TOTAL NONNEGATIVITY AND $(\mathbf{3}+\mathbf{1})$-FREE POSETS

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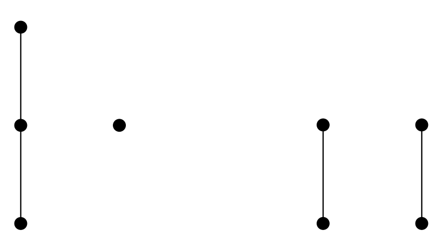
Abstract. We factor the squared antiadjacency matrix A^2 of a $(\mathbf{3}+\mathbf{1})$-free poset as a product of two antiadjacency matrices of unit interval orders. This gives a new combinatorial interpretation for the entries of A^2 in terms of finite planar networks and a proof that the f-vector of a $(\mathbf{3}+\mathbf{1})$-free poset is also the f-vector of a unit interval order. We also state some inequalities satisfied by the components of these f-vectors.

1. Introduction

Much current research in algebraic combinatorics concerns the characterization of f-vectors of simplicial complexes, polytopes, and related combinatorial structures. (See [3], [21, Ch. 2,3].) One interesting source of f-vectors is the class of $(\mathbf{3}+\mathbf{1})$-free posets because the generating polynomials for the corresponding f-vectors are known to have only real zeros [18, Cor. 4.1], [22, Cor. 2.9]. A poset is called $(\mathbf{3}+\mathbf{1})$-free if it contains no induced subposet isomorphic to that shown in Figure 1.1 (a).

An interesting subclass of $(\mathbf{3}+\mathbf{1})$-free posets is the class of those which are also $(\mathbf{2}+\mathbf{2})$-free, i.e. which contain no induced subposet isomorphic to that shown in Figure 1.1 (b). These are often called *unit interval orders* because a well-known



(a) (b)

Figure 1.1

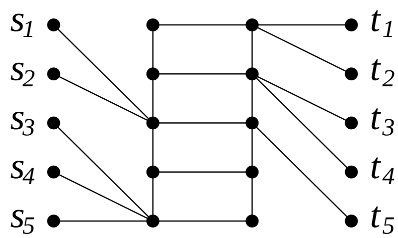


FIGURE 1.2

result [17] characterizes them as the posets P for which there exists a map

$$x \mapsto [q_x, q_x + 1]$$

from P to closed intervals of the real line which satisfies

$$x <_P y \text{ if and only if } q_x < q_y.$$

No analogous result is known to hold for $(\mathbf{3}+\mathbf{1})$-free posets in general.

Since unit interval orders form a proper subclass of $(\mathbf{3}+\mathbf{1})$-free posets, one might be surprised to learn that the containment of the corresponding two sets of f-vectors is not proper. (See Corollary 4.5.) The proof of this fact relies upon the factorization of a totally nonnegative matrix which we will associate to each $(\mathbf{3}+\mathbf{1})$-free poset.

A matrix is called *totally nonnegative* if the determinant of each of its square submatrices is nonnegative. A result often attributed to Lindström [12] describes the most important example of a totally nonnegative matrix in terms of a *planar network*, a planar acyclic directed graph G with $2n$ distinguished boundary vertices labeled counterclockwise as $s_1, \ldots, s_n, t_n, \ldots, t_1$. (See also [10].) Given a planar network G, its *path matrix* $A = [a_{ij}]$, in which a_{ij} counts paths from s_i to t_j, is totally nonnegative. For instance the matrix

$$\begin{bmatrix} 3 & 3 & 2 & 2 & 1 \\ 3 & 3 & 2 & 2 & 1 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \end{bmatrix}$$

is easily verified to be totally nonnegative because it is the path matrix of the planar network in Figure 1.2. When drawing planar networks, we will understand vertical edges to be oriented from bottom to top, and other edges to be oriented toward the right. (See also [7].)

We will finish by stating some inequalities which are satisfied by the f-vectors of all $(\mathbf{3}+\mathbf{1})$-free posets and by posing some open questions.

2. Order ideals and totally nonnegative matrices

Given an n-element poset P whose elements are labeled $1, \ldots, n$, we define the *antiadjacency matrix* [20] of P to be the matrix $A = [a_{ij}]$,

$$a_{ij} = \begin{cases} 0 & \text{if } i <_P j, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, distinct labelings of P can result in distinct antiadjacency matrices. It is easy to see that the antiadjacency matrix of a labeled poset P has no zero entries below the diagonal if and only if P is labeled *naturally* (i.e. each pair i, j of elements satisfying $i <_P j$ also satisfies $i < j$ as integers). Further, it is known that unit interval orders may be labeled so that the corresponding antiadjacency matrices are totally nonnegative [23, Prob. 6.19 (ddd)], and that $(\mathbf{3}+\mathbf{1})$-free posets may be labeled so that the corresponding *squared* antiadjacency matrices are totally nonnegative [18, p. 238].

