THE CLUSTER BASIS OF $\mathbb{Z}[x_{1,1},\ldots,x_{3,3}]$

MARK SKANDERA

Abstract. We show that the set of cluster monomials for the cluster algebra of type D_4 contains a basis of the $\mathbb{Z}$-module $\mathbb{Z}[x_{1,1},\ldots,x_{3,3}]$. We also show that the transition matrices relating this cluster basis to the natural and the dual canonical bases are unitriangular and nonnegative. These results support a conjecture of Fomin and Zelevinsky on the equality of the cluster and dual canonical bases. In the event that this conjectured equality is true, our results also imply an explicit factorization of each dual canonical basis element as a product of cluster variables.

1. Introduction

The coordinate ring $\mathcal{O}(SL(n,\mathbb{C}))$ of polynomial functions in the entries of matrices in $SL(n,\mathbb{C})$ may be realized as a quotient,

$$\mathcal{O}(SL(n,\mathbb{C})) = \mathbb{C}[x_{1,1},\ldots,x_{n,n}]/(\det(x)-1),$$

where $x=(x_{1,1},\ldots,x_{n,n})$ is a matrix of n^2 commuting variables. We will call i the *row index* and j the *column index* of the variable $x_{i,j}$.

Viewing the rings $\mathcal{O}(SL(n,\mathbb{C}))$ and $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ as vector spaces, one often applies the canonical homomorphism to a particular basis of $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ in order to obtain a basis of $\mathcal{O}(SL(n,\mathbb{C}))$. Some bases of $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ which appear in the literature are the natural basis of monomials, the bitableau basis of Désarménien, Kung and Rota [6], and the dual canonical (crystal) basis of Lusztig [23] and Kashiwara [20]. Since the transition matrices relating these bases have integer entries and determinant 1, each is also a basis of the $\mathbb{Z}$-module $\mathbb{Z}[x_{1,1},\ldots,x_{n,n}]$. In the case $n=3$, work of Berenstein, Fomin and Zelevinsky [2, 12, 14, 16] suggests that certain polynomials which arise as *cluster monomials* in the study of cluster algebras may form a basis of $\mathbb{Z}[x_{1,1},\ldots,x_{3,3}]$ and that this basis may be equal to the dual canonical basis. (In fact, unpublished work of these authors [15] implies that these cluster monomials *do* form a basis of $\mathbb{Z}[x_{1,1},\ldots,x_{3,3}]$.) The analogous statement for $n\geq 4$ is known to be false.

After recalling the definition of cluster monomials in Section 2, we will perform rather elementary computations in Section 3 to observe a bijective correspondence between an appropriate set of cluster monomials and 3×3 nonnegative integer matrices. This correspondence will lead to our main theorems in Section 4 which show our set of cluster monomials to form a basis of $\mathbb{Z}[x_{1,1},\ldots,x_{3,3}]$. Using the correspondence

Date: November 25, 2007.

we also relate the cluster basis by unitriangular transition matrices to the natural and dual canonical bases, and give conjectured formulae for the irreducible factorization of dual canonical basis elements.

2. Cluster monomials of type D_4

Fomin and Zelevinsky defined a class of commutative rings called *cluster algebras* [12] in order to study total positivity and dual canonical bases in semisimple algebraic groups. (See also [2], [14], [16].) This definition continued earlier work of the authors with Berenstein [1], [3], [11] and of Lusztig [24]. Further work has revealed connections between cluster algebras and other topics such as Laurent phenomena [13], Teichmüller spaces [8], Poisson geometry [17] and algebraic combinatorics [10].

Each cluster algebra has a distinguished set of generators called *cluster variables* which are grouped into overlapping subsets called *clusters*. Those cluster algebras generated by a finite set of cluster variables enjoy a classification similar to the Cartan-Killing classification of semisimple Lie algebras [14]. We shall consider clusters of the cluster algebra of type D_4, which arises in the study of total nonnegativity within $SL(3, \mathbb{C})$ and $GL(3, \mathbb{C})$. In particular, one may decompose $G = SL(3, \mathbb{C})$ or $GL(3, \mathbb{C})$ as in [11], [24] into a union of intersections of double cosets called *double Bruhat cells* $\{G^{u,v} \,|\, u, v \in S_3\}$. Letting u and v be the longest element w_0 of S_3, we obtain the double Bruhat cell G^{w_0,w_0}, whose coordinate ring $\mathcal{O}(G^{w_0,w_0})$ contains $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$ as a subring and which has a cluster algebra structure of type D_4. More precisely, for $G = GL(3, \mathbb{C})$, the coordinate ring $\mathcal{O}(G^{w_0,w_0})$ is the localization of $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$ at $x_{1,3}$, $x_{3,1}$, $x_{1,2}x_{2,3} - x_{1,3}x_{2,2}$, $x_{2,1}x_{3,2} - x_{2,2}x_{3,1}$ and $\det(x)$. Taking the quotient of this ring modulo $(\det(x) - 1)$, we obtain the analogous coordinate ring corresponding to $SL(3, \mathbb{C})$.

A thorough treatment of the theory of cluster algebras will not be necessary for our purposes. (See [2, Sec. 1] for an introduction to cluster algebras and [2, Sec. 2.4] for more specific information about the the coordinate rings above.) Instead we will merely define certain polynomials in $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$ to be *cluster variables* and *frozen variables*, and will follow [14, Sec. 12.4] in describing sets and products of these polynomials, called *clusters* and *cluster monomials*, in terms of centrally symmetric modified triangulations of an octogon.

Let I and J be subsets of $\{1, 2, 3\}$ with $|I| = |J|$. We define the I, J *submatrix* of x and I, J *minor* of x by

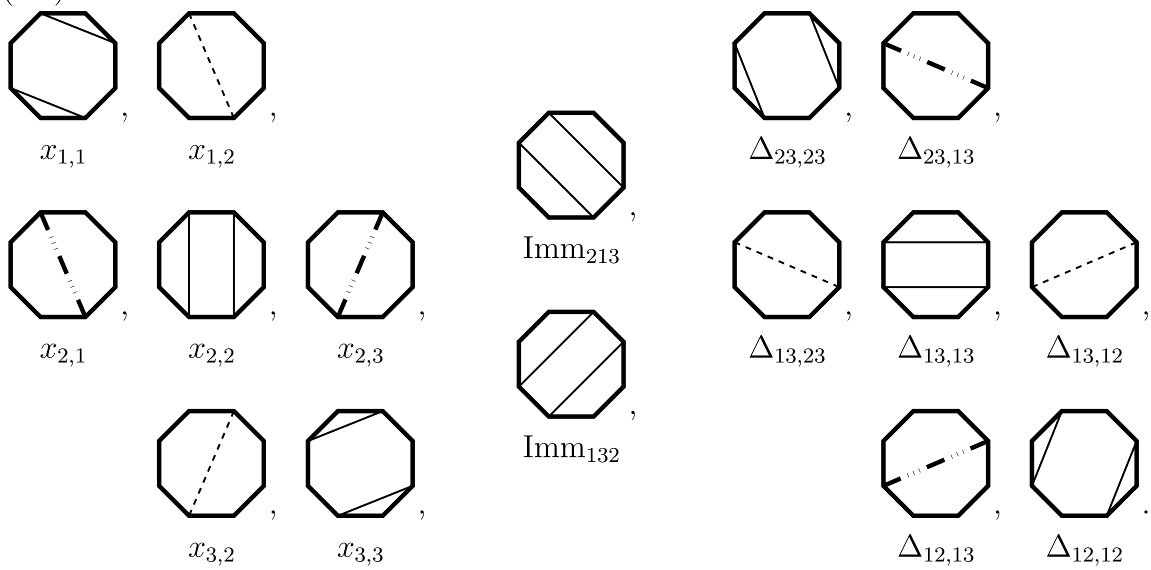
$$x_{I,J} = (x_{i,j})_{i \in I, j \in J}, \quad \Delta_{I,J} = \det(x_{I,J}).$$

When writing $x_{\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}}$ and $\Delta_{\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}}$, we will tacitly assume set elements to satisfy $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. To economize notation, we also may denote the submatrix and minor by $x_{i_1 \cdots i_k, j_1 \cdots j_k}$ and $\Delta_{i_1 \cdots i_k, j_1 \cdots j_k}$.

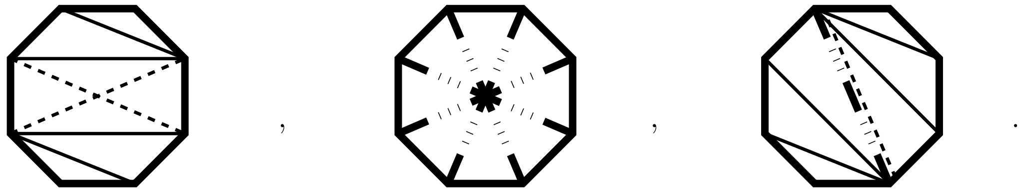
In terms of this notation, our cluster variables are the sixteen polynomials

$$x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,2}, x_{3,3},$$
$$\Delta_{12,12}, \Delta_{12,13}, \Delta_{13,12}, \Delta_{13,13}, \Delta_{13,23}, \Delta_{23,13}, \Delta_{23,23},$$
$$\mathrm{Imm}_{213} \underset{\mathrm{def}}{=} x_{1,2}x_{2,1}x_{3,3} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2} + x_{1,3}x_{2,2}x_{3,1},$$
$$\mathrm{Imm}_{132} \underset{\mathrm{def}}{=} x_{1,1}x_{2,3}x_{3,2} - x_{1,2}x_{2,3}x_{3,1} - x_{1,3}x_{2,1}x_{3,2} + x_{1,3}x_{2,2}x_{3,1}.$$

To each cluster variable we associate a colored diameter of a fixed octogon or a pair of non-diameter diagonals of the octogon by
(2.1)



We define a *centrally symmetric modified triangulation* of the octogon to be a maximal subset of the above diagonals and pairs of diagonals with the property that no two diagonals in the collection cross, unless those diagonals are different diameters of the same color or are different colorings of the same diameter. Three examples and the corresponding sets of cluster variables are



$\{x_{1,1}, \Delta_{13,23}, \Delta_{13,13}, \Delta_{13,12}\}$ $\{x_{2,1}, x_{2,3}, \Delta_{23,13}, \Delta_{12,13}\}$ $\{x_{1,1}, x_{1,2}, x_{2,1}, \mathrm{Imm}_{213}\}$

Our *clusters* are the fifty subsets of cluster variables corresponding to centrally symmetric modified triangulations of the octogon. In other words, clusters are the facets of the simplicial complex in which vertices are cluster variables and faces are sets of cluster variables whose geometric realizations satisfy the noncrossing conditions above. These are shown in the following table, where we have named the clusters in a manner consistent with the naming of the thirty-four clusters shown in [11, Fig. 8]. In the table, we have partitioned the clusters into twelve blocks whose significance will become clear in Observations 3.1 - 3.12.

