AN APPLICATION OF DUMONT'S STATISTIC

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Outline

1. Eulerian permutation statistics
2. The f and h vectors
3. An $f = h$ theorem
4. Proof by descent
5. Proof by Dumont's statistic

Eulerian Statistics

Permutation statistics whose distributions on S_n are given by the nth Eulerian polynomial,

$$A_n(x) = \sum_{k=1}^{n} A(n, k-1)x^k,$$

are known as *Eulerian statistics*.

Two important examples are "des" (descents) and "exc" (excedances).

$$\begin{aligned}\operatorname{des}(\pi) &= \#\{i \,|\, \pi_i > \pi_{i+1}\}.\\ \operatorname{exc}(\pi) &= \#\{i \,|\, \pi_i > i\}.\end{aligned}$$

Example. Let $\pi = 284367951$. Then,

$$\begin{aligned}\operatorname{des}(\pi) &= 4,\\ \operatorname{exc}(\pi) &= 5.\end{aligned}$$

The Eulerian numbers

Entry (n, k) below is the number of permutations in S_n which have $k - 1$ descents (or excedances).

$n\backslash k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

Dumont's statistic counts the number of distinct non-zero letters in the code of a permutation.

Example.

$$\begin{aligned}\pi &= 2\ 8\ 4\ 3\ 6\ 7\ 9\ 5\ 1\\ \mathrm{code}(\pi) &= 1\ 6\ 2\ 1\ 2\ 2\ 2\ 1\ 0\end{aligned}$$

The non-zero letters in $\mathrm{code}(\pi)$ are $LC(\pi) = \{1, 2, 6\}$. Thus, $\mathrm{dmc}(\pi) = 3$.

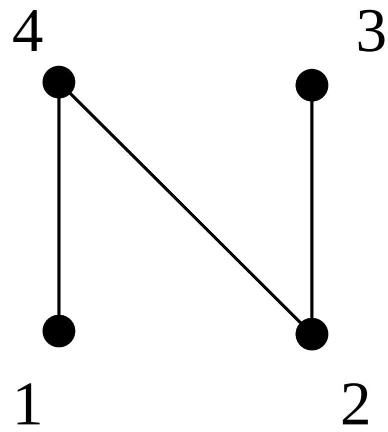
Question: On which subsets $T \subset S_n$ are des, exc, and dmc equidistributed?

(Open.)

Question: What properties of a labelled poset P guarantee that des, exc, and dmc are equidistributed on $\mathcal{L}(P)$, the set of linear extensions of P?

(Open.)

Theorem: (Björner, Wachs) If P is a forest, then it can be labelled so that INV and MAJ are equidistributed on $\mathcal{L}(P)$.



P

$$\begin{array}{rcccccc}
\pi & = & 1234 & 1243 & 2134 & 2143 & 2314 \\
\mathrm{code}(\pi) & = & 0000 & 0010 & 1000 & 1010 & 1100 \\
\mathrm{des}(\pi) & = & 0 & 1 & 1 & 2 & 1 \\
\mathrm{exc}(\pi) & = & 0 & 1 & 1 & 2 & 2 \\
\mathrm{dmc}(\pi) & = & 0 & 1 & 1 & 1 & 1
\end{array}$$

Generalizing permutations on n letters are *words* $w = w_1 \cdots w_m$ on n letters, where letters may be repeated in w.

Given a word w, we define $R(w)$ to be the set of all rearrangements of w.

Example.

$$\begin{aligned} w &= 3\ 2\ 3\ 3\ 1\ 1\ 2\ 1 \\ u &= 1\ 3\ 2\ 1\ 2\ 3\ 1\ 3 \in R(w) \\ v &= 1\ 1\ 1\ 2\ 2\ 3\ 3\ 3 \in R(w) \end{aligned}$$

We define the *non-decreasing rearrangement* of w to be the unique rearrangement $\bar{w}$ satisfying $\bar{w}_1 \leq \cdots \leq \bar{w}_m$.

