SCHUR NONNEGATIVITY AND THE BRUHAT ORDER

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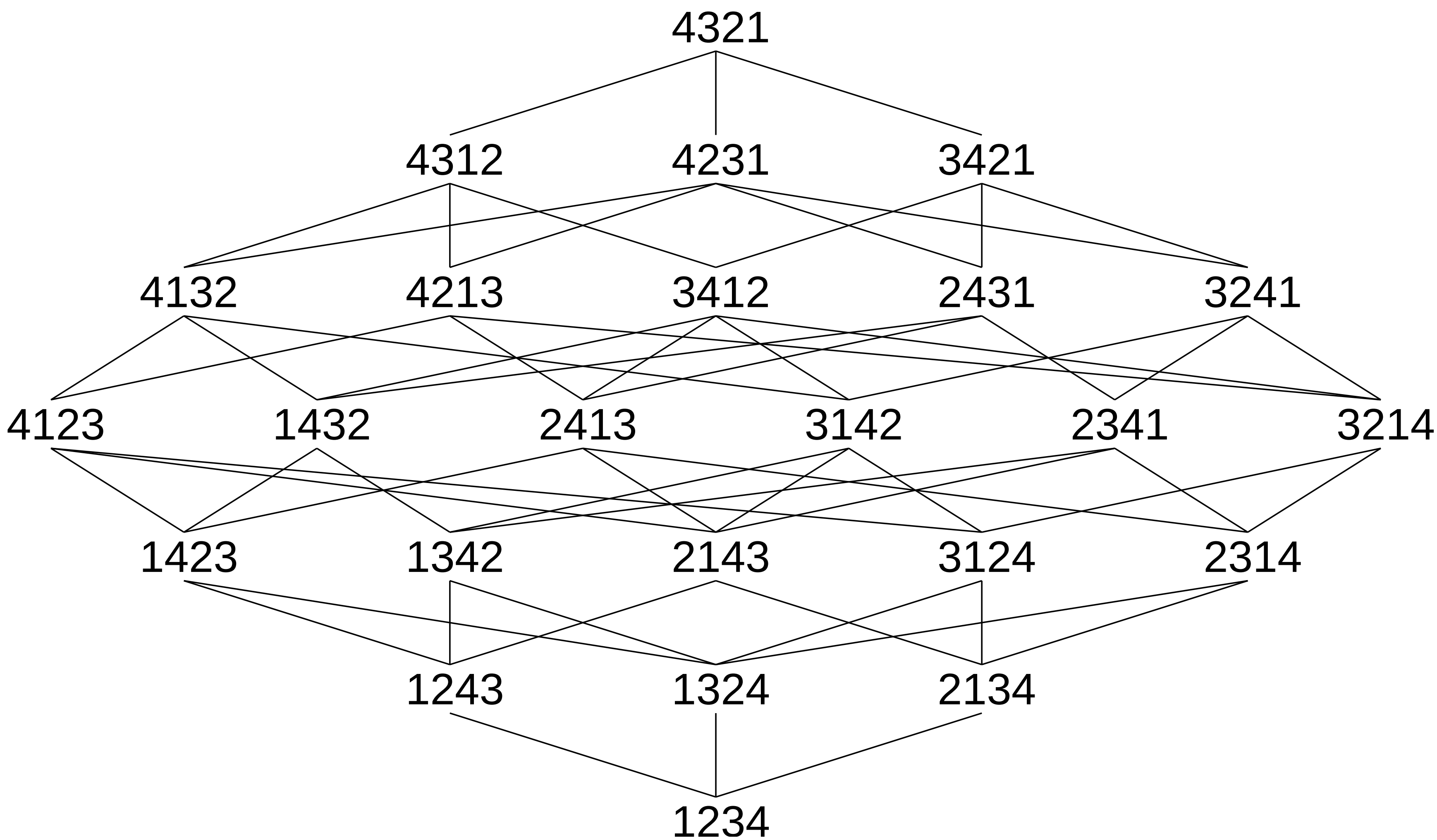
Outline

(1) Defining criteria for the Bruhat order
(2) Schur functions
(3) Schur nonnegative polynomials
(4) A new defining criterion
(5) Open problems



The Bruhat order on S_4 (type A_3).