Cluster name	Cluster variables	Cluster name	Cluster variables
defA	$x_{2,2}, x_{2,3}, x_{3,2}, \Delta_{23,23}$	bcdG	$x_{1,2}, x_{2,1}, x_{2,2}, \Delta_{12,12}$
efgA	$x_{2,3}, x_{3,2}, x_{3,3}, \Delta_{23,23}$	abcG	$x_{1,1}, x_{1,2}, x_{2,1}, \Delta_{12,12}$
cdeA	$x_{2,1}, x_{2,2}, x_{2,3}, \Delta_{23,23}$	bdfG	$x_{1,2}, x_{2,2}, x_{3,2}, \Delta_{12,12}$
ceAB	$x_{2,1}, x_{2,3}, \Delta_{23,23}, \Delta_{23,13}$	bfEG	$x_{1,2}, x_{3,2}, \Delta_{13,12}, \Delta_{12,12}$
egAB	$x_{2,3}, x_{3,3}, \Delta_{23,23}, \Delta_{23,13}$	abEG	$x_{1,1}, x_{1,2}, \Delta_{13,12}, \Delta_{12,12}$
bdfA	$x_{1,2}, x_{2,2}, x_{3,2}, \Delta_{23,23}$	cdeG	$x_{2,1}, x_{2,2}, x_{2,3}, \Delta_{12,12}$
bfAC	$x_{1,2}, x_{3,2}, \Delta_{23,23}, \Delta_{13,23}$	ceFG	$x_{2,1}, x_{2,3}, \Delta_{12,13}, \Delta_{12,12}$
fgAC	$x_{3,2}, x_{3,3}, \Delta_{23,23}, \Delta_{13,23}$	acFG	$x_{1,1}, x_{2,1}, \Delta_{12,13}, \Delta_{12,12}$
ceBF	$x_{2,1}, x_{2,3}, \Delta_{23,13}, \Delta_{12,13}$	bfCE	$x_{1,2}, x_{3,2}, \Delta_{13,23}, \Delta_{13,12}$
acBF	$x_{1,1}, x_{2,1}, \Delta_{23,13}, \Delta_{12,13}$	fgCE	$x_{3,2}, x_{3,3}, \Delta_{13,23}, \Delta_{13,12}$
egBF	$x_{2,3}, x_{3,3}, \Delta_{23,13}, \Delta_{12,13}$	abCE	$x_{1,1}, x_{1,2}, \Delta_{13,23}, \Delta_{13,12}$
aBDF	$x_{1,1}, \Delta_{23,13}, \Delta_{13,13}, \Delta_{12,13}$	gCDE	$x_{3,3}, \Delta_{13,23}, \Delta_{13,13}, \Delta_{13,12}$
gBDF	$x_{3,3}, \Delta_{23,13}, \Delta_{13,13}, \Delta_{12,13}$	aCDE	$x_{1,1}, \Delta_{13,23}, \Delta_{13,13}, \Delta_{13,12}$
bcdA	$x_{1,2}, x_{2,1}, x_{2,2}, \Delta_{23,23}$	defG	$x_{2,2}, x_{2,3}, x_{3,2}, \Delta_{12,12}$
bcAp	$x_{1,2}, x_{2,1}, \Delta_{23,23}, \mathrm{Imm}_{213}$	efGq	$x_{2,3}, x_{3,2}, \Delta_{12,12}, \mathrm{Imm}_{132}$
bACp	$x_{1,2}, \Delta_{23,23}, \Delta_{13,23}, \mathrm{Imm}_{213}$	eFGq	$x_{2,3}, \Delta_{12,13}, \Delta_{12,12}, \mathrm{Imm}_{132}$
cABp	$x_{2,1}, \Delta_{23,23}, \Delta_{23,13}, \mathrm{Imm}_{213}$	fEGq	$x_{3,2}, \Delta_{13,12}, \Delta_{12,12}, \mathrm{Imm}_{132}$
ABCp	$\Delta_{23,23}, \Delta_{23,13}, \Delta_{13,23}, \mathrm{Imm}_{213}$	EFGq	$\Delta_{13,12}, \Delta_{12,13}, \Delta_{12,12}, \mathrm{Imm}_{132}$
gABC	$x_{3,3}, \Delta_{23,23}, \Delta_{23,13}, \Delta_{13,23}$	aEFG	$x_{1,1}, \Delta_{13,12}, \Delta_{12,13}, \Delta_{12,12}$
abcp	$x_{1,1}, x_{1,2}, x_{2,1}, \mathrm{Imm}_{213}$	efgq	$x_{2,3}, x_{3,2}, x_{3,3}, \mathrm{Imm}_{132}$
acBp	$x_{1,1}, x_{2,1}, \Delta_{23,13}, \mathrm{Imm}_{213}$	fgEq	$x_{3,2}, x_{3,3}, \Delta_{13,12}, \mathrm{Imm}_{132}$
abCp	$x_{1,1}, x_{1,2}, \Delta_{13,23}, \mathrm{Imm}_{213}$	egFq	$x_{2,3}, x_{3,3}, \Delta_{12,13}, \mathrm{Imm}_{132}$
aBCp	$x_{1,1}, \Delta_{23,13}, \Delta_{13,23}, \mathrm{Imm}_{213}$	gEFq	$x_{3,3}, \Delta_{13,12}, \Delta_{12,13}, \mathrm{Imm}_{132}$
aBCD	$x_{1,1}, \Delta_{23,13}, \Delta_{13,23}, \Delta_{13,13}$	gDEF	$x_{3,3}, \Delta_{13,13}, \Delta_{13,12}, \Delta_{12,13}$
gBCD	$x_{3,3}, \Delta_{23,13}, \Delta_{13,23}, \Delta_{13,13}$	aDEF	$x_{1,1}, \Delta_{13,13}, \Delta_{13,12}, \Delta_{12,13}$

We define five more polynomials to be *frozen variables*,

$$x_{1,3}, \Delta_{12,23}, \Delta_{123,123}, \Delta_{23,12}, x_{3,1},$$

and define the union of these with any cluster to be an *extended cluster*. We define a *cluster monomial* to be a product of nonnegative powers of cluster variables, and integer powers of frozen variables, all belonging to the same extended cluster. We denote by $\mathcal{M}$ the subset of cluster monomials in which exponents of frozen variables are nonnegative, i.e., the subset of cluster monomials belonging to $\mathbb{Z}[x_{1,1}, \dots, x_{3,3}]$.

In contrast to [11, Fig. 8], we reserve the letters $a, \dots, g, A, \dots, G$ for use as *exponents* of cluster variables rather than using these to denote the cluster variables themselves. We will thus express each cluster monomial having no frozen factors as

$$x_{1,1}^a x_{1,2}^b x_{2,1}^c x_{2,2}^d x_{2,3}^e x_{3,2}^f x_{3,3}^g \Delta_{23,23}^A \Delta_{23,13}^B \Delta_{13,23}^C \Delta_{13,13}^D \Delta_{13,12}^E \Delta_{12,13}^F \Delta_{12,12}^G \mathrm{Imm}_{213}^p \mathrm{Imm}_{312}^q, \tag{2.2}$$

where at most four of the exponents $a,\ldots g, A\ldots, G, p, q$ are positive.

It is worth noting that each extended cluster provides a criterion for testing total positivity of a matrix y in $GL(3,\mathbb{C})$ or $SL(3,\mathbb{C})$. Specifically, y is totally positive (all minors of y are positive) if and only if each element of an (arbitrary) extended cluster evaluates positively on y. (See [2, Sec. 2.4], [11, Fig. 8].) Of course, the inequality $\det(y) > 0$ may be omitted for $y \in SL(3,\mathbb{C})$.

Observe that any permutation of $x_{1,1},\ldots,x_{3,3}$ induces an automorphism of the ring $\mathbb{Z}[x_{1,1},\ldots,x_{3,3}]$. In particular, we will consider three natural permutations and the corresponding involutive automorphisms defined by the usual matrix transposition $x \mapsto x^\top$, by matrix antitransposition $x \mapsto x^\perp$ (transposition across the antidiagonal)

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}^\perp \underset{\text{def}}{=} \begin{bmatrix} x_{3,3} & x_{2,3} & x_{1,3} \\ x_{3,2} & x_{2,2} & x_{1,2} \\ x_{3,1} & x_{2,1} & x_{1,1} \end{bmatrix},$$

and by the composition of these two maps $x \mapsto x^{\top\perp} = x^{\perp\top}$. We will use the same notation for the automorphisms and for induced maps on sets $\mathcal{F}$ of polynomials,

$$f(x)^\top \underset{\text{def}}{=} f(x^\top), \quad f(x)^\perp \underset{\text{def}}{=} f(x^\perp)$$
$$\mathcal{F}^\top \underset{\text{def}}{=} \{f(x)^\top \,|\, f(x) \in \mathcal{F}\}, \quad \mathcal{F}^\perp \underset{\text{def}}{=} \{f(x)^\perp \,|\, f(x) \in \mathcal{F}\}.$$

Observation 2.1. *The maps $\mathcal{C} \mapsto \mathcal{C}^\top$, $\mathcal{C} \mapsto \mathcal{C}^\perp$ are involutions on the set of clusters of $\mathbb{Z}[x_{1,1},\ldots,x_{3,3}]$.*

Proof. The antitransposition map may be interepreted geometrically as a reflection of the octogon in a vertical (equivalently, horizonal) axis. This clearly induces an involution on modified triangulations and therefore on clusters. Specifically, twenty-five pairs $\{\mathcal{C},\mathcal{D}\}$ of clusters satisfy $\mathcal{D} = \mathcal{C}^\perp \neq \mathcal{C}$, and each such pair occupies a single row of the table above. No cluster $\mathcal{C}$ satisfies $\mathcal{C} = \mathcal{C}^\perp$.

