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MONOMIAL NONNEGATIVITY AND THE BRUHAT ORDER

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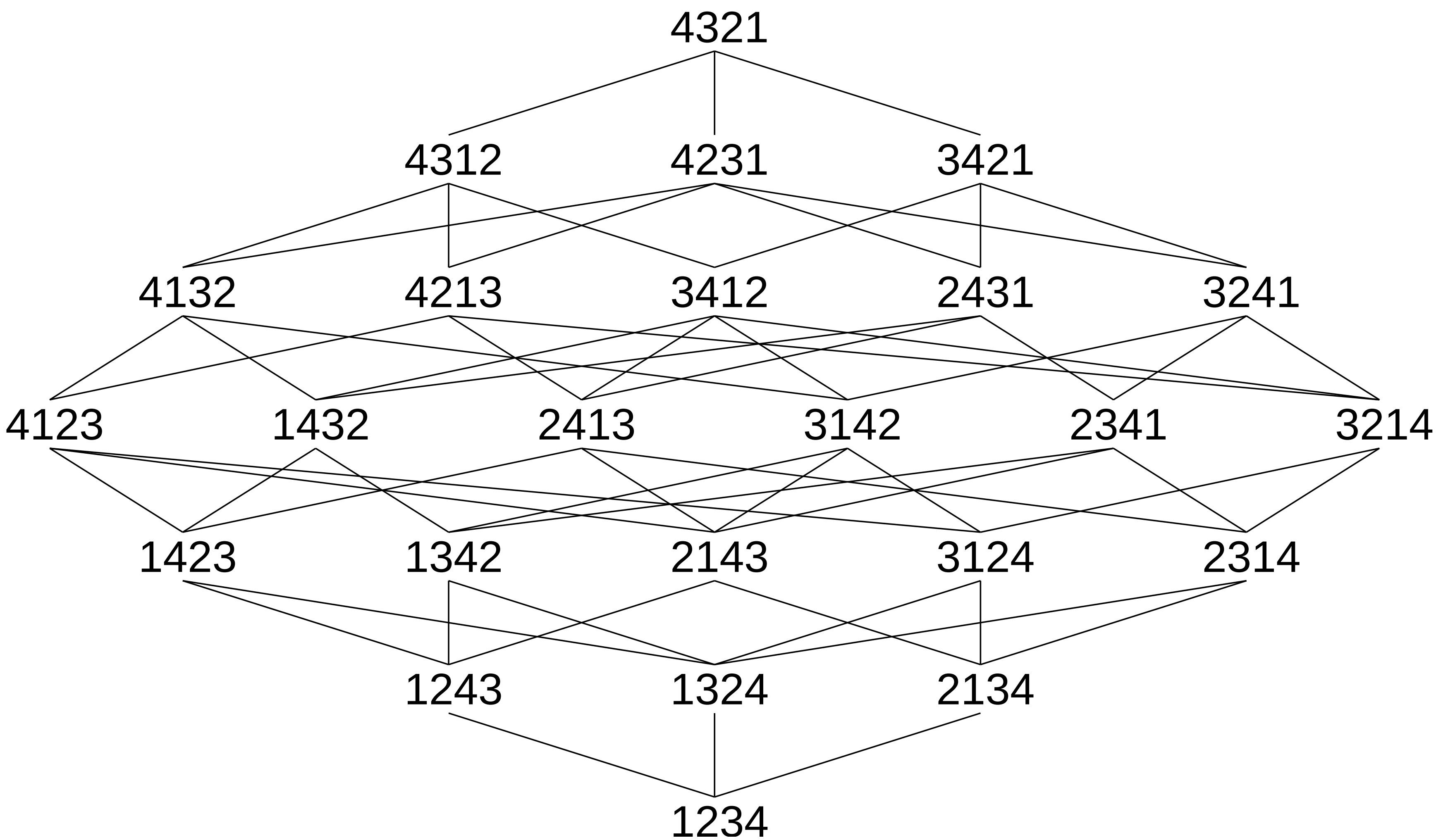
Outline

(1) Defining criteria for the Bruhat order
(2) Symmetric functions
(3) Monomial nonnegative polynomials
(4) A new defining criterion
(5) Open problems



The Bruhat order on S_4 (type A_3).