To characterize the poset labelings which lead to totally nonnegative (squared) antiadjacency matrices, we will use principal order ideals and dual principal order ideals. For any element i in a poset P, we will denote the corresponding principal order ideal and principal dual order ideal by Λ_i and V_i, respectively.

$$\begin{aligned} \Lambda_i &= \{j \in P \,|\, j \leq_P i\}, \\ V_i &= \{j \in P \,|\, j \geq_P i\}. \end{aligned}$$

More precisely, we will *delete* an element i from such ideals and consider the *deleted ideals*

$$\begin{aligned} \Lambda_i^* &= \{j \in P \,|\, j <_P i\}, \\ V_i^* &= \{j \in P \,|\, j >_P i\}. \end{aligned}$$

When discussing the deleted ideals of different posets P and Q, we will use the notation $V_i^*(P)$ and $V_i^*(Q)$ to avoid ambiguity.

For each element i of P, we will define its *altitude* to be the difference in cardinality between its principal order ideal and principal dual order ideal, and will denote this number by $\alpha(i)$,

$$\begin{aligned} \alpha(i) &= |\Lambda_i| - |V_i| \\ &= |\Lambda_i^*| - |V_i^*|. \end{aligned}$$

(See [6, p. 33] for other applications of this function.)

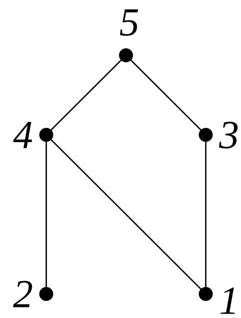


FIGURE 2.1

Example 2.1. Let P be the poset in Figure 2.1. Then we have

$$\begin{array}{lll} \Lambda_1^* = \emptyset & V_1^* = \{3,4,5\} & \alpha(1) = -3 \\ \Lambda_2^* = \emptyset & V_2^* = \{4,5\} & \alpha(2) = -2 \\ \Lambda_3^* = \{1\} & V_3^* = \{5\} & \alpha(3) = 0 \\ \Lambda_4^* = \{1,2\} & V_4^* = \{5\} & \alpha(4) = 1 \\ \Lambda_5^* = \{1,2,3,4\} & V_5^* = \emptyset & \alpha(5) = 4 \end{array}$$

It is easy to verify the following properties of deleted order ideals in $(\mathbf{3}+\mathbf{1})$-free posets and in unit interval orders. (We will use the symbols $\subseteq, \subset$ to denote containment and strict containment, respectively.)

Observation 2.1. *Let i and y be distinct elements of a $(\mathbf{3}+\mathbf{1})$-free poset. The corresponding deleted order ideals satisfy*

$$\Lambda_i^* \subseteq \Lambda_j^* \text{ or } V_i^* \subseteq V_j^*. \tag{2.1}$$

Proof. Left to reader. □

Observation 2.2. *Let i and j be distinct elements of a unit interval order. The corresponding deleted order ideals satisfy*

1. *If $\Lambda_i^* \not\subseteq \Lambda_j^*$, then $\Lambda_j^* \subset \Lambda_i^*$.*
2. *If $V_j^* \not\subseteq V_i^*$, then $V_i^* \subset V_j^*$.*

Proof. Left to reader. □

In $(\mathbf{3}+\mathbf{1})$-free posets, altitude is related to deleted ideals as follows.

Observation 2.3. *Let i and j be elements of a $(\mathbf{3}+\mathbf{1})$-free poset. Then we have*

$$\alpha(i) \leq \alpha(j) \tag{2.2}$$

if and only if we have

$$|\Lambda_i^*| \leq |\Lambda_j^*| \text{ and } V_i^* \supseteq V_j^*, \tag{2.3}$$

or

$$|V_i^*| \geq |V_j^*| \text{ and } \Lambda_i^* \subseteq \Lambda_j^*. \tag{2.4}$$

Proof. First note that the difference $\alpha(j) - \alpha(i)$ is equal to

$$(|\Lambda_j^*| - |\Lambda_i^*|) + (|V_i^*| - |V_j^*|). \tag{2.5}$$

Assume that i and j satisfy (2.2). Then at least one of the two terms in the sum (2.5) is nonnegative. Suppose the first term is nonnegative. If the deleted ideals Λ_i^* and Λ_j^* are equal, then $|V_i^*|$ is greater than or equal to $|V_j^*|$ and the condition (2.4) is satisfied. If on the other hand Λ_j^* and Λ_i^*, are not equal, then Observation 2.1 guarantees that V_y^* is contained in V_i^* and again the condition (2.4) is satisfied. Similarly, when the second term in the sum (2.5) is nonnegative, then the condition (2.3) is satisfied.

Now assume that at least one of the conditions (2.3) (2.4) is satisfied. In either case, we have

$$|\Lambda_i^*| \leq |\Lambda_j^*| \text{ and } |V_i^*| \geq |V_j^*|,$$

which implies (2.2). □

We will say that a labeling of a poset P *respects altitude* if each pair i, j of poset elements satisfying $\alpha(i) < \alpha(j)$ also satisfies $i < j$ (as integers). Note that a labeling which respects altitude is necessarily natural. The following proposition (essentially stated in [25, Sect. 8.2]) shows that the antiadjacency matrices of naturally labeled unit interval orders are totally nonnegative precisely when the labelings respect altitude.