Defining criteria for the Bruhat order

(1) Flag varieties (E 34, K-L 79, P 82)
(2) The coxeter group A_{n-1} (C 55, D 77)
(3) Representations of $\mathfrak{sl}_n$ (K-L 79, P 82)
(4) Tableaux (E 34, P 82, B-B 96)
(5) Yin potential (L-S 96)
(6) Schur nonnegativity (D-G-S 03)

A flag variety criterion

The Bruhat decomposition of GL_n,

$$GL_n = \bigcup_{\pi \in S_n} B\pi B,$$

induces a decomposition of the flag variety GL_n/B into Schubert cells

$$GL_n/B = \bigcup_{\pi \in S_n} (B\pi B)/B = \bigcup_{\pi \in S_n} X_\pi^\circ.$$

Define the Bruhat order on S_n in terms of the closures X_π of these cells by

$$\pi \leq \sigma \qquad \Longleftrightarrow \qquad X_\pi \subset X_\sigma.$$

Ehresmann's flag variety criterion

The flag variety

$$\mathcal{F}(n) = \{E_\bullet = (E_1 \subset \cdots \subset E_n = \mathbb{C}^n) \mid \dim E_i = i\}$$

decomposes as a union of Schubert cells

$$\mathcal{F}(n) = \bigcup_{\pi \in S_n} X_\pi^\circ.$$

Define the Bruhat order on S_n in terms of the closures X_π of these cells by

$$\pi \leq \sigma \qquad \Longleftrightarrow \qquad X_\pi \subset X_\sigma.$$

Fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ and define a reference flag $F_\bullet = (F_1, \ldots, F_n)$ by $F_i = \mathrm{span}\{e_1, \ldots, e_i\}$.

Define the Schubert cell X_π° by

$$\begin{aligned} X_\pi^\circ &= \{E_\bullet \in \mathcal{F}(n) \mid \dim(E_p \cap F_q) = r_{p,q}^\pi\}, \\ r_{p,q}^\pi &= \#\{i \leq p \mid \pi(i) \leq q\}. \end{aligned}$$

Example: $\pi = 3421$.

$$M(\pi) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad R(\pi) = [r_{p,q}^\pi] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

A tableau criterion

For $\pi \in S_n$ and $p, q \in [n]$, let $r^{\pi}_{p,q} = \#\{i \leq p \mid \pi(i) \leq q\}$. Define the Bruhat order by

$$\pi \leq \sigma \quad \Longleftrightarrow \quad r^{\pi}_{p,q} \geq r^{\sigma}_{p,q} \quad \forall p, q.$$

For example, 1423 and 3241 are incomparable.

$$M(1423) = \begin{bmatrix} \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad M(3241) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & \underline{0} & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$R(1423) = \begin{bmatrix} \underline{1} & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \qquad R(3241) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & \underline{2} & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Jacobi-Trudi matrices and Schur functions

Let $h_1, h_2, \ldots$ be the homogeneous symmetric functions. Any square submatrix A of the infinite matrix

$$H = \begin{bmatrix} 1 & h_1 & h_2 & h_3 & h_4 & h_5 & \cdots \\ 0 & 1 & h_1 & h_2 & h_3 & h_4 & \cdots \\ 0 & 0 & 1 & h_1 & h_2 & h_3 & \cdots \\ 0 & 0 & 0 & 1 & h_1 & h_2 & \cdots \\ 0 & 0 & 0 & 0 & 1 & h_1 & \cdots \\ \vdots & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

is called a Jacobi-Trudi matrix. If A consists of consecutive columns of H, the symmetric function $\det(A)$ is called a *Schur function.*

- Schur functions form a basis of Λ.
- Schur functions are monomial nonnegative (MNN).