Defining criteria for the Bruhat order

(1) Flag varieties (E 34, K-L 79, P 82)
(2) The coxeter group A_{n-1} (C 55, D 77)
(3) Representations of $\mathfrak{sl}_n$ (K-L 79, P 82)
(4) Tableaux (E 34, P 82, B-B 96)
(5) Yin potential (L-S 96)
(6) Total nonnegativity, Schur nonnegativity (D-G-S 03)

A tableau criterion

For $\pi \in S_n$ and $p, q \in [n]$, let $r^{\pi}_{p,q} = \#\{i \leq p \mid \pi(i) \leq q\}$. Define the Bruhat order by

$$\pi \leq \sigma \quad \Longleftrightarrow \quad r^{\pi}_{p,q} \geq r^{\sigma}_{p,q} \quad \forall p, q.$$

For example, 1423 and 3241 are incomparable.

$$M(1423) = \begin{bmatrix} \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad M(3241) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & \underline{0} & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$R(1423) = \begin{bmatrix} \underline{1} & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad R(3241) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & \underline{2} & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Nonnegativity properties of symmetric functions

Call a nonnegative linear combination of monomial symmetric functions a *monomial nonnegative* (MNN) symmetric function.

Call a nonnegative linear combination of Schur functions a *Schur nonnegative* (SNN) symmetric function.

TNN matrices, Jacobi-Trudi matrices

A real matrix is called *totally nonnegative* (TNN) if each of its minors is nonnegative.

Let $h_1, h_2, \ldots$ be the homogeneous symmetric functions. Any square submatrix A of the infinite matrix

$$H = \begin{bmatrix} 1 & h_1 & h_2 & h_3 & h_4 & h_5 & \cdots \\ 0 & 1 & h_1 & h_2 & h_3 & h_4 & \cdots \\ 0 & 0 & 1 & h_1 & h_2 & h_3 & \cdots \\ 0 & 0 & 0 & 1 & h_1 & h_2 & \cdots \\ 0 & 0 & 0 & 0 & 1 & h_1 & \cdots \\ \vdots & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

is called a *Jacobi-Trudi* matrix.

Nonnegativity properties of polynomials

A polynomial $p \in \mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ defines a function on $n \times n$ matrices $A = [a_{i,j}]$ by

$$p(A) = p(a_{1,1}, \ldots, a_{n,n}).$$

Definition: Call p a *MNN polynomial* if for every JT matrix A, the symmetric function $p(A)$ is MNN.

Definition: Call p a *SNN polynomial* if for every JT matrix A, the symmetric function $p(A)$ is SNN.

Definition: Call p a *TNN polynomial* if for every TNN matrix A, the number $p(A)$ is nonnegative.

Theorem: (D-G-S 04) The following are equivalent:

$\pi \leq \sigma$ in the Bruhat order.

$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is MNN.

$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is SNN.

$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is TNN.

$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$ is equal to a nonnegative linear combination of Kazhdan-Lusztig immanants.

MNN pf idea: Choose $1 \leq k, \ell \leq n-1$ so that $M(\sigma)_{[k],[\ell]}$ contains $q+1$ ones and $M(\pi)_{[k],[\ell]}$ contains q ones.

Choose b large enough and consider the J-T matrix

$$A = \begin{bmatrix} h_{b+k-1} & \cdots & h_{b+k+\ell-2} & h_{2b+k-1} & \cdots & h_{2b+n+k-\ell-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_b & \cdots & h_{b+\ell-1} & h_{2b} & \cdots & h_{2b+n-1-\ell} \\ h_{n-k-1} & \cdots & h_{n-k+\ell-2} & h_{b+n-k-1} & \cdots & h_{b+2n-k-\ell-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_0 & \cdots & h_{\ell-1} & h_b & \cdots & h_{b+n-\ell-1} \end{bmatrix}.$$

Then $a_{1,\pi(1)} \cdots a_{n,\pi(n)} - a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ is not MNN.