Proposition 2.4. *Let P be a labeled n-element unit interval order with antiadjacency matrix $A = [a_{ij}]$. The labeling of P respects altitude if and only if A satisfies*

$$a_{jk} \geq a_{i\ell} \tag{2.6}$$

for $1 \leq i \leq j \leq n$ and $1 \leq k \leq \ell \leq n$.

Proof. Suppose that A satisfies (2.6). Then for any two indices $g < h$, we have

$$|V_g^*| \geq |V_h^*| \text{ and } |\Lambda_g^*| \leq |\Lambda_h^*|.$$

Combining these inequalities, we obtain $\alpha(g) \leq \alpha(h)$, as desired.

Now suppose that A does not satisfy (2.6). Then for some indices g, h, q with $g < h$ we have

$$a_{qg} < a_{qh} \text{ or } a_{gq} > a_{hq}. \tag{2.7}$$

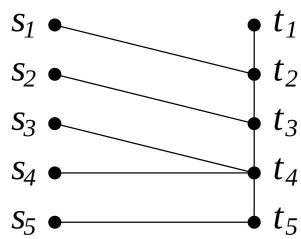


FIGURE 2.2

The first of these two inequalities implies that we have

$$\Lambda_g^* \not\subseteq \Lambda_h^*,$$

which by Observations 2.1 and 2.2 implies that we have

$$V_g^* \subseteq V_h^* \text{ and } \Lambda_h^* \subset \Lambda_g^*,$$

which in turn implies that we have

$$\alpha(h) = |\Lambda_h^*| - |V_h^*| < |\Lambda_g^*| - |V_g^*| = \alpha(g). \tag{2.8}$$

By similar reasoning, the second inequality of (2.7) also implies that we have (2.8). Thus the labeling of P does not respect altitude. □

In a 0-1 matrix satisfying the conditions of Proposition 2.4, the zero entries form a Ferrers shape in the upper right corner of the matrix. For instance, the labeling of the poset in Figure 2.1 respects altitude and the antiadjacency matrix of this poset is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{2.9}$$

One can prove that such a matrix A is totally nonnegative by constructing a planar network whose path matrix is A. (One can also use induction, or appeal to the famous result [1] concerning Toeplitz matrices.) Figure 2.2 shows a planar network whose path matrix is the antiadjacency matrix (2.9) of the poset in Figure 2.1.

A result analogous to Proposition 2.4 holds for the *squared* antiadjacency matrices of $(\mathbf{3}+\mathbf{1})$-free posets. Before stating this result, let us give one interpretation of the entries of these matrices.

Lemma 2.5. *Let P be a labeled n-element $(\mathbf{3}+\mathbf{1})$-free poset with antiadjacency matrix A and define the matrix $B = [b_{ij}] = A^2$. Then we have*

$$b_{ij} = \begin{cases} n - |V_i^*| - |\Lambda_j^*| & \text{if } V_i^* \cap \Lambda_j^* \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the definition of B we have

$$\begin{aligned} b_{ij} &= \sum_{k=1}^{n} a_{ik} a_{kj} \\ &= \#\{k \in [n] \mid a_{ik} = a_{kj} = 1\} \\ &= \#\{k \in P \mid i \not<_P k \not<_P j\} \\ &= n - |V_i^*| - |\Lambda_j^*| + |V_i^* \cap \Lambda_j^*|. \end{aligned}$$

Suppose that b_{ij} is nonzero and let k be an element satisfying $i \not<_P k \not<_P j$. Note that for any element ℓ belonging to the intersection $V_i^* \cap \Lambda_j^*$, the subposet of P induced by $\{i, j, k, \ell\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$. Thus this intersection is empty and we have the desired result. □

Proposition 2.6. *Let P be a labeled n-element $(\mathbf{3}+\mathbf{1})$-free poset with antiadjacency matrix A, and define the matrix $B = [b_{ij}] = A^2$. The labeling of P respects altitude if and only if B satisfies the conditions*

1. *$b_{jk} \geq b_{i\ell}$,*
2. *If $b_{ik} - b_{i\ell} \neq b_{jk} - b_{j\ell}$, then $b_{i\ell} = 0$ and $b_{ik} < b_{jk} - b_{j\ell}$,*

for all integers $1 \leq i \leq j \leq n$ and $1 \leq k \leq \ell \leq n$.

Proof. Assume that B satisfies the conditions above. Then for any two integers $1 \leq i < j \leq n$, any minimal element g, and maximal element h, we have

$$b_{ig} \leq b_{jg} \text{ and } b_{hi} \geq b_{hj}.$$

By the minimality of g and the maximality of h, both Λ_g^* and V_h^* are empty. Therefore by Lemma 2.5 we have

$$|V_i^*| \geq |V_j^*| \text{ and } |\Lambda_i^*| \leq |\Lambda_j^*|. \tag{2.10}$$

Combining the inequalities (2.10), we obtain $\alpha(i) \leq \alpha(j)$, as desired.

