TOTAL NONNEGATIVITY AND $(\mathbf{3}+\mathbf{1})$-FREE POSETS

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Outline

(1) Totally nonnegative matrices
(2) $(\mathbf{3}+\mathbf{1})$-free posets
(3) f-polynomials
(4) A factorization theorem
(5) Open problems

Total nonnegativity

Let $\Delta_{I,J}$ denote the (I, J)-*minor* of A: the determinant of the submatrix corresponding to rows I and columns J.

$$A = \begin{bmatrix} 3 & 3 & 2 & 2 & 1 \\ 3 & 3 & 2 & 2 & 1 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \end{bmatrix}$$

$$\Delta_{\{1,3\},\{2,3\}} = \det \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} = 2.$$

Define a matrix to be *totally nonnegative* (TNN) if each of its minors is nonnegative.

Theorem: The eigenvalues of a TNN matrix are real and nonnegative.

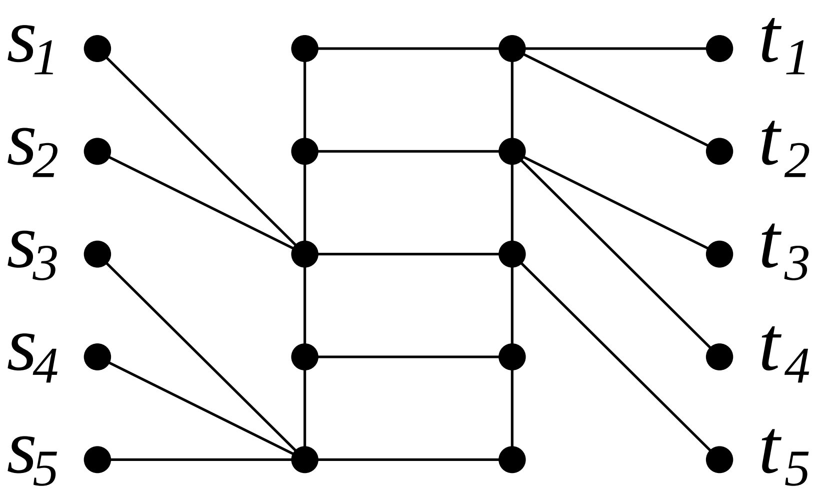
Equivalently, the equation

$$\det(Ax + I) = 0$$

has only real roots.

Planar networks

Define a *planar network of order n* to be a directed acyclic planar graph, in which $2n$ boundary vertices are labeled counterclockwise as $s_1, \ldots, s_n, t_n, \ldots, t_1$ as below.



Define the *path matrix* $A = [a_{ij}]$

$$a_{ij} = \# \text{ paths from } s_i \text{ to } t_j.$$

Theorem: (K-McG '59, L '73) The path matrix of a planar network is always TNN.

Proof idea: $\Delta_{I,J}$ counts families of nonintersecting paths from sources

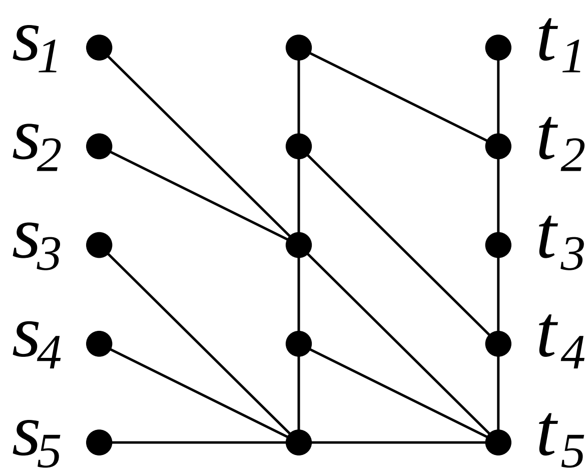
$$S_I = \{s_i \,|\, i \in I\}$$

to sinks

$$T_J = \{t_i \,|\, i \in J\}.$$

Theorem: (W'52, L'55, C'76, B'95) All TNN matrices are essentially path matrices of planar networks.