To each word w we will associate the *biword*

$$\tilde{w} = \begin{pmatrix} \bar{w} \\ w \end{pmatrix}$$

Example. Let $w = 31231121$. Then,

$$\tilde{w} = \begin{pmatrix} \bar{w} \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 1 & 2 & 1 \end{pmatrix}$$

Excedances

We will call position i of w an *excedance* if $w_i \geq \bar{w}_i$.

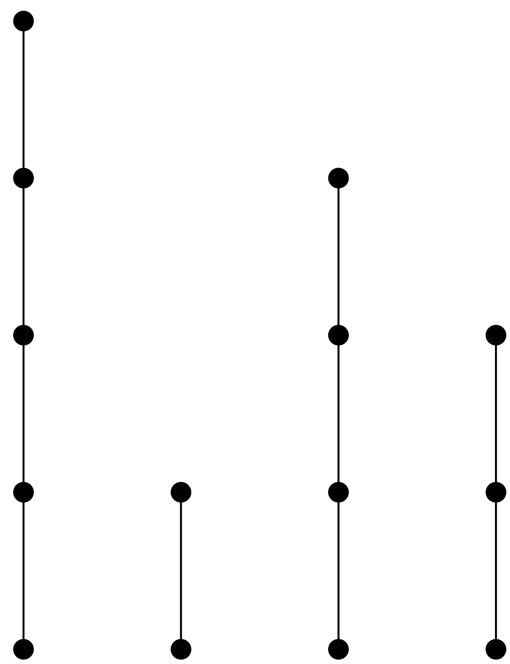
For example, if

$$\tilde{w} = \begin{pmatrix} \bar{w} \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 3 & 1 & 2 & 3 & 1 & 1 & 2 & 1 \end{pmatrix},$$

then the excedance set of w is $E(w) = \{1, 3, 4\}$.

Theorem. For any word w, the word statistics dmc, des, and exc are equidistributed on $R(w)$.

$$\begin{aligned} &\#\{y \in R(w) \,|\, \mathrm{dmc}(y) = k\} \\ &= \#\{y \in R(w) \,|\, \mathrm{des}(y) = k\} \\ &= \#\{y \in R(w) \,|\, \mathrm{exc}(y) = k\}. \end{aligned}$$



Rearrangements of words are linear extensions of posets.

Rearrangements of the word

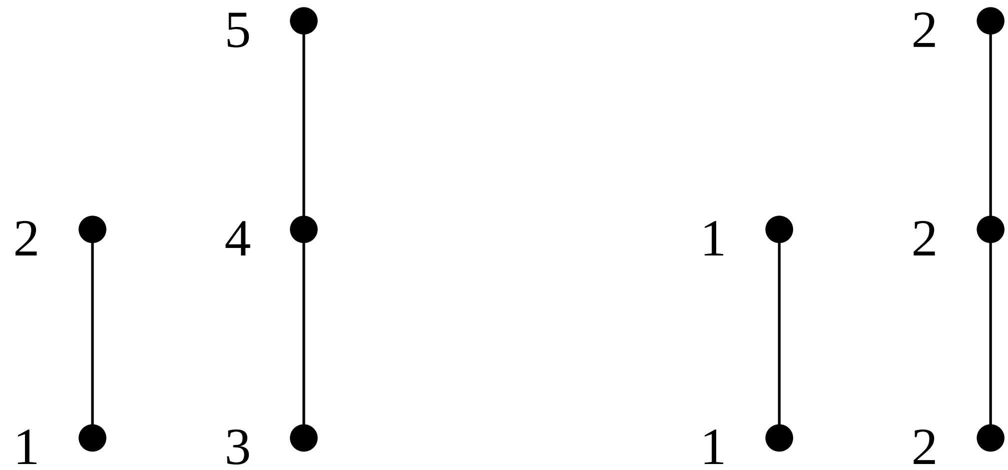
$$11111223333444$$

correspond to linear extensions of the poset

$$\mathbf{5} + \mathbf{2} + \mathbf{4} + \mathbf{3}.$$

One such rearrangement is

$$14433231134112.$$



The extension of exc to words is not as innocent as you might think.

lin. ext.	13452	12221
descents	1345/2	1222/1
code	01110	01110
excedances	$\binom{1\underline{2}\underline{3}\underline{4}5}{13452}$	$\binom{1\underline{1}222}{12221}$

The excedance table

Let $c = c_1 \cdots c_m$ be the code of w, and define

$$\mathrm{etab}(w) = e_1 \cdots e_m$$

to be the unique word satisfying

1. If i is an excedance in w, then $e_i = i$.
2. If $c_i = 0$, then $e_i = 0$.
3. Otherwise, e_i is the c_ith excedance of w having value at least w_i.