Now assume that B does not satisfy the required conditions. By [18, Prop 3.4], B fails to satisfy condition 2 only if it fails to satisfy condition 1. We may assume therefore that for some i, j, g with $i < j$, we have

$$b_{ig} > b_{jg} \text{ or } b_{gi} < b_{gj}. \tag{2.11}$$

Assume that we have $b_{gi} < b_{gj}$. The intersection $V_g^* \cap \Lambda_j^*$ must be empty since b_{gj} is positive.

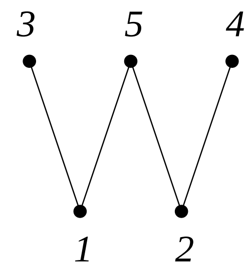


FIGURE 2.3

If the intersection $V_g^* \cap \Lambda_i^*$ is also empty, then by Lemma 2.5 we have

$$|\Lambda_j^*| < |\Lambda_i^*|, \tag{2.12}$$

and Λ_i^* is not contained in Λ_j^*. Observation 2.1 then implies that we have

$$V_i^* \subseteq V_j^*,$$

and therefore

$$|V_i^*| \leq |V_j^*|. \tag{2.13}$$

Combining the inequalities (2.12) and (2.13), we have

$$\alpha(j) < \alpha(i), \tag{2.14}$$

and the labeling does not respect altitude.

If the intersection $V_g^* \cap \Lambda_i^*$ is not empty, then some element h in this intersection does not belong to Λ_j^*. In contrast, there can be no element in Λ_j^* which does not also belong to Λ_k^*, for then the subposet of P induced by this element and g, h, i would be isomorphic to $\mathbf{3} + \mathbf{1}$. Thus Λ_j^* is properly contained in Λ_i^*, and by Observation 2.1 V_i^* is contained in V_j^*. Combining these two containments, we again have (2.14), and the labeling does not respect altitude.

Similarly, the assumption $b_{ig} > b_{jg}$ implies the strict inequality $|V_j^*| < |V_i^*|$, and therefore (2.14). $\square$

As an example of Proposition 2.6, consider the labeled poset in Figure 2.3. This labeling respects altitude. and the corresponding squared antiadjacency matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 3 & 2 & 2 & 1 \\ 3 & 3 & 2 & 2 & 1 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \\ 5 & 5 & 4 & 4 & 3 \end{bmatrix}.$$

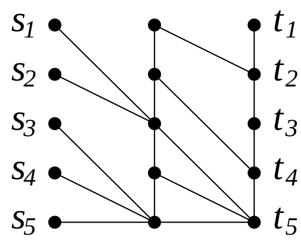


FIGURE 2.4

Again, to prove that such a matrix B is totally nonnegative, it suffices to construct a planar network having path matrix B. Figure 1.2 (a) shows one such planar network which is constructed easily from B. Another possibility in Figure 2.4 is constructed by concatenating two planar networks corresponding to the antiadjacency matrices of unit interval orders [16]. This observation suggests the possibility of factoring the squared antiadjacency matrices of $(\mathbf{3}+\mathbf{1})$-free posets in general. Such a factorization is in fact possible and will be considered further in Sections 3 and 4.

3. A factorization theorem

The planar network in Figure 2.4 is constructed using a factorization of the squared antiadjacency matrix of the poset in Figure 2.3. In general, let A be the antiadjacency matrix corresponding to any labeling of a $(\mathbf{3}+\mathbf{1})$-free poset. To obtain a factorization

$$A^2 = CD,$$

one contstructs C and D from A by "pushing" the zero entries of A to the right and up, respectively. For example, one labeling of the poset in Figure 2.3 gives an antiadjacency matrix whose square factors as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Theorem 3.1. *Let P be a labeled $(\mathbf{3}+\mathbf{1})$-free poset with antiadjacency matrix A. Let C be the matrix obtained from A by permuting the entries of each row into nonincreasing order, and let D be the matrix obtained from A by permuting the entries of each column into nondecreasing order. Then we have*

$$A^2 = CD.$$

Proof. Let n be the cardinality of P and define the matrices $B = [b_{ij}] = A^2$ and $E = [e_{ij}] = CD$. Since the numbers c_{ij} and d_{ij} are given by

$$c_{ij} = \begin{cases} 1 & \text{if } j \leq n - |V_i^*|, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_{ij} = \begin{cases} 1 & \text{if } i \geq |\Lambda_j^*| + 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} e_{ij} &= \sum_{k=1}^{n} c_{ik} d_{kj} \\ &= \#\{k \in [n] \mid c_{ik} = d_{kj} = 1\} \\ &= \#\{|\Lambda_j^*| + 1, \ldots, n - |V_i^*|\} \\ &= \begin{cases} n - |V_i^*| - |\Lambda_j^*| & \text{if } |V_i^*| + |\Lambda_j^*| \leq n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We claim that the inequality