Call a nonnegative linear combination of Schur functions a *Schur nonnegative* (SNN) symmetric function.

These arise in

(1) arbitrary minors of JT matrices.
(2) products of Schur functions.
(3) polynomial representations of GL_n.
(4) zeros of polynomials related to posets (VG 94, RS 95).
(5) products in $H^*(Gr^n(\mathbb{C}^{n+r}))$, and differences of these (F-F-L-P 03).

Monomial nonnegative, Schur nonnegative polynomials

A polynomial $p \in \mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ defines a function on $n \times n$ matrices $A = [a_{i,j}]$ by

$$p(A) = p(a_{1,1}, \ldots, a_{n,n}).$$

Definition: Call p a *MNN polynomial* if for every JT matrix A, the symmetric function $p(A)$ is MNN.

Definition: Call p a *SNN polynomial* if for every JT matrix A, the symmetric function $p(A)$ is SNN.

Question: Which polynomials are MNN? (SNN?)

Fact: (Conj. G-J 89; Pf. CG 91)

$$\mathrm{Imm}_\lambda(x) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \quad \text{is MNN.}$$

Fact: (Conj. JS 91; Pf. MH 92)

$$\mathrm{Imm}_\lambda(x) \quad \text{is SNN.}$$

Problem: (Conj. JS 91)
Show that

$$\mathrm{Imm}_{\phi^\lambda}(x) = \sum_{\sigma \in S_n} \phi^\lambda(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

is SNN (or MNN).

Observation: If $\pi \leq \sigma$ in the Bruhat order, then

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

is Schur nonnegative.

Proof idea: Let A be a JT matrix. If $\sigma = (i,j)\pi$, then

$$\begin{aligned} & a_{1,\pi(1)} \cdots a_{n,\pi(n)} - a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \\ &= \frac{a_{1,\pi(1)} \cdots a_{n,\pi(n)}}{a_{i,\pi(i)} a_{j,\pi(j)}} \left(a_{i,\pi(i)} a_{j,\pi(j)} - a_{i,\pi(j)} a_{j,\pi(i)} \right) \\ &= h_\nu \, s_{\lambda/\mu}. \end{aligned}$$

Proposition: If $\pi \not\leq \sigma$ in the Bruhat order, then

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

is not Schur nonnegative.

Proof idea: There exists a JT matrix A such that

$$a_{1,\pi(1)} \cdots a_{n,\pi(n)} - a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} = h_\lambda - h_\mu,$$

where μ does not dominate λ. Thus,

$$\begin{aligned} h_\lambda &= s_\lambda + \sum_{\nu > \lambda} K_{\nu,\lambda} s_\nu, \\ h_\mu &= s_\mu + \sum_{\nu > \mu} K_{\nu,\mu} s_\nu, \\ h_\lambda - h_\mu &= -s_\mu + \cdots . \end{aligned}$$

Theorem: (D-G-S 03) The Bruhat order on S_n is the Schur nonnegativity order:

$$\pi \leq \sigma \qquad \Longleftrightarrow \qquad \begin{gathered} x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \\ \text{is SNN.} \end{gathered}$$

Question: What is the monomial nonnegativity order on S_n?

It must be an extension of the Bruhat order, since

$$\text{SNN} \Rightarrow \text{MNN}.$$

Open questions

Fact: Certain polynomials

$$\mathrm{Imm}_\tau(x) = \sum_{\sigma \geq \tau} f_\tau(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)},$$

which arise in the study of Temperley-Lieb algebras and cluster algebras, are MNN.