Example. Let w be the word 431431421, and let c be its code.

$$\begin{pmatrix} w' \\ w \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 \\ \bar{4} & \bar{3} & 1 & \bar{4} & 3 & 1 & 4 & 2 & 1 \\ 6 & 4 & 0 & 4 & 3 & 0 & 2 & 1 & 0 \end{pmatrix}$$

Since w has excedances at positions $1, 2$, and 4, we write these values into e.

$$\begin{pmatrix} w \\ e \end{pmatrix} = \begin{pmatrix} \bar{4} & \bar{3} & 1 & \bar{4} & 3 & 1 & 4 & 2 & 1 \\ 1 & 2 & & 4 & & & & & \end{pmatrix}.$$

For non-excedances i, we set e_i equal to the c_ith excedance having value at least w_i,

$$\begin{pmatrix} w \\ e \end{pmatrix} = \begin{pmatrix} \bar{4} & \bar{3} & 1 & \bar{4} & 3 & 1 & 4 & 2 & 1 \\ 1 & 2 & & 4 & 4 & & 4 & 1 & \end{pmatrix}.$$

We place zeros elsewhere.

$$d = \begin{pmatrix} 1 & 2 & 0 & 4 & 4 & 0 & 4 & 1 & 0 \end{pmatrix}$$

Let $\theta : S_n \to S_n$ be the bijection proving the equidistribution of dmc and exc on $R(w)$:

1. Define the biword $y = \binom{w}{\mathrm{etab}(w)}$.
2. Let $y' = \binom{w'}{d}$ be the unique rearrangement of y satisfying $\mathrm{code}(w') = d$.
3. Set $\theta(w) = w'$.

Example. Rearranging

$$\begin{pmatrix} w \\ \mathrm{etab}(w) \end{pmatrix} = \begin{pmatrix} 431431421 \\ 120440410 \end{pmatrix},$$

we have

$$\begin{pmatrix} w' \\ \mathrm{code}(w') \end{pmatrix} = \begin{pmatrix} 314413241 \\ 404402110 \end{pmatrix}.$$

We therefore set

$$\theta(431431421) = 314413241.$$

f-vectors and h-vectors

The f vector of a poset P is

$$f_P = (f_{-1}, f_0, f_1, \ldots, f_{d-1}),$$

where f_i counts $(i-1)$-element chains in P.

The f vector of a $(d-1)$-dimensional simplicial complex Σ is

$$f_\Sigma = (f_{-1}, f_0, f_1, \ldots, f_{d-1}),$$

where f_i counts i-faces in Σ.

Define the polynomial $f(x)$ by

$$f(x) = f_{-1} + f_0 x + \cdots + f_{d-1} x^d.$$

The h-vector of a poset or simplicial complex

$$h = (h_0, \ldots, h_d),$$

is defined in terms of the f-vector,

$$h(x) = (1-x)^d f\left(\frac{x}{1-x}\right).$$

Theorem. (Stanley) The h-vector of the distributive lattice $J(P)$ counts linear extensions of P by descent.

Conjecture. (Neggers-Stanley) Let $J(P)$ be a finite distributive lattice. Then the chain polynomial $f_{J(P)}(x)$ has only real zeros.