$$|V_i^*| + |\Lambda_j^*| \leq n \tag{3.1}$$

holds if and only if the intersection $V_i^* \cap \Lambda_j^*$ is empty. If the intersection is empty, then (3.1) is clear. Suppose therefore that the intersection is not empty. Then P contains some element ℓ which satisfies

$$i <_P \ell <_P j$$

and we have

$$|V_i^* \cup \Lambda_j^*| \leq |V_i^*| + |\Lambda_j^*| - 1. \tag{3.2}$$

If some element lies outside of the union $V_i^* \cup \Lambda_j^*$ above, then it is incomparable to i, ℓ, and j, contradicting the fact that P is $(\mathbf{3} + \mathbf{1})$-free . Thus the cardinality of this union is n and the inequality (3.2) gives

$$n \leq |V_i^*| + |\Lambda_j^*| - 1,$$

contradicting (3.1).

We therefore obtain the expression

$$e_{ij} = \begin{cases} n - |V_i^*| - |\Lambda_j^*| & \text{if } V_i^* \cap \Lambda_j^* \text{ is empty,} \\ 0 & \text{otherwise,} \end{cases}$$

which is identical to that for b_{ij} given in Lemma 2.5. □

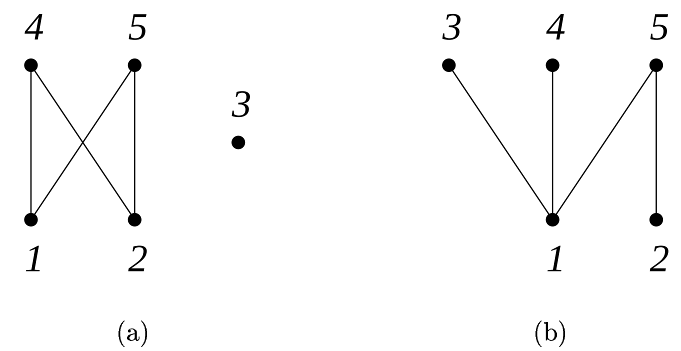


FIGURE 3.1

In the event that the labeling of P in Theorem 3.1 respects altitude, the matrices C and D in the theorem are the antiadjacency matrices corresponding to altitude respecting labelings of unit interval orders. Note however that the implied map from $(\mathbf{3}+\mathbf{1})$-free posets to pairs of unit interval orders is neither injective nor surjective.

Corollary 3.2. *Let P be a labeled $(\mathbf{3}+\mathbf{1})$-free poset with antiadjacency matrix A. If the labeling of P respects altitude then there are labeled unit interval orders Q_1 and Q_2 whose antiadjacency matrices are the matrices C and D defined in Theorem 3.1. Furthermore the labelings of Q_1 and Q_2 respect altitude.*

Proof. Let P be an n-element $(\mathbf{3}+\mathbf{1})$-free poset with an altitude respecting labeling. By Observation 2.3, the sequence $(|V_1^*|, \ldots, |V_n^*|)$ weakly decreases and the sequence $(|\Lambda_1^*|, \ldots, |\Lambda_n^*|)$ weakly increases. Thus the zero entries of C and D form Ferrers shapes in the upper right corners of these matrices. By Proposition 2.4, the corresponding poset labelings respect altitude. □

Since the class of totally nonnegative matrices is closed under multiplication (see e.g. [2]), Corollary 3.2 gives an easy proof of the total nonnegativity of the squared antiadjacency matrix A^2 of a labeled $(\mathbf{3}+\mathbf{1})$-free poset whose labeling respects altitude. It also allows one to combinatorially interpret A^2 without computing it.

Take for example the poset in Figure 2.3 labeled as shown. Its squared antiadjacency matrix factors as

$$\begin{bmatrix} 1&1&0&1&0\\ 1&1&1&0&0\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{bmatrix}^2 = \begin{bmatrix} 1&1&1&0&0\\ 1&1&1&0&0\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{bmatrix} \begin{bmatrix} 1&1&0&0&0\\ 1&1&1&1&0\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{bmatrix},$$

and counts paths in the planar network shown in Figure 2.4. This factorization also associates to the poset the two unit interval orders shown in Figure 3.1.

4. f-vectors of $(\mathbf{3}+\mathbf{1})$-free posets

The relationship between the unit interval orders and the $(\mathbf{3}+\mathbf{1})$-free poset in Corollary 3.2 extends beyond the factorization stated in Theorem 3.1. We will show that for any k, there is a bijective correspondence between the k-element chains in any two of these three posets.

A k-element *chain* in a poset P is a sequence of elements $(x_1, \ldots, x_k)$ of P which satisfy

$$x_1 <_P \cdots <_P x_k.$$

The *chain polynomial* or *f-polynomial* of P is the polynomial

$$f_P(t) = 1 + a_1 t + \cdots + a_m t^m,$$

where a_i is the number of i-element chains in P. The sequence of numbers

$$f_P = (1, a_1, \ldots, a_m)$$

is called the *f-vector* of P and is often written as $(1, f_0, \ldots, f_{m-1})$.