Proof idea: factorization corresponds to concatenation.



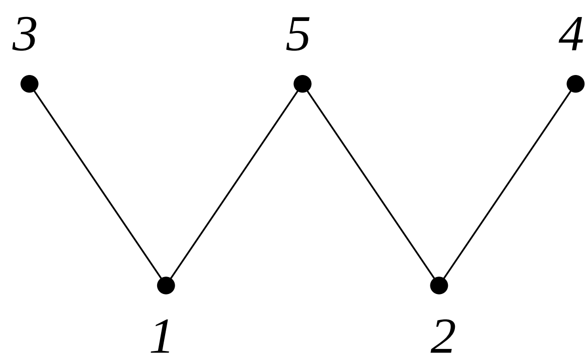
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 2 & 2 & 1 \\ 3 & 3 & 2 & 2 & 1 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \end{bmatrix}.$$

Antiadjacency matrices

Given a labeled poset P, define its **antiadjacency matrix** $A = [a_{ij}]$ by

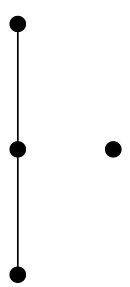
$$a_{ij} = \begin{cases} 0 & i <_P j \\ 1 & \text{otherwise.} \end{cases}$$

Example:

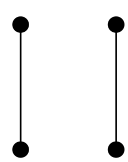


$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

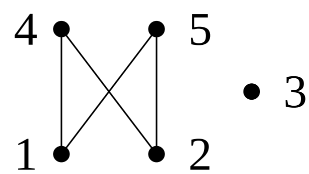
Theorem: (Dean-Keller '68) The antiadjacency matrix of a poset P is TNN only if P is a **unit interval order** (contains no induced subposet isomorphic to $\mathbf{3}+\mathbf{1}$ or $\mathbf{2}+\mathbf{2}$).



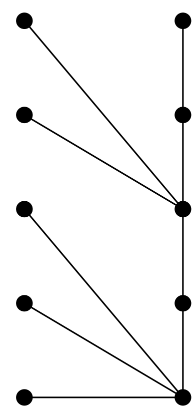
3 + 1



2 + 2

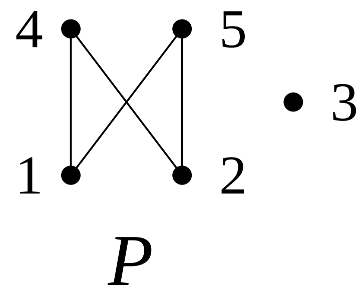


P



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Definition: The **f-polynomial** $f_P(z)$ of a poset P counts chains by cardinality.



Above, $f_P(z) = 1 + 5z + 4z^2$.

Theorem: (Stanley, '80) Let P have anti-adjacency matrix A. The f-polynomial of P is $f_P(z) = \det(Az + I)$.

Corollary: The f-polynomial of a unit interval order has only real zeros.

Theorem: (Stanley, Gasharov, S.) The f-polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros.

polynomials
$$1 + a_1 z + \cdots + a_n z^n \in \mathbb{N}[z]$$
having only real zeros

$$\bigcup$$

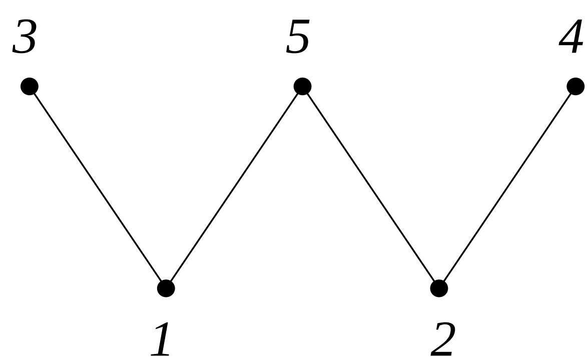
f-polynomials of
$(\mathbf{3}+\mathbf{1})$-free posets

$$\bigcup$$

f-polynomials of
$(\mathbf{3}+\mathbf{1})$-free, $(\mathbf{2}+\mathbf{2})$-free posets

