Two new criteria for comparison in the Bruhat order

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Abstract

We give two new criteria by which pairs of permutations may be compared in defining the Bruhat order (of type A). One criterion uses totally nonnegative polynomials and the other uses Schur functions.

RÉSUMÉ. Nous donons deux critères nouveaux avec lesquels on peut comparer couples de permutations en definant l'order de Bruhat (de type A). Un critère utilise les polynômes totallement nonnegatifs et l'autre utilise les fonctions symétriques de Schur.

The Bruhat order on S_n is often defined by comparing two permutations $\pi = \pi(1)\cdots\pi(n)$ and $\sigma = \sigma(1)\cdots\sigma(n)$ according to the following criterion: $\pi \leq \sigma$ if σ is obtainable from π by a sequence of transpositions (i, j) where $i < j$ and i appears to the left of j in π. (See e.g. [7, p. 119].) A second well-known criterion compares permutations in terms of their defining matrices. Let $M(\pi)$ be the matrix whose (i, j) entry is 1 if $j = \pi(i)$ and zero otherwise. Defining $[i] = \{1, \ldots, i\}$, and denoting the submatrix of $M(\pi)$ corresponding to rows I and columns J by $M(\pi)_{I,J}$, we have the following.

Theorem 1 *Let π and σ be two permutations in S_n. Then π is less than or equal to σ in the Bruhat order if and only if for all $1 \leq i, j \leq n-1$, the number of ones in $M(\pi)_{[i],[j]}$ is greater than or equal to the number of ones in $M(\sigma)_{[i],[j]}$.*

(See [1], [2], [3], [6, pp. 173-177], [8] for more criteria.) Using Theorem 1 and our defining criterion we will state and prove the validity of two more criteria.

Our first new criterion defines the Bruhat order in terms of totally nonnegative polynomials. A matrix A is called *totally nonnegative* (TNN) if the determinant of each square submatrix of A is nonnegative. (See e.g. [5].) A polynomial in n^2 variables $f(x_{1,1}, \ldots, x_{n,n})$ is called *totally nonnegative* (TNN) if $f(a_{1,1}, \ldots, a_{n,n})$ is nonnegative for each TNN matrix $A = (a_{i,j})$. Some recent interest in TNN polynomials is motivated by problems in the study of canonical bases. (See [10].)

Theorem 2 *Let π and σ be two permutations in S_n. Then π is less than or equal to σ in the Bruhat order if and only if the polynomial*

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \tag{1}$$

is totally nonnegative.

Proof: ($\Rightarrow$) Suppose that π is less than σ in the Bruhat order. If π differs from σ by a single transposition (i, j) with $i < j$, then we have $\pi(i) = \sigma(j) < \pi(j) = \sigma(i)$, and the polynomial (1) is equal to

$$\frac{x_{1,\pi(1)} \cdots x_{n,\pi(n)}}{x_{i,\pi(i)} x_{j,\pi(j)}} (x_{i,\pi(i)} x_{j,\pi(j)} - x_{i,\pi(j)} x_{j,\pi(i)}) \tag{2}$$

which is clearly TNN. If π differs from σ by a sequence of transpositions, then the polynomial (1) is equal to a sum of polynomials of the form (2) and again is TNN.

($\Leftarrow$) Suppose that π is not less than or equal to σ in the Bruhat order. By Theorem 1 we may choose indices $1 \leq k, \ell \leq n-1$ such that $M(\sigma)_{[k],[\ell]}$ contains $q+1$ ones and $M(\pi)_{[k],[\ell]}$ contains q ones. Now define the matrix $A = (a_{i,j})$ by

$$a_{i,j} = \begin{cases} 2 & \text{if } i \leq k \text{ and } j \leq \ell, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that A is TNN, since all square submatrices of A have determinant equal to $0, 1$, or 2. Applying the polynomial (1) to A we have

$$a_{1,\pi(1)} \cdots a_{n,\pi(n)} - a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} = -2^q,$$