Sums of these have the form

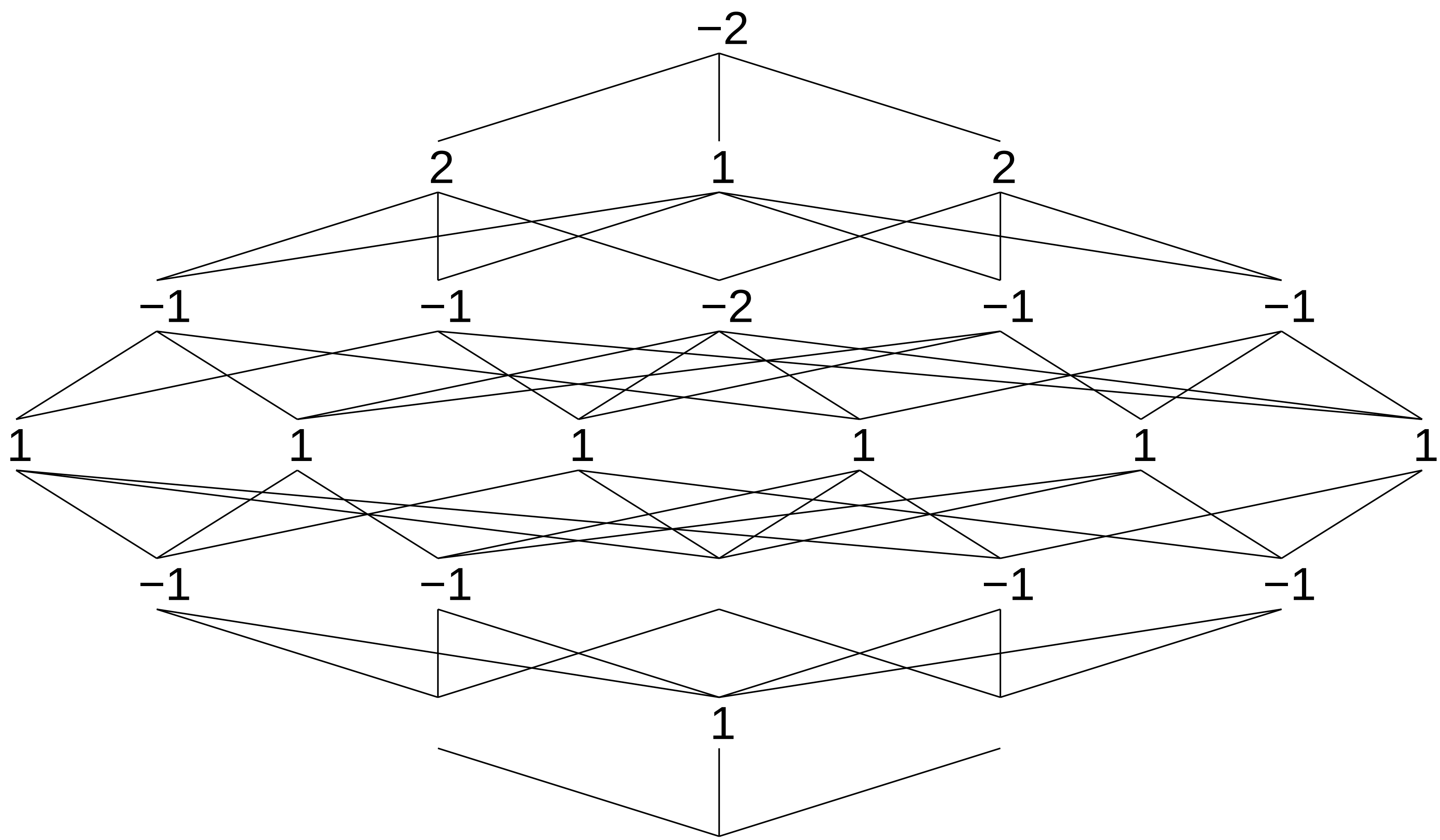
$$p(x) = \Delta_{J,J'}(x)\Delta_{L,L'}(x) - \Delta_{I,I'}(x)\Delta_{K,K'}(x),$$

so that

$$p(A) = s_{\lambda/\alpha} s_{\rho/\beta} - s_{\mu/\gamma} s_{\nu/\delta},$$

if A is a JT matrix.

Question: Are these polynomials SNN?



The coefficients of $\mathrm{Imm}_{1324}(x)$.