THE CLUSTER BASIS OF $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$

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Outline

(1) Bases of $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ and factorization results
(2) Centrally symmetric triangulations of the octagon
(3) The cluster basis of $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$
(4) Parametrization of the cluster basis by 3×3 matrices
(5) A factorization formula
(6) Open questions

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Bases of $\mathbb{Z}[x_{1,1},\ldots,x_{n,n}]$

Natural basis: $\{x^A \mid A = (a_{i,j}) \in \mathrm{Mat}_{n\times n}(\mathbb{N})\}$,

$$x^A \underset{\mathrm{def}}{=} \prod_{i,j=1}^{n} x_{i,j}^{a_{i,j}}.$$

Dual canonical basis: $\{Z_A \mid A = (a_{i,j}) \in \mathrm{Mat}_{n\times n}(\mathbb{N})\}$,

$$Z_A \underset{\mathrm{def}}{=} x^A + \sum_{\substack{B>A \\ r(B)=r(A) \\ c(B)=c(A)}} \mathrm{sgn}(A,B) Q_{A,B}(1) x^B,$$

$Q_{A,B}(q) =$ a double-parabolic Kazhdan-Lusztig polynomial.

Question: How do DCB elements factor?

Special case $n = 1$: $\mathbb{Z}[x] = \mathbb{Z}[x_{1,1}]$.

$$\text{DCB} = \{1, x_{1,1}, x_{1,1}^2, x_{1,1}^3, \ldots\}.$$

Special case $n = 2$: $\mathbb{Z}[x] = \mathbb{Z}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]$.

Theorem: (D '92, Z '04)

$$\begin{aligned}\text{DCB} &= \text{all products of } \{x_{1,1}, x_{1,2}, x_{2,1}, \det(x)\} \\ &\cup \text{ all products of } \{x_{2,2}, x_{1,2}, x_{2,1}, \det(x)\}.\end{aligned}$$

In particular,

$$Z_A = \begin{cases} x_{1,1}^a x_{1,2}^b x_{2,1}^c \det(x)^d & \text{if } A = \left[\begin{smallmatrix} a+d & b \\ c & d \end{smallmatrix}\right], \\ x_{2,2}^e x_{1,2}^b x_{2,1}^c \det(x)^d & \text{if } A = \left[\begin{smallmatrix} d & b \\ c & d+e \end{smallmatrix}\right]. \end{cases}$$

$$Z_{\left[\begin{smallmatrix} 7 & 2 \\ 8 & 3 \end{smallmatrix}\right]} = Z_{\left[\begin{smallmatrix} 4 & 0 \\ 0 & 0 \end{smallmatrix}\right]} Z_{\left[\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right]} Z_{\left[\begin{smallmatrix} 0 & 0 \\ 8 & 0 \end{smallmatrix}\right]} Z_{\left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]} = x_{1,1}^4 x_{1,2}^2 x_{2,1}^8 \det(x)^3.$$

Special case $n = 3$: $\mathbb{Z}[x] = \mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$.

Theorem: (D '92, Z '04) If $A = E + \begin{bmatrix} c & b & a \\ d & c & b \\ e & d & c \end{bmatrix}$, with $E \in \mathrm{Mat}_{3\times 3}(\mathbb{N})$, $a, b, c, d, e \in \mathbb{N}$, then

$$Z_A = Z_E \cdot x_{3,1}^e \Delta_{23,12}^d \det(x)^c \Delta_{12,23}^b x_{1,3}^a.$$

Question: How does Z_E factor, for E of the forms

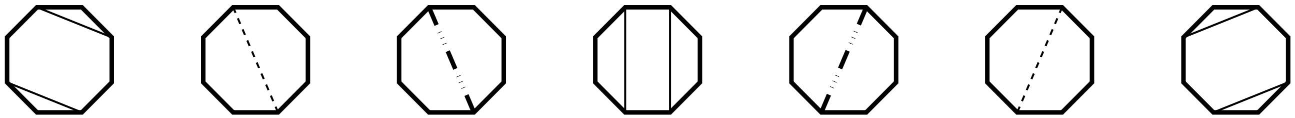
$$\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ * & 0 & * \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} 0 & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \ldots ?$$

Conjecture: (F-Z) Z_E is a product of cluster variables belonging to some (type D_4) cluster $\mathcal{C}_i$

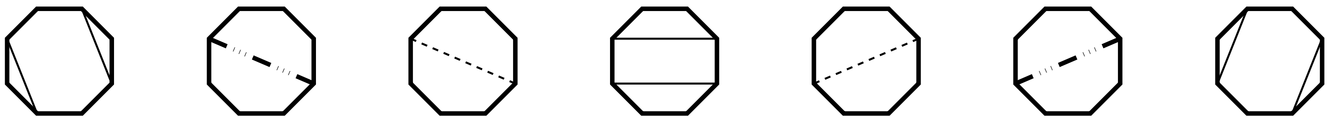
$$\mathcal{C}_1, \ldots, \mathcal{C}_{50} \subset \mathbb{Z}[x_{1,1}, \ldots, x_{3,3}] \subset \mathbb{C}[GL_3^{w_0, w_0}],$$

and all such products are DCB elements.