and the polynomial (1) is not TNN. $\square$

Our second criterion defines the Bruhat order in terms of Schur functions. (See [9, Ch. 7] for definitions.) Any finite submatrix of the infinite matrix $H = (h_{j-i})_{i,j \geq 0}$, where h_k is the kth complete homogeneous symmetric function and $h_k = 0$ for $k < 0$, is called a *Jacobi-Trudi matrix*. Let us define a polynomial in n^2 variables $f(x_{1,1}, \ldots, x_{n,n})$ to be *Schur-nonnegative* (SNN) if the symmetric function $f(a_{1,1}, \ldots, a_{n,n})$ is equal to a nonnegative linear combination of Schur functions for each Jacobi-Trudi matrix $A = (a_{i,j})$. Some recent interest in SNN polynomials is motivated by problems in algebraic geometry [4, Conj. 2.8, Conj. 5.1].

Theorem 3 *Let π and σ be two permutations in S_n. Then π is less than or equal to σ in the Bruhat order if and only if the polynomial*

$$x_{1,\pi(1)} \cdots x_{n,\pi(n)} - x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \tag{3}$$

is Schur-nonnegative.

Proof: ($\Rightarrow$) Let A be an $n \times n$ Jacobi-Trudi matrix and suppose that π is less than σ in the Bruhat order. If π differs from σ by a single transposition (i, j), then for some partition ν and some $k, \ell, m \geq 0$, the evaluation of the polynomial (3) at A is equal to

$$h_\nu(h_{k+\ell}h_{k+m} - h_{k+\ell+m}h_k), \tag{4}$$

and (3) is clearly SNN. If π differs from σ by a sequence of transpositions, then the evaluation of (3) at A is equal to a sum of polynomials of the form (4) and again is SNN.

($\Leftarrow$) Suppose that π is not less than or equal to σ in the Bruhat order. By Theorem 1 we may choose indices $1 \leq k, \ell \leq n-1$ such that $M(\sigma)_{[k],[\ell]}$ contains $q+1$ ones and $M(\pi)_{[k],[\ell]}$ contains q ones. Now define the nonnegative number $r = (k-q)(n+k-\ell-2)$ and consider the Jacobi-Trudi matrix B defined by the skew shape $(n-1+2r)^k(n-1+r)^{n-k}/r^\ell$,

$$B = \begin{bmatrix} h_{n-1+r} & \cdots & h_{n+\ell-2+r} & h_{n+\ell-1+2r} & \cdots & h_{2n-2+2r} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{n-k+r} & \cdots & h_{n-k+\ell-1+r} & h_{n-k+\ell+2r} & \cdots & h_{2n-k-1+2r} \\ h_{n-k-1} & \cdots & h_{n-k+\ell-2} & h_{n-k+\ell-1+r} & \cdots & h_{2n-k-2+r} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_0 & \cdots & h_{\ell-1} & h_{\ell+r} & \cdots & h_{n-1+r} \end{bmatrix}.$$

The polynomial (3) applied to B may be expressed as $h_\lambda - h_\mu$ for some appropriate partitions λ, μ depending on π, σ, respectively. We claim that λ is incomparable to or greater than μ in the dominance order. Since $M(\pi)_{[k],[\ell+1,n]}$ contains $k-q$ ones we have that

$$\lambda_1 + \cdots + \lambda_{k-q} \geq (k-q)(n-k+\ell+2r). \tag{5}$$

Similarly, we have

$$\mu_1 + \cdots + \mu_{k-q} \leq (k-q-1)(2n-2+2r) + \max\{n+\ell-2+r, 2n-k-2+r\}. \tag{6}$$

Subtracting (6) from (5), we obtain

$$(\lambda_1 + \cdots + \lambda_{k-q}) - (\mu_1 + \cdots + \mu_{k-q}) \geq n - \max\{\ell, n-k\} > 0,$$

as desired.

Recall that the Schur expansion of h_μ is

$$h_\mu = s_\mu + \sum_{\nu > \mu} K_{\nu,\mu} s_\nu,$$

where the comparison of partitions is in the dominance order and the nonnegative *Kostka numbers* $K_{\nu,\mu}$ count Young tableaux of shape ν and content μ. (See e.g. [9, Prop. 7.10.5, Cor. 7.12.4].) It follows that the coefficient of s_μ in the Schur expansion of $h_\lambda - h_\mu$ is -1 and the polynomial (3) is not SNN. $\square$

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