The transposition map may be interpreted geometrically as a swapping of colors on diameters which fixes all pairs of non-diameter diagonals. Again, this clearly induces an involution on modified triangulations and therefore on clusters. Specifically, fifteen pairs $\{\mathcal{C},\mathcal{D}\}$ of clusters satisfy $\mathcal{D} = \mathcal{C}^\top \neq \mathcal{C}$. Eleven of these pairs occupy the consecutive rows of the table containing clusters cdeA, . . . , gBDF and four more such pairs are

$$\{\text{bACp}, \text{cABp}\}, \{\text{acBp}, \text{abCp}\}, \{\text{eFGq}, \text{fEGq}\}, \{\text{egFq}, \text{fgEq}\}.$$

The twenty clusters $\mathcal{C}$ not included in these fifteen pairs satisfy $\mathcal{C} = \mathcal{C}^\top$. □

Using the diagrams (2.1) and the definition of modified triangulations of the octogon, we can identify certain pairs of cluster variables which never appear together in a single cluster, and therefore never appear together in a cluster monomial. In particular, we shall use the following facts.

Observation 2.2. *A product $x_{i_1,j_1}x_{i_2,j_2}$ of cluster variables is a cluster monomial if and only if we have $(i_1 - i_2)(j_1 - j_2) \leq 0$.*

By (2.2), the lowercase letters a, . . . , g in a cluster name correspond to cluster variables which are single matrix entries. Observation 2.2 therefore says that the pairs of letters in this range which appear together in a cluster name are precisely

ab, ac, bc, bd, bf, cd, ce, de, df, ef, eg, fg.

Observation 2.3. *Each product $\Delta_{\{i_1,i_2\},\{j_1,j_2\}}x_{i_1,j_1}$ and $\Delta_{\{i_1,i_2\},\{j_1,j_2\}}x_{i_2,j_2}$ of cluster variables is a cluster monomial.*

By (2.2), the capital letters A, . . . , G in a cluster name correspond to cluster variables which are 2×2 minors. Observation 2.3 therefore says that the pairs of cluster variables satisfying the claimed conditions are precisely those corersponding to the pairs of letters

aD, aE, aF, aG, bC, cB, dA, dG, eF, fE, gA, gB, gC, gD.

3. A correspondence between cluster monomials and matrices

Let $\mathcal{M}$ be the set of cluster monomials of $\mathbb{Z}[x_{1,1}, \dots, x_{3,3}]$. Let $\mathrm{Mat}_3(\mathbb{N})$ be the set of 3×3 matrices with entries in $\mathbb{N}$, and let $E_{i,j} \in \mathrm{Mat}_3(\mathbb{N})$ be the matrix whose (i, j) entry is 1 and whose other entries are 0. Let $\phi : \mathcal{M} \to \mathrm{Mat}_3(\mathbb{N})$ be the map defined on cluster variables by

$$\begin{aligned} \phi(\Delta_{\{i_1,\dots,i_k\},\{j_1,\dots,j_k\}}(x)) &= E_{i_1,j_1} + \cdots + E_{i_k,j_k}, \\ \phi(\mathrm{Imm}_{213}(x)) &= E_{1,2} + E_{2,1} + E_{3,3}, \\ \phi(\mathrm{Imm}_{132}(x)) &= E_{1,1} + E_{2,3} + E_{3,2}, \end{aligned}$$

and extended to cluster monomials in $\mathbb{Z}[x_{1,1}, \dots, x_{3,3}]$ by

$$\phi(z_1^{\epsilon_1} \cdots z_k^{\epsilon_k}) = \epsilon_1\phi(z_1) + \cdots + \epsilon_k\phi(z_k).$$

Employing a sequence of rather benign observations and propositions, we will show in Theorem 3.17 that ϕ is a bijection.

By definition we have $\phi(1) = 0$, and it is clear that ϕ maps each cluster monomial of degree r in $x_{1,1}, \dots, x_{3,3}$ to a matrix whose entries sum to r. It is also clear that ϕ commutes with the transposition and antitransposition maps,

$$\phi(x^{\top}) = \phi(x)^{\top}, \quad \phi(x^{\perp}) = \phi(x)^{\perp}.$$

To begin to establish that ϕ is a bijection, we partition the fifty clusters into twelve blocks defined in terms of ϕ. Specifically, each block consists of the clusters $\mathcal{C} = \{z_1, z_2, z_3, z_4\}$ with the property that for every cluster monomial $Z = z_1^{\epsilon_1}z_2^{\epsilon_2}z_3^{\epsilon_3}z_4^{\epsilon_4}$ (which contains no frozen factors) the matrix $\phi(Z)$ has five specific entries which are equal to zero.

Two blocks of clusters produce cluster monomials Z for which the matrices $\phi(Z)$ have the forms

$$\begin{bmatrix} 0&0&0\\ 0&*&*\\ 0&*&* \end{bmatrix}, \quad \begin{bmatrix} *&*&0\\ *&*&0\\ 0&0&0 \end{bmatrix}.$$

Observation 3.1. *Applying ϕ to the cluster monomials*

$$x_{2,2}^d x_{2,3}^e x_{3,2}^f \Delta_{23,23}^A, \quad x_{2,3}^e x_{3,2}^f x_{3,3}^g \Delta_{23,23}^A, \tag{3.1}$$

we obtain the matrices

$$\begin{bmatrix} 0&0&0\\ 0&d+A&e\\ 0&f&A \end{bmatrix}, \quad \begin{bmatrix} 0&0&0\\ 0&A&e\\ 0&f&A+g \end{bmatrix}.$$

Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ satisfies $p_{1,1} = p_{1,2} = p_{1,3} = p_{2,1} = p_{3,1} = 0$, then the unique cluster monomial Z appearing in (3.1) and satisfying $\phi(Z) = P$ is given by the formula

$$Z = \begin{cases} x_{2,2}^d x_{2,3}^e x_{3,2}^f \Delta_{23,23}^A & \text{if } p_{3,3} \le p_{2,2}, \\ x_{2,3}^e x_{3,2}^f x_{3,3}^g \Delta_{23,23}^A & \text{if } p_{2,2} < p_{3,3}, \end{cases}$$

where

$$A = \min\{p_{2,2}, p_{3,3}\}, \quad d = p_{2,2} - A, \quad g = p_{3,3} - A, \quad e = p_{2,3}, \quad f = p_{3,2}.$$

Note that Observation 3.1 does not assert the existence of a unique cluster monomial Z satifying $\phi(Z) = P$ for a matrix P of the stated form, except when Z is assumed to appear on the list (3.1). We in fact will make the stronger assertion in Theorem 3.17.

Observation 3.2. *Applying ϕ to the cluster monomials*

$$x_{1,2}^b x_{2,1}^c x_{2,2}^d \Delta_{12,12}^G, \quad x_{1,1}^a x_{1,2}^b x_{2,1}^c \Delta_{12,12}^G, \tag{3.2}$$

we obtain matrices P satisfying $p_{1,3} = p_{2,3} = p_{3,1} = p_{3,2} = p_{3,3} = 0$. Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ has the stated form then there is a unique cluster monomial Z in (3.2) satisfying $\phi(Z) = P$.

Proof. Apply the antitransposition map to Observation 3.1, or use straightforward computation. □

Four blocks of clusters produce cluster monomials Z for which the matrices $\phi(Z)$ have the forms

$$\begin{bmatrix} 0&0&0\\ *&*&*\\ 0&0&* \end{bmatrix}, \quad \begin{bmatrix} *&*&0\\ 0&*&0\\ 0&*&0 \end{bmatrix}, \quad \begin{bmatrix} 0&*&0\\ 0&*&0\\ 0&*&* \end{bmatrix}, \quad \begin{bmatrix} *&0&0\\ *&*&*\\ 0&0&0 \end{bmatrix}.$$

Observation 3.3. *Applying ϕ to the cluster monomials*

$$x_{2,1}^c x_{2,2}^d x_{2,3}^e \Delta_{23,23}^A, \quad x_{2,1}^c x_{2,3}^e \Delta_{23,23}^A \Delta_{23,13}^B, \quad x_{2,3}^e x_{3,3}^g \Delta_{23,23}^A \Delta_{23,13}^B, \tag{3.3}$$

we obtain the matrices

$$\begin{bmatrix} 0&0&0\\ c&d+A&e\\ 0&0&A \end{bmatrix}, \quad \begin{bmatrix} 0&0&0\\ c+B&A&e\\ 0&0&A+B \end{bmatrix}, \quad \begin{bmatrix} 0&0&0\\ B&A&e\\ 0&0&g+A+B \end{bmatrix}.$$

Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ satisfies $p_{1,1} = p_{1,2} = p_{1,3} = p_{3,1} = p_{3,2} = 0$, then the unique cluster monomial Z appearing in (3.3) and satisfying $\phi(Z) = P$ is given by the formula

$$Z = \begin{cases} x_{2,1}^c x_{2,2}^d x_{2,3}^e \Delta_{23,23}^A & \text{if } p_{3,3} \leq p_{2,2}, \\ x_{2,1}^c x_{2,3}^e \Delta_{23,23}^A \Delta_{23,13}^B & \text{if } p_{2,2} < p_{3,3} \leq p_{2,1} + p_{2,2}, \\ x_{2,3}^e x_{3,3}^g \Delta_{23,23}^A \Delta_{23,13}^B & \text{if } p_{2,1} + p_{2,2} < p_{3,3}, \end{cases}$$

where

$$A = \min\{p_{2,2}, p_{3,3}\}, \quad B = \min\{p_{3,3} - A, p_{2,1}\}, \quad e = p_{2,3},$$
$$c = p_{2,1} - B, \quad d = p_{2,2} - A, \quad g = p_{3,3} - A - B.$$

Observation 3.4. *Applying ϕ to the cluster monomials*

$$x_{1,2}^b x_{2,2}^d x_{3,2}^f \Delta_{12,12}^G, \quad x_{1,2}^b x_{3,2}^f \Delta_{13,12}^E \Delta_{12,12}^G, \quad x_{1,1}^a x_{1,2}^b \Delta_{13,12}^E \Delta_{12,12}^G, \tag{3.4}$$

we obtain matrices P satisfying $p_{1,3} = p_{2,1} = p_{2,3} = p_{3,1} = p_{3,3} = 0$. Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ has the stated form then there is a unique cluster monomial Z in (3.4) satisfying $\phi(Z) = P$.