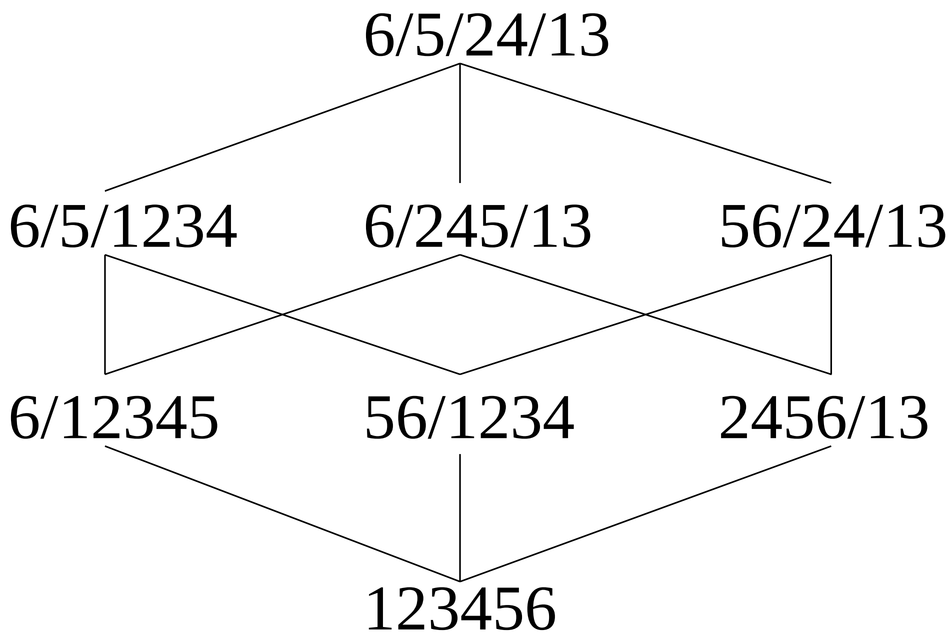
Theorem. (Simion) Let P be a disjoint sum of chains. Then the chain polynomial $f_{J(P)}(x)$ has only real zeros.

Question. If L is a *modular lattice*, does the chain polynomial $f_L(x)$ have only real zeros? (Open)

Sometimes the h-vector of one simplicial complex is the f-vector of another.

Theorem. (Stanley) If Σ is a balanced Cohen Macauley complex, then its h-vector is the f-vector of some balanced simplicial complex.

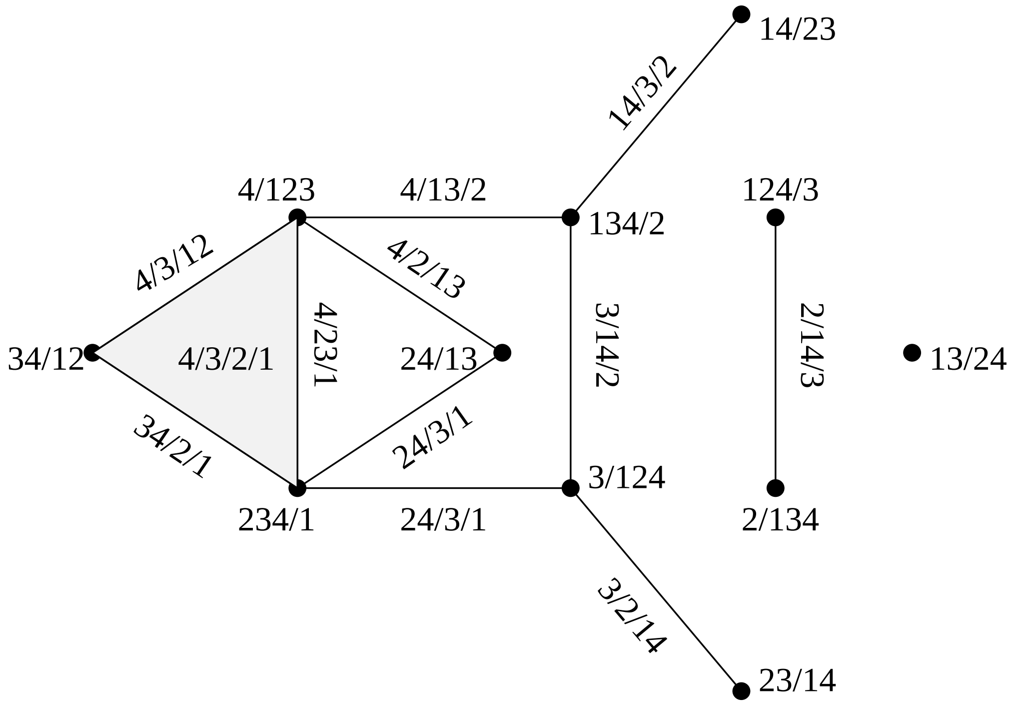
Special case. If $J(P)$ is a finite distributive lattice, then its h-vector is the f-vector of some balanced simplicial complex.



