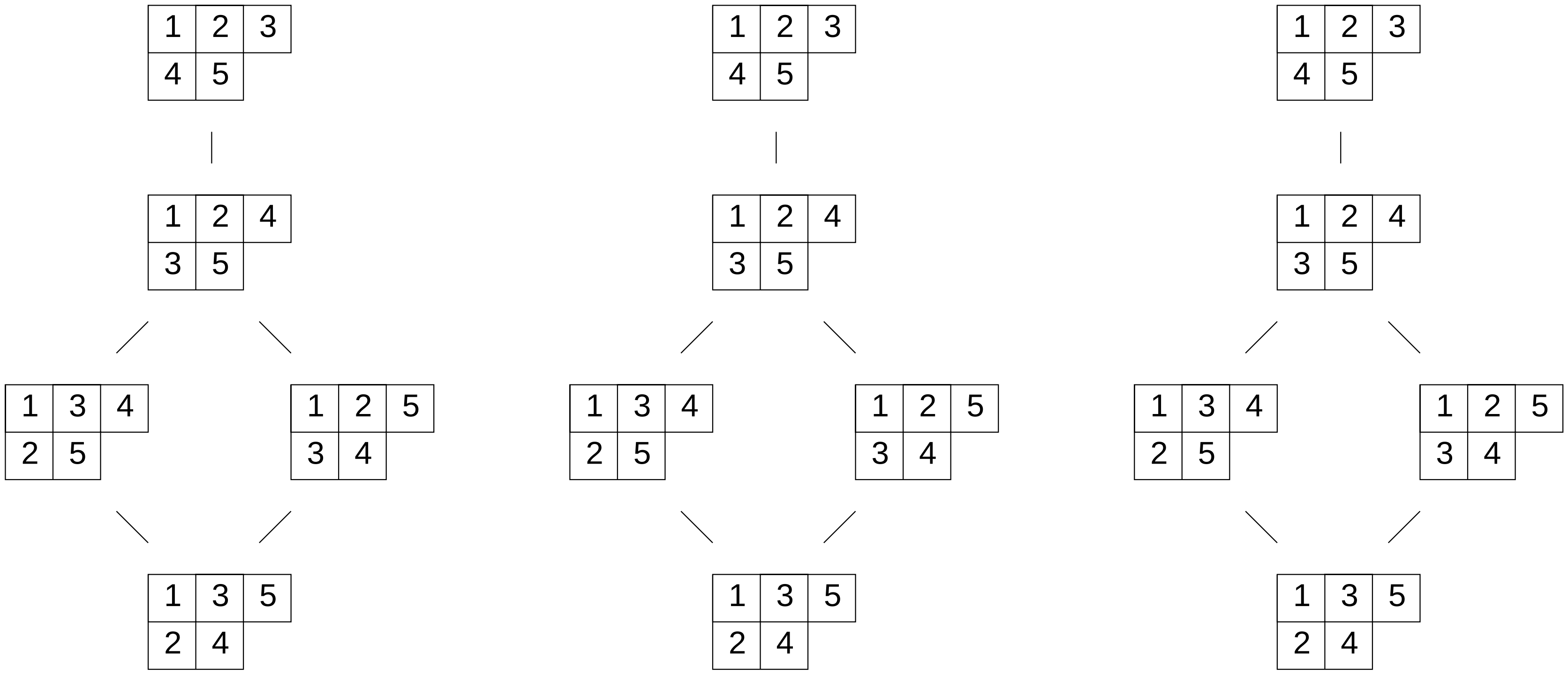
Ex: $T = \begin{matrix} 1 & 3 & 4 \\ 2 & & \end{matrix}$ and $U = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$ are incomparable, since

$$\text{shape}(T_2) < \text{shape}(U_2),$$
$$\text{shape}(T_4) > \text{shape}(U_4).$$

The iterated dominance order on SYT of shape (3,2)



The iterated dominance order on SYT of shape (3,2)