To better describe the relationship between the posets mentioned above, we will define a self-map ϕ on the set of all naturally labeled n-element posets. Given such a poset P with antiadjacency matrix A, we define $\phi(P)$ as follows.

1. If all rows of A are weakly decreasing, define $\phi(P) = P$.
2. Otherwise,
   (a) Let j be the greatest integer in $[n-1]$ for which we have

$$0 = a_{i,j} < a_{i,j+1} = 1, \tag{4.1}$$

   for some index i in $[n]$.
   (b) Define A' to be the matrix obtained from A by exchanging the entries $a_{i,j}$ and $a_{i,j+1}$ for each index i satisfying (4.1).
   (c) Let $\phi(P)$ be the poset whose antiadjacency matrix is A'.

It is easy to see that the map ϕ is well defined, for the entries of $A' = [a'_{g,h}]$ satisfy $a'_{g,h} = 1$ whenever the integer g is greater than the integer h, and $a'_{f,h} = 0$ whenever $a'_{f,g}$ and $a'_{g,h}$ are both zero.

An equivalent definition of the map ϕ in terms of posets is as follows.

1. Let j be the greatest integer such that we have

$$i <_P j \text{ and } i \not<_P j+1 \tag{4.2}$$

   for some i.

2. For each element x satisfying $x <_P j$,
   (a) If $x \not<_P j+1$, then define $x <_{\phi(P)} j+1$.
   (b) Otherwise, define $x <_{\phi(P)} j$.
3. For each pair (x,y) of elements, if y is not equal to j, then define $x <_{\phi(P)} y$.

It is not difficult to construct the Hasse diagram for the poset $\phi(P)$ from that of P. Assuming the existence of an element j for which the antiadjacency marix of P satisfies (4.1), let I be the set of elements of P covered by j and not comparable to $j+1$. For each element i in I, replace the edge (i,j) by the edge $(i,j+1)$. For each element h covered by an element in I and not covered by any element in $\Lambda_j^* \cap \Lambda_{j+1}^*$, introduce the new edge (h,j).

While the map ϕ does not in general preserve altitude, it does preserve $|V_i^*|$ for all elements i in P. Further, if P is a $(\mathbf{3}+\mathbf{1})$-free poset which satisfies

$$|V_1^*| \geq \cdots \geq |V_n^*| \tag{4.3}$$

we can infer several interesting things about P and $\phi(P)$. Note that a labeling of a $(\mathbf{3}+\mathbf{1})$-free poset which respects altitude necessarily satisfies the condition (4.3), and that a labeling of a $(\mathbf{3}+\mathbf{1})$-free poset which satisfies the condition (4.3) is necessarily natural.

Lemma 4.1. *Let P be an n-element $(\mathbf{3}+\mathbf{1})$-free which satisfies (4.3) and let j be any integer in $[n]$ which satisfies (4.1) for some i. Then the four deleted ideals $V_j^*(P), V_{j+1}^*(P), V_j^*(\phi(P)), V_{j+1}^*(\phi(P))$ are equal. Furthermore, each element i satisfying (4.1) is covered by j in P and is covered by $j+1$ in $\phi(P)$.*

Proof. Note that $\Lambda_j^*(P)$ is not contained in $\Lambda_{j+1}^*(P)$. Therefore by Observation 2.1, $V_j^*(P)$ must be contained in $V_{j+1}^*(P)$. Since the cardinality of $V_j^*(P)$ is no smaller than that of $V_{j+1}^*(P)$, these sets must be equal. Since j and $j+1$ do not belong to these deleted ideals, it follows that any element greater than j and $j+1$ in P is again greater than j and $j+1$ in $\phi(P)$.

Suppose that some element i satisfying (4.1) is not covered by j. Then there is an element h satisfying

$$i <_P h <_P j.$$

By our choice of i, we cannot have $i <_P j+1$. Neither can we have $j+1 <_P j$ since P is natural. Thus $j+1$ is incomparable in P to i, h, and j, contradicting the fact that P is a $(\mathbf{3}+\mathbf{1})$-free poset.

Suppose that some element i satisfying (4.1) is not covered by $j+1$ in $\phi(P)$. Then for some element h we have

$$i <_{\phi(P)} h <_{\phi(P)} j+1,$$

and since h is not equal to $j+1$ we have

$$i <_P h.$$

By the previous argument, i must be covered by j in P. We therefore have

$$h \not<_P j.$$

It follows that we have

$$h <_P j+1,$$

and by transitivity we have

$$i <_P j+1,$$

contradicting our assumption that i satisfies (4.1). □

Furthermore it is not difficult to see that the f-vectors of P and $\phi(P)$ are equal.