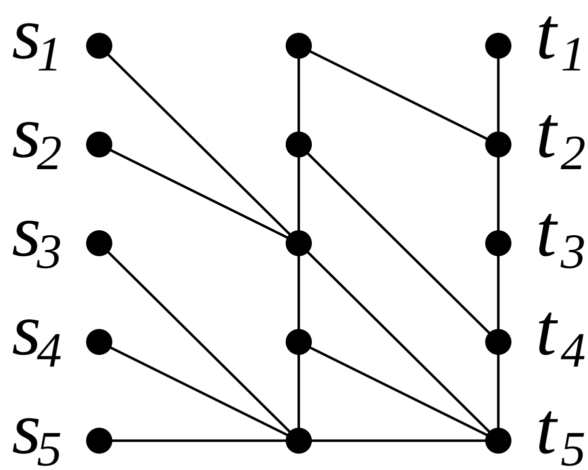
Theorem: (S-R '02) The f-polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset is the f-polynomial of a unit interval order.

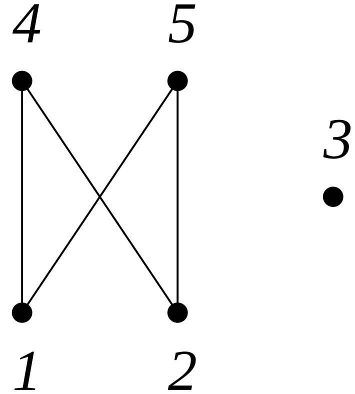
Proof (1st idea): the squared antiadjacency matrix has a nice factorization.

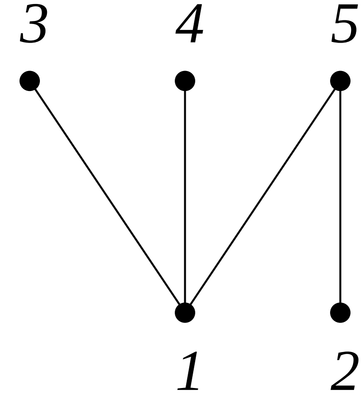


$$\begin{bmatrix}1&1&0&1&0\\1&1&1&0&0\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{bmatrix}^2 = \begin{bmatrix}1&1&1&0&0\\1&1&1&0&0\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{bmatrix}\begin{bmatrix}1&1&0&0&0\\1&1&1&1&0\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{bmatrix}.$$

This allows us to associate several new combinatorial objects to a $(\mathbf{3}+\mathbf{1})$-free poset: a planar network and two unit interval orders.







Proof (2nd idea): the f-polynomials of the unit interval orders are equal to that of our $(\mathbf{3}+\mathbf{1})$-free poset.

In our example, all three f-polynomials are

$$1 + 5z + 4z^2.$$

(Negative) comment: the chromatic symmetric functions of the two unit interval orders are not equal to that of our $(\mathbf{3}+\mathbf{1})$-free poset.

Open problems

The factorization of the squared antiadjacency matrix of a $(\mathbf{3}+\mathbf{1})$-free poset has at least two combinatorial interpretations.

Problem: Use this factorization to list the $(\mathbf{3}+\mathbf{1})$-free posets on n elements.

Problem: Use this factorization to count the $(\mathbf{3}+\mathbf{1})$-free posets on n elements.

Problem: Show that the chromatic symmetric function of a $(\mathbf{3}+\mathbf{1})$-free poset is e-positive.

Open problems

Let P_n be the set of all polynomials

$$1 + a_1 z + \cdots + a_n z^n \in \mathbb{N}[z]$$

having only real zeros.

Problem: Find a larger set of combinatorially defined polynomials belonging to P_n.

Theorem: (Bell-S '02) Every polynomial in P_n is the f-polynomial of a multicomplex on n variables.

Problem: Find a smaller set of combinatorially defined polynomials containing P_n.

Question: Is every polynomial in P_n the f-polynomial of a simplicial complex on n vertices?