Proof. Apply the antitransposition map to Observation 3.3. □

Observation 3.5. *Applying ϕ to the cluster monomials*

$$x_{1,2}^b x_{2,2}^d x_{3,2}^f \Delta_{23,23}^A, \quad x_{1,2}^b x_{3,2}^f \Delta_{23,23}^A \Delta_{13,23}^C, \quad x_{3,2}^f x_{3,3}^g \Delta_{23,23}^A \Delta_{13,23}^C, \tag{3.5}$$

we obtain matrices P satisfying $p_{1,1} = p_{1,3} = p_{2,1} = p_{2,3} = p_{3,1} = 0$. Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ has the stated form then there is a unique cluster monomial Z in (3.5) satisfying $\phi(Z) = P$.

Proof. Apply the transposition map to Observation 3.3. □

Observation 3.6. *Applying ϕ to the cluster monomials*

$$x_{2,1}^c x_{2,2}^d x_{2,3}^e \Delta_{12,12}^G, \quad x_{2,1}^c x_{2,3}^e \Delta_{12,13}^F \Delta_{12,12}^G, \quad x_{1,1}^a x_{2,1}^c \Delta_{12,13}^F \Delta_{12,12}^G, \tag{3.6}$$

we obtain matrices P satisfying $p_{1,2} = p_{1,3} = p_{3,1} = p_{3,2} = p_{3,3} = 0$. Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ has the stated form then there is a unique cluster monomial Z in (3.6) satisfying $\phi(Z) = P$.

Proof. Apply the transposition and antitransposition maps to Observation 3.3. □

Two blocks of clusters produce cluster monomials Z for which the matrices $\phi(Z)$ have the forms

$$\begin{bmatrix} * & 0 & 0 \\ * & 0 & * \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & * & * \end{bmatrix}.$$

Observation 3.7. *Applying ϕ to the cluster monomials*

$$x_{2,1}^c x_{2,3}^e \Delta_{23,13}^B \Delta_{12,13}^F, \quad x_{1,1}^a x_{2,1}^c \Delta_{23,13}^B \Delta_{12,13}^F, \quad x_{2,3}^e x_{3,3}^g \Delta_{23,13}^B \Delta_{12,13}^F,$$
$$x_{1,1}^a \Delta_{23,13}^B \Delta_{13,13}^D \Delta_{12,13}^F, \quad x_{3,3}^g \Delta_{23,13}^B \Delta_{13,13}^D \Delta_{12,13}^F, \tag{3.7}$$

we obtain the matrices

$$\begin{bmatrix} F & 0 & 0 \\ c+B & 0 & e+F \\ 0 & 0 & B \end{bmatrix}, \quad \begin{bmatrix} a+F & 0 & 0 \\ c+B & 0 & F \\ 0 & 0 & B \end{bmatrix}, \quad \begin{bmatrix} F & 0 & 0 \\ B & 0 & e+F \\ 0 & 0 & g+B \end{bmatrix},$$
$$\begin{bmatrix} a+D+F & 0 & 0 \\ B & 0 & F \\ 0 & 0 & B+D \end{bmatrix}, \quad \begin{bmatrix} D+F & 0 & 0 \\ B & 0 & F \\ 0 & 0 & g+B+D \end{bmatrix}.$$

Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ satisfies $p_{1,2} = p_{1,3} = p_{2,2} = p_{3,1} = p_{3,2} = 0$, then the unique cluster monomial Z appearing in (3.7) and satisfying $\phi(Z) = P$ is given by the formula

$$Z = \begin{cases} x_{2,1}^c x_{2,3}^e \Delta_{23,13}^B \Delta_{12,13}^F & \text{if } p_{1,1} - p_{2,3}, p_{3,3} - p_{2,1} \le 0, \\ x_{1,1}^a x_{2,1}^c \Delta_{23,13}^B \Delta_{12,13}^F & \text{if } p_{3,3} - p_{2,1} \le 0 < p_{1,1} - p_{2,3}, \\ x_{2,3}^e x_{3,3}^g \Delta_{23,13}^B \Delta_{12,13}^F & \text{if } p_{1,1} - p_{2,3} \le 0 < p_{3,3} - p_{2,1}, \\ x_{1,1}^a \Delta_{23,13}^B \Delta_{13,13}^D \Delta_{12,13}^F & \text{if } 0 < p_{3,3} - p_{2,1} \le p_{1,1} - p_{2,3}, \\ x_{3,3}^g \Delta_{23,13}^B \Delta_{13,13}^D \Delta_{12,13}^F & \text{if } 0 < p_{1,1} - p_{2,3} < p_{3,3} - p_{2,1}, \end{cases}$$

where

$$B = \min\{p_{2,1}, p_{3,3}\}, \quad F = \min\{p_{1,1}, p_{2,3}\}, \quad D = \min\{p_{1,1} - F, p_{3,3} - B\},$$
$$e = p_{2,3} - F, \quad c = p_{2,1} - B, \quad a = p_{1,1} - D - F, \quad g = p_{3,3} - B - D.$$

Observation 3.8. *Applying ϕ to the cluster monomials*

$$x_{1,2}^b x_{3,2}^f \Delta_{13,23}^C \Delta_{13,12}^E, \quad x_{3,2}^f x_{3,3}^g \Delta_{13,23}^C \Delta_{13,12}^E, \quad x_{1,1}^a x_{1,2}^b \Delta_{13,23}^C \Delta_{13,12}^E,$$
$$x_{3,3}^g \Delta_{13,23}^C \Delta_{13,13}^D \Delta_{13,12}^E, \quad x_{1,1}^a \Delta_{13,23}^C \Delta_{13,13}^D \Delta_{13,12}^E, \tag{3.8}$$

we obtain matrices P satisfying $p_{1,3} = p_{2,1} = p_{2,2} = p_{2,3} = p_{3,1} = 0$. Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ has the stated form then there is a unique cluster monomial Z in (3.8) satisfying $\phi(Z) = P$.

Proof. Apply the transposition or antitransposition map to Observation 3.7. □

Two blocks of clusters produce cluster monomials Z for which the matrices $\phi(Z)$ have the forms

$$\begin{bmatrix} 0 & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & 0 \end{bmatrix}.$$

Observation 3.9. *Applying ϕ to the clusters*

$$x_{1,2}^b x_{2,1}^c x_{2,2}^d \Delta_{23,23}^A, \quad x_{1,2}^b x_{2,1}^c \Delta_{23,23}^A \mathrm{Imm}_{213}^p, \quad x_{1,2}^b \Delta_{23,23}^A \Delta_{13,23}^C \mathrm{Imm}_{213}^p,$$
$$x_{2,1}^c \Delta_{23,23}^A \Delta_{23,13}^B \mathrm{Imm}_{213}^p, \quad \Delta_{23,23}^A \Delta_{23,13}^B \Delta_{13,23}^C \mathrm{Imm}_{213}^p, \quad x_{3,3}^g \Delta_{23,23}^A \Delta_{23,13}^B \Delta_{13,23}^C, \tag{3.9}$$

we obtain the matrices

$$\begin{bmatrix}0 & b & 0\\ c & d+A & 0\\ 0 & 0 & A\end{bmatrix}, \quad \begin{bmatrix}0 & b+p & 0\\ c+p & A & 0\\ 0 & 0 & A+p\end{bmatrix}, \quad \begin{bmatrix}0 & b+C+p & 0\\ p & A & 0\\ 0 & 0 & p+A+C\end{bmatrix},$$
$$\begin{bmatrix}0 & p & 0\\ c+B+p & A & 0\\ 0 & 0 & p+A+B\end{bmatrix}, \quad \begin{bmatrix}0 & p+C & 0\\ p+B & A & 0\\ 0 & 0 & p+A+B+C\end{bmatrix},$$
$$\begin{bmatrix}0 & C & 0\\ B & A & 0\\ 0 & 0 & g+A+B+C\end{bmatrix}.$$

Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ *satisfies* $p_{1,1} = p_{1,3} = p_{2,3} = p_{3,1} = p_{3,2} = 0$*, then the unique cluster monomial* Z *appearing in (3.9) and satisfying* $\phi(Z) = P$ *is given by the formula*

$$Z = \begin{cases} x_{1,2}^b x_{2,1}^c x_{2,2}^d \Delta_{23,23}^A & \textit{if } p_{3,3} \leq p_{2,2} \\ x_{1,2}^b x_{2,1}^c \Delta_{23,23}^A \mathrm{Imm}_{213}^p & \textit{if } p_{2,2} < p_{3,3} \leq p_{1,2} + p_{2,2}, p_{2,1} + p_{2,2} \\ x_{1,2}^b \Delta_{23,23}^A \Delta_{13,23}^C \mathrm{Imm}_{213}^p & \textit{if } p_{2,1} + p_{2,2} < p_{3,3} \leq p_{1,2} + p_{2,2} \\ x_{2,1}^c \Delta_{23,23}^A \Delta_{23,13}^B \mathrm{Imm}_{213}^p & \textit{if } p_{1,2} + p_{2,2} < p_{3,3} \leq p_{2,1} + p_{2,2} \\ \Delta_{23,23}^A \Delta_{23,13}^B \Delta_{13,23}^C \mathrm{Imm}_{213}^p & \textit{if } p_{1,2} + p_{2,2}, p_{2,1} + p_{2,2} < p_{3,3} \leq p_{1,2} + p_{2,1} + p_{2,2} \\ x_{3,3}^g \Delta_{23,23}^A \Delta_{23,13}^B \Delta_{13,23}^C & \textit{if } p_{1,2} + p_{2,1} + p_{2,2} < p_{3,3}, \end{cases}$$

where

$$A = \min\{p_{2,2}, p_{3,3}\}, \quad b = \max\{p_{1,2} + A - p_{3,3}, 0\}, \quad c = \max\{p_{2,1} + A - p_{3,3}, 0\},$$
$$p = \max\{p_{1,2} - b + p_{2,1} - c + A - p_{3,3}, 0\}, \quad B = \min\{p_{3,3} - A, p_{2,1}\} - p,$$
$$C = \min\{p_{3,3} - A, p_{1,2}\} - p, \quad d = p_{2,2} - A, \quad g = p_{3,3} - A - B - C - p.$$