Proposition 4.2. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset whose labeling satisfies (4.3). Then the f-vector of $\phi(P)$ is equal to that of P.*

Proof. If $\phi(P) = P$ then there is nothing to prove. Assume therefore that $\phi(P)$ does not equal P, and that j is the index satisfying (4.1) in the definition of the map ϕ.

It will suffice to show that for any ℓ, the sequences $(x_1, \ldots, x_\ell)$ which are chains in P and not chains in $\phi(P)$ are in bijective correspondence with the sequences which are chains in $\phi(P)$ and not chains in P.

A sequence which is a chain in P and not a chain in $\phi(P)$ necessarily has the form

$$(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_\ell).$$

Further, the sequence defined from this by

$$(x_1, \ldots, x_{i-1}, j+1, x_{i+1}, \ldots, x_\ell)$$

is a chain in $\phi(P)$ and is *not* a chain in P.

Conversely, a sequence which is a chain in $\phi(P)$ and not a chain in P necessarily has the form

$$(y_1, \ldots, y_{h-1}, j+1, y_{h+1}, \ldots, y_\ell),$$

and the sequence defined from this by

$$(y_1, \ldots, y_{h-1}, j, y_{h+1}, \ldots, y_\ell)$$

is not a chain in P or in $\phi(P)$.

It follows that the map

$$(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_\ell) \mapsto (x_1, \ldots, x_{i-1}, j+1, x_{i+1}, \ldots, x_\ell)$$

induced by ϕ is our desired bijection of chains.

□

In addition to preserving the f-vector of a poset, the map ϕ also preserves $\mathbf{3}+\mathbf{1}$ avoidance.

Proposition 4.3. *Let P be a labeled $(\mathbf{3}+\mathbf{1})$-free poset which satisfies (4.3). Then $\phi(P)$ is $(\mathbf{3}+\mathbf{1})$-free.*

Proof. Suppose that $\phi(P)$ is not $(\mathbf{3}+\mathbf{1})$-free. Then there are four elements $\{x_1, x_2, x_3, x_4\}$ related as

$$x_1 <_{\phi(P)} x_2 <_{\phi(P)} x_3,$$
$$x_4 \text{ incomparable to } x_1, x_2, x_3 \text{ in } \phi(P).$$

Since the same four elements do not induce a subposet of P isomorphic to $\mathbf{3}+\mathbf{1}$, we have at least one of the following comparisons in P.

1. $x_1 \not<_P x_2$.
2. $x_2 \not<_P x_3$.
3. $x_1 <_P x_4$.
4. $x_4 <_P x_3$.

Let j be the index refered to in the definition of ϕ.

Assume that comparison (1) is true. Then we have

$$x_1 <_P j$$

and x_2 is equal to $j+1$. Since the deleted ideals $V^*_{j+1}(\phi(P))$ and $V^*_j(P)$ are equal, the relation $x_2 <_{\phi(P)} x_3$ (i.e. $j+1 <_{\phi(P)} x_3$) implies that we have

$$j <_P x_3,$$

and the relation $x_4 \not<_{\phi(P)} x_3$ implies that x_4 is not equal to j. Thus the relations $x_1 \not<_{\phi(P)} x_4$ and $x_4 \not<_{\phi(P)} x_3$ imply that we have

$$x_1 \not<_P x_4 \text{ and } x_4 \not<_P x_3,$$

contradicting the fact that P is $(\mathbf{3}+\mathbf{1})$-free.

Assume therefore that comparison (1) is false and that comparison (2) is true. Then x_3 is equal to $j+1$ and we have

$$x_1 <_P x_2 <_P j.$$

The comparison $x_4 \not<_{\phi(P)} j+1$ implies that we have

$$x_4 \not<_P j.$$

Since x_1 is not covered by j in P, we must have

$$x_1 <_{\phi(P)} j,$$

which implies that x_4 is not equal to j. Thus the relation $x_1 \not<_{\phi(P)} x_4$ implies that we have

$$x_1 \not<_P x_4,$$

contradicting the fact that P is $(\mathbf{3}+\mathbf{1})$-free.

Assume therefore that comparisons (1) and (2) are false, while comparison (3) is true. Then we have

$$x_1 <_P x_2 <_P x_3,$$
$$x_1 \not<_P j+1,$$

and x_4 is equal to j. Since $V_j^*(\phi(P))$ and $V_{j+1}^*(P)$ are equal, the comparison $x_4 \not<_{\phi(P)} x_3$ (i.e. $j \not<_{\phi(P)} x_3$) implies that we have

$$j+1 \not<_P x_3,$$

contradicting the fact that P is $(\mathbf{3}+\mathbf{1})$-free.

Finally, assume that comparisons (1)-(3) are false, and that comparison (4) is true. Then we have

$$x_1 <_P x_2,$$
$$x_1 \not<_P x_4,$$

and x_3 is equal to j. The comparison $x_2 <_{\phi(P)} j$ then implies that we have

$$x_2 <_P j+1,$$

and the comparisons $x_4 <_P x_3$ and $x_4 \not<_{\phi(P)} x_3$ imply that we have

$$x_4 \not<_P j+1,$$

contradicting the fact that P is $(\mathbf{3}+\mathbf{1})$-free. □

Thus by applying several iterations of the map ϕ to a $(\mathbf{3}+\mathbf{1})$-free poset P whose labeling respects altitude, we obtain the poset Q_1 from Corollary 3.2, and find that the f-vector of Q_1 is equal to that of P.