Proof idea.

For any poset P we can construct a simplicial complex Γ satisfying $f_\Gamma = h_{J(P)}$ by associating a $(k-1)$-simplex to each linear extension of P which has k descents.

Descent removal preserves membership in $\mathcal{L}(P)$.



Example. If P is the antichain $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$, then Γ is the complex shown above.

$$f_\Gamma(x) = h_{J(P)}(x) = 1 + 11x + 11x^2 + x^3.$$

Conjecture. If P is any finite poset, then the h-vector of $J(P)$ is the f-vector of some *poset*.

Theorem. If P is a disjoint sum of chains, then the h-vector of $J(P)$ is the f-vector of some poset.

Let $C(w)$ be the set of codes of rearrangements of the word w. The following basic operations on codes preserve membership in $C(w)$:

1. Replace 41 with 13.
2. Replace $45 \cdots 51$ with $15 \cdots 53$.

Example.

$$466220110 \in C(w),$$
$$462520110 \in C(w),$$
$$266320110 \in C(w).$$

The operations move letters to the right and reduce them.

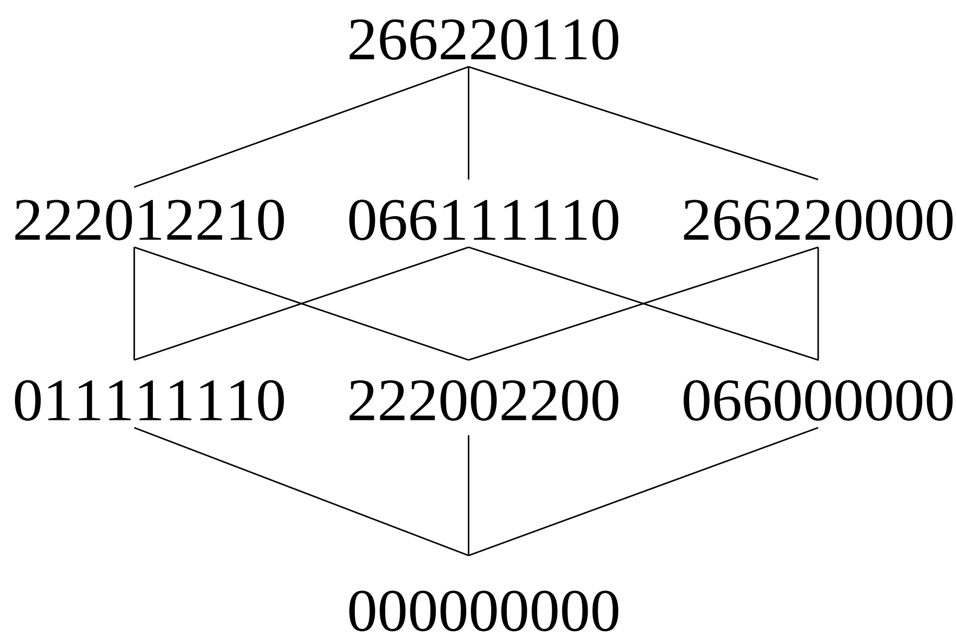
The following operation on codes preserve membership in $C(w)$:

1. Move and reduce each 6 until it becomes 2.
2. Replace all occurrences of the smallest letter with 0.

Example.

$$\begin{aligned} 266220110 &\in C(w), \\ 225520110 &\in C(w), \\ 222440110 &\in C(w), \\ 222033110 &\in C(w), \\ 222012210 &\in C(w). \end{aligned}$$

Denote the unique one-letter codes resulting from these operations by $\psi_6(c), \psi_2(c), \psi_1(c)$.