Observation 3.10. *Applying* ϕ *to the cluster monomials*

$$x_{2,2}^d x_{2,3}^e x_{3,2}^f \Delta_{12,12}^G, \quad x_{2,3}^e x_{3,2}^f \Delta_{12,12}^G \mathrm{Imm}_{132}^q, \quad x_{3,2}^f \Delta_{13,12}^E \Delta_{12,12}^G \mathrm{Imm}_{132}^q,$$
$$x_{2,3}^e \Delta_{12,13}^F \Delta_{12,12}^G \mathrm{Imm}_{132}^q, \quad \Delta_{13,12}^E \Delta_{12,13}^F \Delta_{12,12}^G \mathrm{Imm}_{132}^q, \quad x_{1,1}^a \Delta_{13,12}^E \Delta_{12,13}^F \Delta_{12,12}^G, \tag{3.10}$$

we obtain matrices P *satisfying* $p_{1,2} = p_{1,3} = p_{2,1} = p_{3,1} = p_{3,3} = 0$*. Conversely, if* $P \in \mathrm{Mat}_3(\mathbb{N})$ *has the stated form then there is a unique cluster monomial* Z *in (3.10) satisfying* $\phi(Z) = P$*.*

Proof. Apply the antitransposition map to Observation 3.9. □

Two blocks of clusters produce cluster monomials Z for which the matrices $\phi(Z)$ have the forms

$$\begin{bmatrix}* & * & 0\\ * & 0 & 0\\ 0 & 0 & *\end{bmatrix}, \quad \begin{bmatrix}* & 0 & 0\\ 0 & 0 & *\\ 0 & * & *\end{bmatrix}.$$

Observation 3.11. *Applying ϕ to the cluster monomials*

$$\begin{gathered} x_{1,1}^a x_{1,2}^b x_{2,1}^c \mathrm{Imm}_{213}^p, \quad x_{1,1}^a x_{2,1}^c \Delta_{23,13}^B \mathrm{Imm}_{213}^p, \quad x_{1,1}^a x_{1,2}^b \Delta_{13,23}^C \mathrm{Imm}_{213}^p, \\ x_{1,1}^a \Delta_{23,13}^B \Delta_{13,23}^C \mathrm{Imm}_{213}^p, \quad x_{1,1}^a \Delta_{23,13}^B \Delta_{13,23}^C \Delta_{13,13}^D, \quad x_{3,3}^g \Delta_{23,13}^B \Delta_{13,23}^C \Delta_{13,13}^D, \end{gathered} \tag{3.11}$$

we obtain the matrices

$$\begin{bmatrix} a & b+p & 0 \\ c+p & 0 & 0 \\ 0 & 0 & p \end{bmatrix}, \quad \begin{bmatrix} a & p & 0 \\ c+B+p & 0 & 0 \\ 0 & 0 & p+B \end{bmatrix}, \quad \begin{bmatrix} a & b+C+p & 0 \\ p & 0 & 0 \\ 0 & 0 & C+p \end{bmatrix},$$

$$\begin{bmatrix} a & C+p & 0 \\ B+p & 0 & 0 \\ 0 & 0 & B+C+p \end{bmatrix}, \begin{bmatrix} a+D & C & 0 \\ B & 0 & 0 \\ 0 & 0 & B+C+D \end{bmatrix}, \begin{bmatrix} D & C & 0 \\ B & 0 & 0 \\ 0 & 0 & g+B+C+D \end{bmatrix}.$$

Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ satisfies $p_{1,3} = p_{2,2} = p_{2,3} = p_{3,1} = p_{3,2} = 0$, then the unique cluster monomial Z appearing in (3.11) and satisfying $\phi(Z) = P$ is given by the formula

$$Z = \begin{cases} x_{1,1}^a x_{1,2}^b x_{2,1}^c \mathrm{Imm}_{213}^p & \text{if } p_{3,3} \le p_{1,2}, p_{2,1} \\ x_{1,1}^a x_{2,1}^c \Delta_{23,13}^B \mathrm{Imm}_{213}^p & \text{if } p_{1,2} < p_{3,3} \le p_{2,1} \\ x_{1,1}^a x_{1,2}^b \Delta_{13,23}^C \mathrm{Imm}_{213}^p & \text{if } p_{2,1} < p_{3,3} \le p_{1,2} \\ x_{1,1}^a \Delta_{23,13}^B \Delta_{13,23}^C \mathrm{Imm}_{213}^p & \text{if } p_{1,2}, p_{2,1} < p_{3,3} \le p_{1,2} + p_{2,1} \\ x_{1,1}^a \Delta_{23,13}^B \Delta_{13,23}^C \Delta_{13,13}^D & \text{if } p_{1,2} + p_{2,1} < p_{3,3} \le p_{1,1} + p_{1,2} + p_{2,1} \\ x_{3,3}^g \Delta_{23,13}^B \Delta_{13,23}^C \Delta_{13,13}^D & \text{if } p_{1,1} + p_{1,2} + p_{2,1} < p_{3,3}, \end{cases}$$

where

$$\begin{gathered} b = \max\{p_{1,2} - p_{3,3}, 0\}, \quad c = \max\{p_{2,1} - p_{3,3}, 0\}, \quad g = \max\{p_{3,3} - p_{1,1} - p_{1,2} - p_{2,1}, 0\}, \\ D = \max\{p_{3,3} - p_{1,2} - p_{2,1} - g, 0\}, \quad a = p_{1,1} - D, \quad B = p_{3,3} - p_{1,2} + b - g - D \\ C = p_{3,3} - p_{2,1} + c - g - D \quad p = p_{3,3} - g - D - B - C. \end{gathered}$$

Observation 3.12. *Applying the map ϕ to the cluster monomials*

$$\begin{gathered} x_{2,3}^e x_{3,2}^f x_{3,3}^g \mathrm{Imm}_{132}^q, \quad x_{3,2}^f x_{3,3}^g \Delta_{13,12}^E \mathrm{Imm}_{132}^q, \quad x_{2,3}^e x_{3,3}^g \Delta_{12,13}^F \mathrm{Imm}_{132}^q, \\ x_{3,3}^g \Delta_{13,12}^E \Delta_{12,13}^F \mathrm{Imm}_{132}^q, \quad x_{3,3}^g \Delta_{13,13}^D \Delta_{13,12}^E \Delta_{12,13}^F, \quad x_{1,1}^a \Delta_{13,13}^D \Delta_{13,12}^E \Delta_{12,13}^F, \end{gathered} \tag{3.12}$$

we obtain matrices P satisfying $p_{1,2} = p_{1,3} = p_{2,1} = p_{2,2} = p_{3,1} = 0$. Conversely, if $P \in \mathrm{Mat}_3(\mathbb{N})$ has the stated form then there is a unique cluster monomial Z in (3.12) satisfying $\phi(Z) = P$.

Proof. Apply the antitransposition map to Observation 3.11. □

Combining Observations 3.1 - 3.12, we now have the following.

Proposition 3.13. *Let $Z = z_1^{\epsilon_1} \cdots z_4^{\epsilon_4}$ be a cluster monomial having no frozen factors. Then the matrix $P = \phi(Z)$ has at most four nonzero entries and satisfies*

$$p_{1,3} = p_{1,2}p_{2,3} = p_{1,1}p_{2,2}p_{3,3} = p_{2,1}p_{3,2} = p_{3,1} = 0. \tag{3.13}$$

Conversely, every matrix P satisfying (3.13) is equal to $\phi(Z)$ for some cluster monomial Z, and this cluster monomial has no frozen factors. If P has exactly four nonzero entries, then Z is unique.

Proof. By Observations 3.1 - 3.12, $\phi(Z)$ clearly satisfies (3.13). Conversely, let P be any matrix in $\mathrm{Mat}_3(\mathbb{N})$ which satisfies (3.13). Then P has at most four nonzero entries, which appear in positions as described preceding Observations 3.1, 3.3, 3.7, 3.9 and 3.11. Therefore at least one of Observations 3.1 - 3.12 gives a cluster monomial Z having no frozen factors and satisfying $\phi(Z) = P$. If P has exactly four nonzero entries, then exactly one of the observations gives such a cluster monomial Z, and this cluster monomial is unique. □

Now we consider matrices P satisfying (3.13) and having fewer than four nonzero entries. We will show in the following propositions that P still uniquely determines a cluster monomial Z satisfying $\phi(Z) = P$.

Proposition 3.14. *Let $P \in \mathrm{Mat}_3(\mathbb{N})$ satisfy (3.13) and have at most two positive entries. Then there is a unique cluster monomial Z satisfying $\phi(Z) = P$.*

Proof. If $P = 0$ then $Z = 1$. If P has a single nonzero entry $p_{i,j}$, then $Z = x_{i,j}^{p_{i,j}}$. Let us assume therefore that P has exactly two nonzero entries p_{i_1,j_1}, p_{i_2,j_2}, with indices satisfying $i_1 \leq i_2$.

If the indices of these two entries satisfy $i_1 < i_2$ and $j_1 < j_2$, then by Observation 2.3, $\Delta_{\{i_1,i_2\},\{j_1,j_2\}}$ is a cluster variable which appears in at least one cluster with x_{i_1,j_1}, and in at least one cluster with x_{i_2,j_2}. On the other hand, by Observation 2.2, the two cluster variables x_{i_1,j_1}, x_{i_2,j_2} never appear together in a cluster. Thus we have

$$Z = \begin{cases} x_{i_2,j_2}^{p_{i_2,j_2}-p_{i_1,j_1}} \Delta_{\{i_1,i_2\},\{j_1,j_2\}}^{p_{i_1,j_1}} & \text{if } p_{i_1,j_1} < p_{i_2,j_2}, \\ x_{i_1,j_1}^{p_{i_1,j_1}-p_{i_2,j_2}} \Delta_{\{i_1,i_2\},\{j_1,j_2\}}^{p_{i_2,j_2}} & \text{otherwise.} \end{cases}$$

Now suppose that the indices satisfy $i_1 = i_2$ or $j_1 \geq j_2$. Since there is no cluster variable Z for which $\phi(Z)$ has nonzero entries in the required positions, we must have that Z is a product of powers of x_{i_1,j_1} and x_{i_2,j_2}. Furthermore, by Observation 2.2, these two cluster variables appear together in at least one cluster. Thus we have

$$Z = x_{i_1,j_1}^{p_{i_1,j_1}} x_{i_2,j_2}^{p_{i_2,j_2}}.$$

□

If P is a matrix satisfying (3.13) and having exactly three nonzero entries, then the positions of these entries must be one of the twenty-four three-element subsets $S = \{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}$ of $\{(1, 1), \ldots, (3, 3)\}$ not containing

$$\{(1, 3)\}, \quad \{(3, 1)\}, \quad \{(1, 2), (2, 3)\}, \quad \{(2, 1), (3, 2)\}, \quad \{(1, 1), (2, 2), (3, 3)\}.$$

Propositions 3.15 and 3.16 treat these subsets S several at a time.