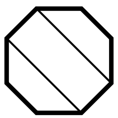
Cluster variables as diagonals of an octogon



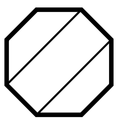
$x_{1,1}$ $x_{1,2}$ $x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{3,2}$ $x_{3,3}$



$\Delta_{23,23}$ $\Delta_{23,13}$ $\Delta_{13,23}$ $\Delta_{13,13}$ $\Delta_{13,12}$ $\Delta_{12,13}$ $\Delta_{12,12}$



$$\mathrm{Imm}_{213} = x_{1,2}\Delta_{23,13} - x_{1,3}\Delta_{23,12}$$



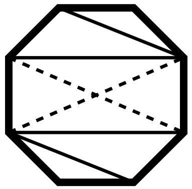
$$\mathrm{Imm}_{132} = x_{2,3}\Delta_{13,12} - x_{1,3}\Delta_{23,12}$$

Clusters = centrally symmetric triangulations

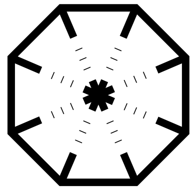
Diagonals may not cross, except for

(1) different diameters of the same color,

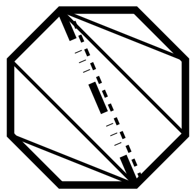
(2) different colorings of the same diameter.



$\{x_{1,1}, \Delta_{13,23}, \Delta_{13,13}, \Delta_{13,12}\}$



$\{x_{2,1}, x_{2,3}, \Delta_{23,13}, \Delta_{12,13}\}$



$\{x_{1,1}, x_{1,2}, x_{2,1}, \mathrm{Imm}_{213}\}$

Cluster monomials

Frozen variables: polynomials in the set

$$\mathcal{F} = \{x_{1,3}, \Delta_{12,23}, \det(x), \Delta_{23,12}, x_{3,1}\}.$$

Cluster monomials: products of polynomials in $\mathcal{C}_i \cup \mathcal{F}$ for some i.

Theorem: (F-Z '07) The (infinite) set $\mathcal{M}$ of all such products is linearly independent.

Theorem: (F-Z unpublished) These products also span $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$.

Call $\mathcal{M}$ the *cluster basis* of $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$.

The cluster basis

Theorem: (S '07) Cluster basis elements may be parametrized by 3×3 matrices $\mathcal{M} = \{Y_A \mid A \in \mathrm{Mat}_{3\times 3}(\mathbb{N})\}$ so that they expand with respect to the DCB as

$$Y_A = Z_A + \sum_{\substack{B>A \\ r(B)=r(A) \\ c(B)=c(A)}} d_{A,B} Z_B.$$

Corollary: If the two bases are equal then

(1) $Z_A = Y_A$,
(2) the factorization of Z_A is encoded by A,
(3) Z_A has a combinatorial interpretation.

Matrices encode cluster monomials

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \quad \cdots \quad \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

$$x_{1,1} \quad x_{1,2} \quad x_{2,1} \quad x_{2,2} \quad \quad x_{3,3} \quad \mathrm{Imm}_{213}$$

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&0&0\\1&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix} \quad \cdots \quad \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$\Delta_{23,23} \quad \Delta_{23,13} \quad \Delta_{13,23} \quad \Delta_{13,13} \quad \quad \Delta_{12,12} \quad \mathrm{Imm}_{132}$$

$\Delta^6_{23,23}\Delta^3_{23,13}x^2_{2,3}x_{2,1} = Y_A$, where

$$A = \begin{bmatrix}0&0&0\\0&6&0\\0&0&6\end{bmatrix} + \begin{bmatrix}0&0&0\\3&0&0\\0&0&3\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&2\\0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} = \begin{bmatrix}0&0&0\\4&6&2\\0&0&9\end{bmatrix}.$$

One of five factorization formulae

If $E = \begin{bmatrix} 0 & 0 & 0 \\ e_{2,1} & e_{2,2} & e_{2,3} \\ 0 & 0 & e_{3,3} \end{bmatrix}$, then we have

$$Z_E \stackrel{?}{=} Y_E = \Delta_{23,23}^{a} \Delta_{23,13}^{b} x_{2,3}^{c} x_{2,1}^{d} x_{2,2}^{f} x_{3,3}^{g},$$

where

$$a = \min\{e_{2,2}, e_{3,3}\}, \quad b = \min\{e_{3,3} - a, e_{2,1}\}, \quad c = e_{2,3},$$
$$d = e_{2,1} - b, \quad f = e_{2,2} - a, \quad g = e_{3,3} - a - b.$$

Example: If $E = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 6 & 2 \\ 0 & 0 & 9 \end{bmatrix}$, then we have

$$(a, b, c, d, f, g) = (6, 3, 2, 1, 0, 0)$$

and

$$Z_E \stackrel{?}{=} Y_E = \Delta_{23,23}^{6} \Delta_{23,13}^{3} x_{2,3}^{2} x_{2,1}.$$

Future research

Question: Do appropriately defined cluster monomials form a basis of $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ for $n > 3$?

Answer: No.

Fact: If $A = E + \begin{bmatrix} d & c & b & a \\ e & d & c & b \\ f & e & d & c \\ g & f & e & d \end{bmatrix}$, with $E \in \mathrm{Mat}_{4\times 4}(\mathbb{N})$, $a, b, c, d, e, f, g \in \mathbb{N}$, then

$$Z_A = Z_E \cdot x_{4,1}^g \Delta_{34,12}^f \Delta_{234,123}^e \det(x)^d \Delta_{123,234}^c \Delta_{12,34}^b x_{1,4}^a.$$

Problem: Describe the irreducible elements in the DCB of $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$.