The iterated dominance order on S_n

Define iterated dominance on standard bitableaux by $(R, S) \leq_I (T, U)$ if

$$R \leq_I T, \quad S \leq_I U.$$

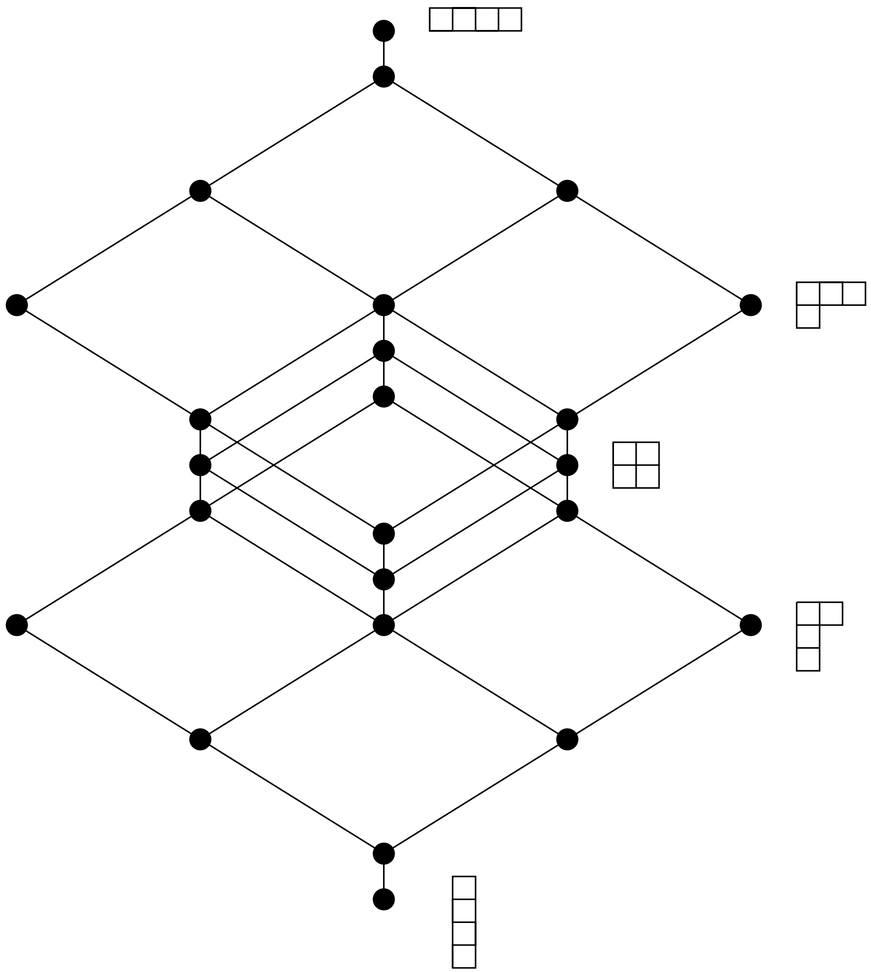
Define iterated dominance on permutations by $v \leq_I w$ if

$$(T(v), U(v)) \leq_I (T(w), U(w)),$$

where

$$w \underset{\substack{\text{RSK} \\ \text{col}}}{\longmapsto} (T(w), U(w)).$$

The iterated dominance order on S_4



The transition matrix between the bases

Denote each D-K-R basis element by

$$R_w(x) = [U(w) : T(w)](x),$$

where RSK col : $w \mapsto (T(w), U(w))$.

Thm: (R-S, '06) For some $d_{v,w} \in \mathbb{N}$, $d_{w,w} = 1$, we have

$$R_w(x) = \sum_{v \leq_I w} d_{v,w} \mathrm{Imm}_v(x).$$

Problem: Find a formula for $d_{v,w}$.

This would give an alternative formula for the numbers $P_{v,w}(1)$.

The transition matrix for $n = 3$

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}.$$

$$\begin{bmatrix} R_{123}(x) \\ R_{132}(x) \\ R_{231}(x) \\ R_{312}(x) \\ R_{213}(x) \\ R_{321}(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Imm}_{123}(x) \\ \mathrm{Imm}_{132}(x) \\ \mathrm{Imm}_{231}(x) \\ \mathrm{Imm}_{312}(x) \\ \mathrm{Imm}_{213}(x) \\ \mathrm{Imm}_{321}(x) \end{bmatrix}.$$