First we show that any matrix having one of the six forms

$$\begin{bmatrix} * & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{bmatrix}, \quad \begin{bmatrix} 0 & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

is equal to $\phi(Z)$ for a unique cluster monomial Z.

Proposition 3.15. *Let $P \in \mathrm{Mat}_3(\mathbb{N})$ satisfy (3.13) and have exactly three positive entries p_{i_1,j_1}, p_{i_2,j_2}, p_{i_3,j_3} with indices satisfying $i_1 \leq i_2 \leq i_3$ and $j_1 \geq j_2 \geq j_3$. Then there is a unique cluster monomial Z satisfying $\phi(Z) = P$.*

Proof. Note that for no indices $k_1 \leq k_2$ and $\ell_1 \geq \ell_2$ is there a cluster variable Z satisfying $\phi(Z) = E_{k_1,\ell_1} + E_{k_2,\ell_2}$. Thus, Z must be a product of three cluster variables which are matrix entries. The six triples of variables satisfying the index conditions correspond to cluster names containing

abc, efg, bcd, edf, bdf, cde.

Since each of these appears in at least one cluster (actually two), we have

$$Z = x_{i_1,j_1}^{p_{i_1,j_1}} x_{i_2,j_2}^{p_{i_2,j_2}} x_{i_3,j_3}^{p_{i_3,j_3}}.$$

□

Now we consider matrices having one of the remaining eighteen forms

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.14}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ * & 0 & * \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & * & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ * & 0 & * \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.15}$$

$$\begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \tag{3.16}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & * & 0 \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.17}$$

$$\begin{bmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix}. \tag{3.18}$$

Proposition 3.16. *Let $P \in \mathrm{Mat}_3(\mathbb{N})$ satisfy (3.13) and have exactly three positive entries whose indices do not satisfy the conditions of Proposition 3.15. Then there is a unique cluster monomial Z satisfying $\phi(Z) = P$.*

Proof. First we prove the proposition for matrices P having the forms appearing first on the five lists (3.14) - (3.18). In each of these five cases, Observations 3.1 - 3.12 imply that there are at most two distinct cluster monomials Z satisfying $\phi(Z) = P$.

If P is of the first form listed in (3.14), i.e., if $p_{2,2}, p_{2,3}, p_{3,3} > 0$, then only Observations 3.1 and 3.3 give a formula for Z. Moreover, these two observations give the same formula,

$$Z = \begin{cases} \Delta^A_{23,23} x^e_{2,3} x^g_{3,3} & \text{if } p_{2,2} \leq p_{3,3}, \\ \Delta^A_{23,23} x^e_{2,3} x^d_{2,2} & \text{otherwise,} \end{cases} \tag{3.19}$$

and Z is therefore unique. Similarly, if P is of the first form listed in (3.15), then Observations 3.3 and 3.7 give a unique formula for Z, if P is of the first form listed in (3.16), then Observations 3.7 and 3.11 give a unique formula for Z, if P is of the first form listed in (3.17), then Observations 3.3 and 3.9 give a unique formula for Z, and if P is of the first form listed in (3.18), then Observations 3.9 and 3.11 give a unique formula for Z.

Now to each of the formulas $\phi(Z) = P$ we have obtained, we apply the transpose and antitranspose maps to obtain formulas for matrices of all the remaining forms listed in (3.14) - (3.18). □

Combining Propositions 3.13-3.16, we have that for any matrix P satisfying (3.13), there is a unique cluster monomial Z satisfying $\phi(Z) = P$. Now we remove the requirement that P satisfy (3.13).

Theorem 3.17. *For any $P \in \mathrm{Mat}_3(\mathbb{N})$, there is a unique cluster monomial $Z = z_1^{\epsilon_1} \cdots z_k^{\epsilon_k}$ with $\epsilon_1, \ldots, \epsilon_k > 0$ satisfying $\phi(Z) = P$.*

Proof. If P satisfies (3.13), then it has at most four nonzero entries and we may apply Propositions 3.13 - 3.16 to reach the desired conclusion.

Suppose therefore that P does not satisfy (3.13). By definition, every cluster monomial Z factors uniquely as $Z = XY$ where X has no frozen factors and Y has only frozen factors, say

$$Y = x_{1,3}^i x_{3,1}^h \Delta_{12,23}^H \Delta_{23,12}^I \det{}^J, \tag{3.20}$$

for some integers $h, i, H, I, J \in \mathbb{N}$. Then we have

$$\phi(Y) = \begin{bmatrix} J & H & i \\ I & J & H \\ h & I & J \end{bmatrix}$$

and $\phi(Z) = \phi(X) + \phi(Y)$.

Choosing the exponents h, i, H, I, J in Y to be

$$\begin{gathered} i = p_{1,3}, \quad h = p_{3,1}, \quad H = \min\{p_{1,2}, p_{2,3}\}, \quad I = \min\{p_{2,1}, p_{3,2}\}, \\ J = \min\{p_{1,1}, p_{2,2}, p_{3,3}\}, \end{gathered} \tag{3.21}$$

and subtracting $\phi(Y)$ from P, we obtain a matrix $Q = P - \phi(Y)$ which satisfies (3.13). By Propositions 3.13 - 3.16, the matrix Q uniquely determines X above and we have $\phi(Z) = \phi(XY) = P$.

Now suppose that an arbitrary cluster monomial Z' satisfies $\phi(Z') = P$ and factors uniquely as $Z' = X'Y'$ with X' having only nonfrozen factors and Y' having only frozen factors,

$$Y' = x_{1,3}^{i'} x_{3,1}^{h'} \Delta_{12,23}^{H'} \Delta_{23,12}^{I'} \det{}^{J'}.$$

If any of the components of the sequence (h', i', H', I', J') is greater than the corresponding compoent of (h, i, H, I, J), then $P - \phi(Y')$ has a negative entry and cannot be equal to $\phi(X')$. On the other hand, if any of the components of (h', i', H', I', J')

is less than the corresponding component of (h, i, H, I, J), then the matrix $P - \phi(Y')$ does not satisfy (3.13) and again cannot be equal to $\phi(X')$. We conclude that $Y' = Y$ and therefore that $X' = X$. □

We now have a well-defined map $\phi^{-1} : \mathrm{Mat}_3(\mathbb{N}) \to \mathcal{M}$. To compute $\phi^{-1}(P)$, we first define numbers h, i, H, I, J as in (3.21) and define a cluster monomial Y as in (3.20). Then we define the matrix

$$Q = P - \begin{bmatrix} J & H & i \\ I & J & H \\ h & I & J \end{bmatrix}$$

and use Observations 3.1 - 3.12 to compute a second cluster monomial $X = \phi^{-1}(Q)$. Finally, we have $\phi^{-1}(P) = XY$.

4. Comparison of three bases of $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$

The bijection given in Theorem 3.17 is closely related to a multigrading of the ring $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$ and to two bases known as the *natural* basis and *dual canonical* basis. Using a well-known description of the natural basis in terms of $\mathrm{Mat}_3(\mathbb{N})$, we will show in Theorem 4.3 that the set $\mathcal{M}$ of cluster monomials is a basis as well. (The fact that $\mathcal{M}$ is a basis also follows from a more general unpublished result of Fomin and Zelevinsky [15]. Linear independence was established in [16, Thm. 11.2].) Furthermore, we will show in Theorems 4.2, 4.3, 4.4 that all three bases are related by unitriangular transition matrices, and will state a conjectured formula relating the cluster and dual canonical bases in Corollary 4.5.

Of course $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$ has a natural grading by degree,

$$\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}] = \bigoplus_{r \geq 0} \mathcal{A}_r,$$

where $\mathcal{A}_r$ is the $\mathbb{Z}$-span of all monomials of total degree r, and the natural basis $\{x_{1,1}^{a_{1,1}} \cdots x_{3,3}^{a_{3,3}} \,|\, (a_{1,1}, \ldots, a_{3,3}) \in \mathbb{N}^9\}$ of $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$ is a disjoint union

$$\bigcup_{r \geq 0} \{x_{1,1}^{a_{1,1}} \cdots x_{3,3}^{a_{3,3}} \,|\, a_{1,1} + \cdots + a_{3,3} = r\}$$

of bases of $\{\mathcal{A}_r \,|\, r \geq 0\}$. Partially ordering monomials by weighting the variables $x_{1,2}$, $x_{2,3}$, $x_{2,1}$, $x_{3,2}$ more heavily than $x_{1,3}$, $x_{3,1}$ and weighting $x_{1,1}$, $x_{2,2}$, $x_{3,3}$ more heavily still, one observes the leading term in the natural expansion of each cluster monomial Z in $\mathcal{A}_r$ to have coefficient 1 and exponents equal to the entries of $\phi(Z)$. This fact then implies $\mathcal{A}_r \cap \mathcal{M}$ to be a $\mathbb{Z}$-basis of $\mathcal{A}_r$ and provides a unitriangular transition matrix relating this basis to the natural basis. Rather than using this point of view however, we will consider a coarser (than $\mathbb{N}^9$) grading of $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$ and a different partial order on monomials which will illuminate connections between the cluster basis, dual canonical basis and symmetric groups.