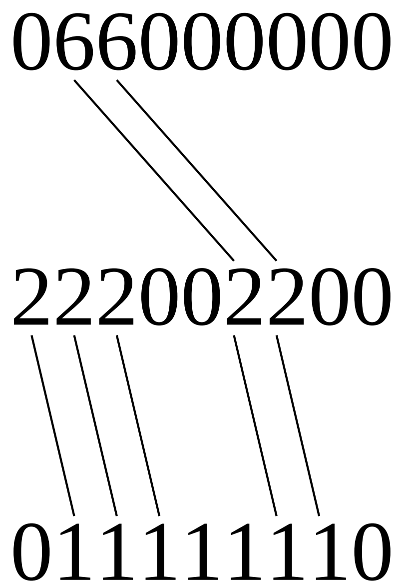
Proof idea.

For any disjoint sum of chains P we can construct a poset Q satisfying $f_Q = h_{J(P)}$ by associating a $(k-1)$-element chain to each linear extension of P with k letters in its code.

Let Q be the subset of one-letter codes in $C(w)$, and let c and c' be codes in Q whose letters are ℓ and ℓ', respectively.

Define $c <_Q c'$ if

1. $\ell < \ell'$.
2. The multiplicity of ℓ in c is strictly *greater* than that of ℓ' in c'.
3. For each position i such that $c'_i = \ell'$, we have $c_{i+\ell'-\ell} = \ell$.

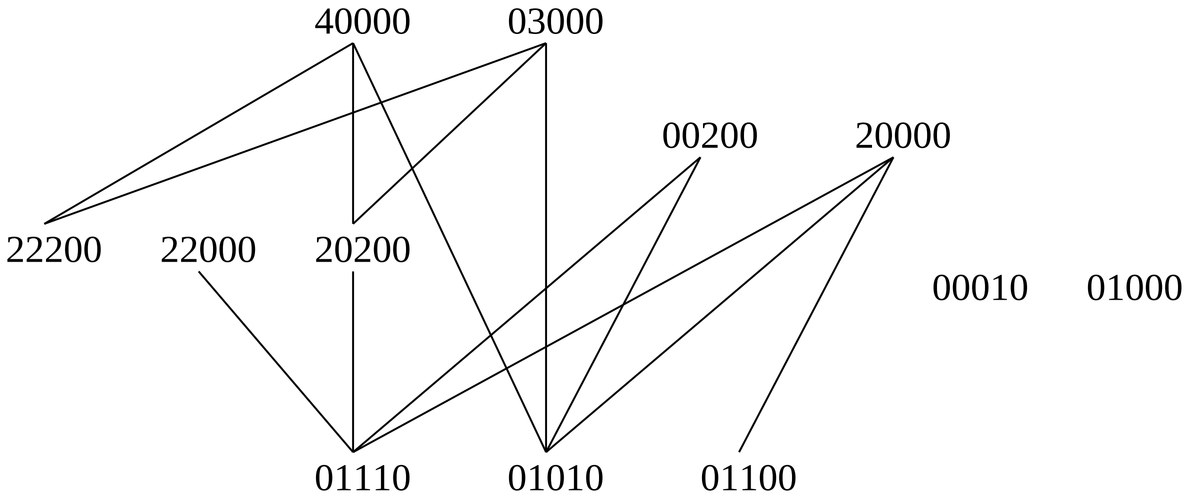


Example.

$$066000000 >_Q 222002200 >_Q 011111110.$$

Theorem. Let c, c', c'' be codes in $C(w)$ on the letters ℓ, ℓ', and ℓ'' respectively, and suppose that $c <_Q c' <_Q c''$. Then there exists a code d in $C(w)$ on letters $\{\ell, \ell', \ell''\}$ which satisfies

$$\begin{aligned} \psi_\ell(d) &= c, \\ \psi_{\ell'}(d) &= c', \\ \psi_{\ell''}(d) &= c''. \end{aligned}$$



For any poset P which is a disjoint sum of chains, we can construct a poset Q satisfying $f_Q = h_{J(P)}$, by associating a k-element chain to each linear extension of P which has k letters in its code.

Example. If P is the sum of chains $\mathbf{2}+\mathbf{2}+\mathbf{1}$, then Q is the poset shown above.

$$f_Q(x) = h_{J(P)}(x) = 1 + 12x + 15x^2 + 2x^3.$$