Each homogeneous component $\mathcal{A}_r$ of $\mathbb{Z}[x_{1,1},\dots,x_{3,3}]$ may be further decomposed by considering multisets $M=(m(1),\dots,m(r))$, $N=(n(1),\dots,n(r))$ of integers

$$\begin{aligned}1\le m(1)\le\cdots\le m(r)\le 3,\\ 1\le n(1)\le\cdots\le n(r)\le 3.\end{aligned}$$

In particular we have the multigrading

$$\mathcal{A}_r=\bigoplus_{\substack{M,N\\|M|=|N|=r}}\mathcal{A}_{M,N},$$

where $\mathcal{A}_{M,N}$ is the linear span of monomials whose row indices and column indices are given by the multisets M and N, respectively. We will say that an element of $\mathcal{A}_{M,N}$ is *homogeneous of multidegree* (M,N). If $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$ are the multiplicities with which 1, 2, 3 appear in M and N respectively, we will sometimes write $M=1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}$, $N=1^{\beta_1}2^{\beta_2}3^{\beta_3}$. Just as the $\mathbb{Z}$-graded components $\mathcal{A}_r$ and $\mathcal{A}_s$ satisfy $\mathcal{A}_r\mathcal{A}_s\subset\mathcal{A}_{r+s}$, the multigraded components $\mathcal{A}_{M,N}$ and $\mathcal{A}_{M',N'}$ satisfy

$$\mathcal{A}_{M,N}\mathcal{A}_{M',N'}\subset\mathcal{A}_{M\Cup M',N\Cup N'},$$

where $\Cup$ denotes the *multiset union* of two multisets,

$$1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\Cup 1^{\alpha'_1}2^{\alpha'_2}3^{\alpha'_3}\underset{\text{def}}{=}1^{\alpha_1+\alpha'_1}2^{\alpha_2+\alpha'_2}3^{\alpha_3+\alpha'_3}. \tag{4.1}$$

Given two multisets

$$\begin{aligned}M=(m(1),\dots,m(r))=1^{\alpha_1}2^{\alpha_2}3^{\alpha_3},\\ N=(n(1),\dots,n(r))=1^{\beta_1}2^{\beta_2}3^{\beta_3},\end{aligned}\tag{4.2}$$

define the Young subgroups S_α and S_β of the symmetric group S_r by

$$\begin{aligned}S_\alpha&=S_{[1,\alpha_1]}\times S_{[\alpha_1+1,\alpha_1+\alpha_2]}\times S_{[\alpha_1+\alpha_2+1,r]},\\ S_\beta&=S_{[1,\beta_1]}\times S_{[\beta_1+1,\beta_1+\beta_2]}\times S_{[\beta_1+\beta_2+1,r]},\end{aligned}$$

where $S_{[i,j]}$ is the subgroup of S_r generated by the adjacent transpositions $s_i,\dots,s_{j-1}$. These Young subgroups decompose S_r as a disjoint union of double cosets of the form $S_\alpha vS_\beta$. Letting $\Lambda(r,\alpha,\beta)$ be the set of Bruhat maximal representatives, i.e., permutations $v\in S_r$ satisfying $uv<v$ for all $u\in S_\alpha$ and $v>vw$ for all $w\in S_\beta$, we may write

$$S_r=\bigcup_{v\in\Lambda(r,\alpha,\beta)}S_\alpha vS_\beta.$$

(See [4], [18] for more information on the Bruhat order of S_r.) Given a permutation $v=v_1\cdots v_r$ in S_r and an $r\times r$ matrix $y=(y_{1,1},\dots,y_{r,r})$, we define

$$y^v=y_{1,v_1}\cdots y_{r,v_r}.$$

In particular, we will apply this notation to the *generalized submatrix* $x_{M,N}$ of x, defined by

$$x_{M,N} \underset{\text{def}}{=} \begin{bmatrix} x_{m(1),n(1)} & \cdots & x_{m(1),n(r)} \\ x_{m(2),n(1)} & \cdots & x_{m(2),n(r)} \\ \vdots & & \vdots \\ x_{m(r),n(1)} & \cdots & x_{m(r),n(r)} \end{bmatrix},$$

to express the natural basis of $\mathcal{A}_{M,N}$ as $\{(x_{M,N})^v \,|\, v \in \Lambda(r,\alpha,\beta)\}$.

It is well known that double cosets $\{S_\alpha v S_\beta \,|\, v \in \Lambda(3,\alpha,\beta)\}$ correspond bijectively to matrices in $\mathrm{Mat}_3(\mathbb{N})$ having row sums α and column sums β. (See, e.g., [19], [27, Sec. 7.11].) In particular, given a permutation $v = v_1 \cdots v_r$ and r-element multisets M, N as in (4.2) define the matrix $D_{M,N}(v) = (d_{1,1},\ldots,d_{3,3})$ by letting $d_{i,j}$ be the number of letters appearing in positions $\alpha_1 + \cdots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \cdots + \alpha_i$ of v and having value in the range $\beta_1 + \cdots + \beta_{j-1} + 1, \ldots, \beta_1 + \cdots + \beta_j$. The coset $S_\alpha v S_\beta$ consists of all permutations w with $D_{M,N}(w) = D_{M,N}(v)$. In this case, it is easy to see that $(x_{M,N})^w = (x_{M,N})^v$. Conversely, given a matrix $P \in \mathrm{Mat}_3(\mathbb{N})$ having row sums α and column sums β, let $r = \alpha_1+\alpha_2+\alpha_3 = \beta_1+\beta_2+\beta_3$ and define $\psi(P)$ to be the maximal element of $\Lambda(r,\alpha,\beta)$.

The homogeneity of the cluster variables ensures that each cluster monomial too is homogeneous with respect to our multigrading of $\mathbb{Z}[x_{1,1},\ldots,x_{3,3}]$. For example, the cluster monomial

$$x_{2,1}^2 x_{2,3} \Delta_{23,13} \Delta_{12,13}^2 x_{1,3}^3 \Delta_{123,123},$$

which is a product of cluster variables belonging to the cluster ceBF and of frozen variables $x_{1,3}$, $\Delta_{123,123}$, belongs to $\mathcal{A}_{M,N}$, where $M = 1^6 2^7 3^2$, $N = 1^6 2 3^8$. Furthermore, we have the following.

Proposition 4.1. *Let the cluster monomial Z belong to $\mathcal{A}_{M,N}$ for some r-element multisets $M = 1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}$, $N = 1^{\beta_1}2^{\beta_2}3^{\beta_3}$. Then the row sums of $\phi(Z)$ and column sums of $\phi(Z)$ are α and β, respectively.*

Proof. Let the cluster monomial Z factor as $Z = z_1^{\epsilon_1} \cdots z_k^{\epsilon_k}$, where each cluster variable z_ℓ belongs to $\mathcal{A}_{I_\ell,J_\ell}$ for some subsets $I_1,\ldots,I_k$, $J_1,\ldots,J_k$ of $\{1,2,3\}$.

Since Z belongs to $\mathcal{A}_{M,N}$, we may use the multigrading (4.1) of $\mathbb{Z}[x_{1,1},\ldots,x_{3,3}]$ to see that

$$M = \underbrace{I_1 \uplus \cdots \uplus I_1}_{\epsilon_1} \uplus \cdots \uplus \underbrace{I_k \uplus \cdots \uplus I_k}_{\epsilon_k},$$
$$N = \underbrace{J_1 \uplus \cdots \uplus J_1}_{\epsilon_1} \uplus \cdots \uplus \underbrace{J_k \uplus \cdots \uplus J_k}_{\epsilon_k}.$$

Thus α and β are given by

$$\alpha_i = \sum_{\substack{1\le \ell \le k \\ i \in I_\ell}} \epsilon_\ell, \qquad \beta_j = \sum_{\substack{1\le \ell \le k \\ j \in J_\ell}} \epsilon_\ell. \tag{4.3}$$

On the other hand, for each index ℓ, and for $i \in I_\ell$ and $j \in J_\ell$, we have that row i and column j of the matrix $\phi(z_\ell^{\epsilon_\ell})$ both sum to ϵ_ℓ. The definition of the map ϕ thus implies that row and column sums of $\phi(Z)$ are given by (4.3). □

We now express cluster monomials in terms of the natural basis. In the proof of the following proposition, we will say that a word $a_1 \cdots a_d$ consisting of distinct letters in $\mathbb{N}$ *matches the pattern* of a permutation $b = b_1 \cdots b_d$ in S_d if the relative order of letters in the word is the same as that in the permutation. For instance, 5246 matches the pattern 3124 since 5, 2, 4, 6 are respectively the third smallest, smallest, second smallest, and fourth smallest letters in the word.

Theorem 4.2. *Let the cluster monomial Z belong to $\mathcal{A}_{M,N}$ for some r-element multisets $M = 1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}$, $N = 1^{\beta_1}2^{\beta_2}3^{\beta_3}$, and define the permutation $v = \psi(\phi(Z))$ in S_r. Then we have*

$$Z = \sum_{\substack{w \in \Lambda(r,\alpha,\beta) \\ w \geq v}} c_{v,w}(x_{M,N})^w, \tag{4.4}$$

where $\{c_{v,w}\}$ are integers with $c_{v,v} = 1$.

Proof. Since $\mathcal{A}_{M,N} = \mathrm{span}\{(x_{M,N})^w \,|\, w \in \Lambda(r,\alpha,\beta)\}$, it is clear that all permutations w in the sum in Equation (4.4) may be chosen so that they belong to $\Lambda(r,\alpha,\beta)$. To justify the remaining aspects of the formula, let $Z = z_1^{\epsilon_1} \cdots z_n^{\epsilon_n}$. We proceed by induction on the sum $\epsilon_1 + \cdots + \epsilon_n$, which we will call the *cluster degree* of Z.

When the cluster degree of Z is 1, then Z is equal to a single cluster variable. This cluster variable is a matrix minor or is Imm_{213} or Imm_{132} and we have

$$Z = \sum_{\substack{u \in S_d \\ u \geq t}} (-1)^{\ell(u)-\ell(t)}(x_{I,J})^u, \tag{4.5}$$

where I and J are row and column sets (without repetition) having cardinality d, $1 \leq d \leq 3$. The permutation $t \in S_d$ is e, 213, or 132 and the coefficient of $(x_{I,J})^t$ is 1, as required.

Now suppose that Equation (4.4) holds for cluster monomials of cluster degree $1, \ldots, \ell$. Let Z be a cluster monomial of cluster degree $\ell + 1$, and let i be the least index for which ϵ_i is positive. Then we may write $Z = z_i Z'$, where the cluster variable z_i has the form (4.5) and we have by induction that

$$Z = z_i \left(\sum_{\substack{w' \in \Lambda(r',\alpha',\beta') \\ w' \geq v'}} c'_{v',w'}(x_{M',N'})^{w'} \right),$$

where $r' = r - d$, $v' \in S_{r'}$, α', β' are defined from $\phi(Z')$ as before, and $c'_{v',v'} = 1$.

Each term of Z is a monomial in x of the form $(x_{M,N})^w$, where for some indices $j_1, \ldots, j_d$, the d letters $w_{j_1}, \ldots, w_{j_d}$ of the one-line notation of w match the pattern of some permutation $u = u_1 \cdots u_d$ in S_d and the remaining $r-d$ letters $w_{k_1}, \cdots, w_{k_{r-d}}$ match the pattern of some permutation $w' = w'_1 \cdots w'_{r-d}$ in S_{r-d}.

We necessarily have that w is greater than or equal to the permutation in S_r obtained from w by rearranging the letters $w_{j_1}, \dots, w_{j_d}$ to match the pattern t. This permutation in turn is greater than or equal to the permutation obtained from it by rearranging the letters $w_{k_1}, \dots, w_{k_{r-d}}$ to match the pattern v'. Since this last permutation is v, it follows that $w \geq v$. Since the coefficient of $(x_{I,J})^t$ in the expansion of z_i is 1, it follows that $c_{v,v} = c'_{v',v'} = 1$. □

We now can deduce that the set $\mathcal{M}$ of cluster monomials of $\mathbb{Z}[x_{1,1},\dots,x_{3,3}]$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}[x_{1,1},\dots,x_{3,3}]$. More specifically, we have the following.

Theorem 4.3. *Let $M = 1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}$, $N = 1^{\beta_1}2^{\beta_2}3^{\beta_3}$ be r-element multisets of $\{1,2,3\}$. Then the set of cluster monomials Z for which $\phi(Z)$ has row sums α and column sums β forms a basis for $\mathcal{A}_{M,N}$.*

Proof. By Proposition 4.1 these cluster monomials belong to $\mathcal{A}_{M,N}$, and by Theorem 3.17 they are in bijective correspondence with the elements of the natural basis of $\mathcal{A}_{M,N}$. Expanding each cluster monomial in terms of the natural basis of $\mathcal{A}_{M,N}$ as in Theorem 4.2, we obtain a unitriangular matrix of coefficients $\{c_{u,v} \mid u, v \in \Lambda(r,\alpha,\beta)\}$. Thus our collection of cluster monomials also forms a basis of $\mathcal{A}_{M,N}$. □

Another basis of $\mathbb{Z}[x_{1,1},\dots,x_{3,3}]$ which arises often in physics and representation theory is the *dual canonical* or *crystal* basis, introduced independently by Lusztig [23] and Kashiwara [20]. Popular because it facilitates the construction of irreducible modules for quantum groups, the basis unfortunately has no known elementary description. We shall express this basis as

$$\bigcup_{r\geq 0} \bigcup_{M,N} \{\mathrm{Imm}_v(x_{M,N}) \mid v \in \Lambda(r,\alpha,\beta)\},$$

where M, N are r-element multisets of $\{1,2,3\}$ and α, β are related to these as before.

Du [7] has shown that the basis elements expand in terms of the natural basis as

$$\mathrm{Imm}_v(x_{M,N}) = \sum_{\substack{w\in\Lambda(r,\alpha,\beta)\\ w\geq v}} (-1)^{\ell(w)-\ell(v)} Q^{\alpha,\beta}_{v,w}(1) x_{1,1}^{p_{1,1}} \cdots x_{3,3}^{p_{3,3}}, \tag{4.6}$$

where $\{Q^{\alpha,\beta}_{v,w} \mid v, w \in \Lambda(r,\alpha,\beta)\}$ are polynomials in a single variable, with integer coefficients, and which are identically 1 when $v = w$. These in turn are equal to alternating sums of *inverse Kazhdan-Lusztig polynomials*, defined recursively in [21] or by alternating sums of other polynomials in [5]. Alternatively, we have the formula [26]

$$\mathrm{Imm}_v(x_{M,N}) = \sum_{\substack{w\in S_r\\ w\geq v}} (-1)^{\ell(w)-\ell(v)} Q_{v,w}(1)(x_{M,N})^w,$$

which also relies on the inverse Kazhdan-Lusztig polynomials $\{Q_{v,w} \mid v, w \in S_r\}$.

Since combinatorial interpretations of cluster monomials follow immediately from [22] and [25], it would be interesting to have exact formulae for the entries of the transition matrix relating the cluster and dual canonical bases. The following result gives evidence in favor of a simple formula.

Theorem 4.4. *Let* $M = 1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}$, $N = 1^{\beta_1}2^{\beta_2}3^{\beta_3}$ *be* r*-element multisets of* $\{1, 2, 3\}$*. Then the transition matrix relating the cluster basis and dual canonical basis of* $\mathcal{A}_{M,N}$ *is unitriangular. In particular we have*

$$Z = \sum_{\substack{w \in \Lambda(r,\alpha,\beta) \\ w \geq v}} d_{v,w} \mathrm{Imm}_w(x_{M,N}),$$

where $v = v(Z) = \psi(\phi(Z))$ *and the coefficients* $\{d_{v,w} \,|\, v, w \in \Lambda(r, \alpha, \beta)\}$ *are nonnegative integers satisfying* $d_{v,v} = 1$*.*

Proof. By Theorem 4.3 the transition matrix expressing cluster monomials in terms of the natural basis is unitriangular and has integer entries. By Du's formula (4.6), the transition matrix expressing natural basis elements in terms of the dual canonical basis is unitriangular and has integer entries. Furthermore, the ordering of basis elements (in terms of the Bruhat order) can be chosen to be the same in both formulae. Thus the coefficients $d_{v,w}$ above are integers and satisfy $d_{v,v} = 1$.

Since every cluster variable is itself a dual canonical basis element, a result of Lusztig [24] implies that each cluster monomial expands nonnegatively in terms of the dual canonical basis. Thus we have the desired result. □

An exact formula for the coefficients $d_{v,w}$ in the above theorem is not known. Fomin and Zelevinsky [9] conjecture that we have $d_{v,w} = 0$ for $v \neq w$, equivalently, that the cluster basis and dual canonical basis for $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$ are the same. If this conjecture is true, then we have an explicit formula for the factorization of dual canonical basis elements as products of cluster variables.

Corollary 4.5. *If the dual canonical basis and cluster basis of* $\mathbb{Z}[x_{1,1}, \ldots, x_{3,3}]$ *are equal, then for each pair* (M, N) *of* r*-element subsets of* $\{1, \ldots, n\}$ *and each permutation* $v \in \Lambda(r, \alpha, \beta)$ *we have*

$$\mathrm{Imm}_v(x_{M,N}) = \phi^{-1}(D_{M,N}(v)). \tag{4.7}$$

As we have mentioned, the obvious generalization of Theorem 4.3 to the ring $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ is known to be false for $n \geq 4$. While it would be interesting to understand the failure of this generalization, the relevant cluster algebras present something of an obstacle: for each $n \geq 4$, the appropriate sets of cluster variables and clusters are infinite. Nevertheless, it may be possible to use cluster monomials and other polynomials to describe the irreducible factorization of elements of the dual canonical basis of $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ for some fixed $n \geq 4$.

5. Acknowledgments

The author is grateful to Jie Du, Sergey Fomin, Richard Stanley, Andrei Zelevinsky, Stephen Anthony, Muhammed Ashraf, Lauren Berner, Nathaniel Brooks, Cheryl Kaplun, Matthew Parks, Brittany Powers, Olivia Vellios, Ariel Weissman, Christopher Yale, Adam Yusko and Garrett Zambrows for helpful conversations, and to an anonymous referee for helpful comments.

References

[1] A. Berenstein, S. Fomin, and A. Zelevinsky. Parametrizations of canonical bases and totally positive matrices. *Adv. Math.*, **122**, 1 (1996) pp. 49–149.

[2] A. Berenstein, S. Fomin, and A. Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, **126**, 1 (2005) pp. 1–52.

[3] A. Berenstein and A. Zelevinsky. Total positivity in Schubert varieties. *Comment. Math. Helv.*, **72**, 1 (1997) pp. 128–166.

[4] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, vol. 231 of *Graduate Texts in Mathmatics*. Springer, New York (2005).

[5] F. Brenti. A combinatorial formula for Kazhdan-Lusztig polynomials. *Invent. Math.*, **118**, 2 (1994) pp. 371–394.

[6] J. Désarménien, J. P. S. Kung, and G.-C. Rota. Invariant theory, Young bitableaux, and combinatorics. *Advances in Math.*, **27**, 1 (1978) pp. 63–92.

[7] J. Du. Canonical bases for irreducible representations of quantum GL_n. *Bull. London Math. Soc.*, **24**, 4 (1992) pp. 325–334.

[8] V. Fock and A. Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, **103** (2006) pp. 1–211.

[9] S. Fomin (2002). Personal communication.

[10] S. Fomin and N. Reading. Generalized cluster complexes and Coxeter combinatorics. *Int. Math. Res. Not.*, **44** (2005) pp. 2709–2757.

[11] S. Fomin and A. Zelevinsky. Double Bruhat cells and total positivity. *J. Amer. Math. Soc.*, **12** (1999) pp. 335–380.

[12] S. Fomin and A. Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, **15**, 2 (2002) pp. 497–529 (electronic).

[13] S. Fomin and A. Zelevinsky. The Laurent phenomenon. *Adv. in Appl. Math.*, **28**, 2 (2002) pp. 119–144.

[14] S. Fomin and A. Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, **154**, 1 (2003) pp. 63–121.

[15] S. Fomin and A. Zelevinsky (2007). Personal communication.

[16] S. Fomin and A. Zelevinsky. Cluster algebras. IV. Coefficients. *Compos. Math.*, **143**, 1 (2007) pp. 112–164.

[17] M. Gekhtman, M. Shapiro, and A. Vainshtein. Cluster algebras and Poisson geometry. *Mosc. Math. J.*, **3**, 3 (2003) pp. 899–934, 1199. {Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday}.

[18] J. E. Humphreys. *Reflection groups and Coxeter groups*. Cambridge University Press (1990).

[19] G. James and A. Kerber. *The representation theory of the symmetric group*, vol. 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass. (1981).

[20] M. Kashiwara. On crystal bases of the Q-analog of universal enveloping algebras. *Duke Math. J.*, **63** (1991) pp. 465–516.

[21] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Inv. Math.*, **53** (1979) pp. 165–184.

[22] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, **5** (1973) pp. 85–90.

[23] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, **3** (1990) pp. 447–498.

[24] G. Lusztig. Total positivity in reductive groups. In *Lie Theory and Geometry: in Honor of Bertram Kostant*, vol. 123 of *Progress in Mathematics*. Birkhäuser, Boston (1994), pp. 531–568.

[25] B. Rhoades and M. Skandera. Temperley-Lieb immanants. *Ann. Comb.*, **9**, 4 (2005) pp. 451–494.

[26] M. Skandera. On the dual canonical and Kazhdan-Lusztig bases and 3412, 4231-avoiding permutations (2006). To appear in *J. Pure Appl. Algebra*.

[27] R. Stanley. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge (1999).

Author's address: Lehigh University Department of Mathematics, 14 East Packer Street, Bethlehem, PA 18015