Theorem 4.4. *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset and let A be the antiadjacency matrix corresponding to an altitude-respecting labeling of P. Define the matrices C and D as in Theorem 3.1, and let Q_1 and Q_2 be the two labeled unit interval orders whose antiadjacency matrices are C and D. Then the f-vectors of all three posets are equal.*

Proof. If P is a unit interval order, then $P = Q_1 = Q_2$ and we are done. Suppose therefore that P is not a unit interval order. Then for some number k we have $\phi^k(P) = Q_1$, which by Proposition 4.2 implies that the f-vectors of P and Q_1 are equal. Applying the same argument to the dual poset of P, we find that the f-vectors of P and Q_2 are equal as well. □

Thus although the set of $(\mathbf{3}+\mathbf{1})$-free posets on n elements strictly contains the set of unit interval orders on n elements (for $n > 3$), the corresponding containment of sets of f-vectors is not strict.

Corollary 4.5. *The set of f-vectors of $(\mathbf{3}+\mathbf{1})$-free posets on n elements is equal to the set of f-vectors of unit interval orders on n elements.*

As is the case with many interesting classes of f-vectors, no characterization of the f-vectors of $(\mathbf{3}+\mathbf{1})$-free posets is known. On the other hand, it is not too difficult to prove some inequalities that must be satisfied by the components of these f-vectors. Somewhat surprisingly, the inequalities below are satisfied also by pure O-sequences [8], by the h-vectors of matroid complexes [4], and by the coefficients of the Poincaré polynomials of singluar Schubert varieties [26]. (See also [9], [19, Cor. 2.4].)

Proposition 4.6. *Let $a(t) = a_0+a_1t+\cdots+a_mt^m$ be the f-polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset. Then for $i = 0, \ldots, \lfloor \frac{m-1}{2} \rfloor$ we have*

$$a_i \leq a_{i+1}, \tag{4.4}$$

$$a_i \leq a_{m-i}. \tag{4.5}$$

Proof. Let P be an n-element $(\mathbf{3}+\mathbf{1})$-free poset.

If P is a chain we have

$$a(t) = (1+t)^m.$$

Clearly $a(t)$ satisfies the inequalities (4.4) and (4.5). Assume therefore that P is not a chain, and choose an element x not belonging to some m-element chain in P. Now define the induced subposets Q and R of P by

$$Q = P \smallsetminus x,$$
$$R = \Lambda_x^* \cup V_x^*,$$

and let $b(t)$ and $c(t)$ be the f-polynomials of Q and R respectively. These are related to $a(t)$ by

$$a(t) = b(t) + tc(t). \tag{4.6}$$

The longest chain of Q clearly has m elements. Therefore we will write

$$b(t) = b_0 + b_1t + \cdots + b_mt^m,$$

where $b_0, \ldots, b_m$ are positive. It is clear that the longest chain in R has at most $m-1$ elements. Since x is incomparable to at most two elements of any m-element chain in Q the longest chain in R must have at least $m-2$ elements. Therefore we will write

$$c(t) = c_0 + c_1t + \cdots + c_{m-1}t^{m-1},$$

where $c_0, \cdots, c_{m-2}$ are positive and c_{m-1} is nonnegative. Let us define c_{-1} to be zero, so that Equation (4.6) becomes

$$a_i = b_i + c_{i-1}, \quad \text{for } i = 0, \ldots, m. \tag{4.7}$$

Assume by induction that the proposition is true for $(\mathbf{3}+\mathbf{1})$-free posets having fewer than n elements, and note that Q and R are two such posets. Applying this inductive hypothesis to (4.4) we see that $b(t)$ satisfies

$$b_i \leq b_{i+1}, \quad \text{for } i = 0, \ldots, \lfloor \tfrac{m-1}{2} \rfloor. \tag{4.8}$$

and $c(t)$ satisfies

$$c_i \leq c_{i+1}, \quad \text{for } \begin{cases} i = -1, \ldots, \lfloor \frac{m-3}{2} \rfloor & \text{if } c_{m-1} \text{ is zero,} \\ i = -1, \ldots, \lfloor \frac{m-2}{2} \rfloor & \text{if } c_{m-1} \text{ is positive.} \end{cases} \tag{4.9}$$

Rewriting (4.9) we have

$$c_{i-1} \leq c_i, \quad \text{for } \begin{cases} i = 0, \ldots, \lfloor \frac{m-1}{2} \rfloor & \text{if } c_{m-1} \text{ is zero,} \\ i = 0, \ldots, \lfloor \frac{m}{2} \rfloor & \text{if } c_{m-1} \text{ is positive.} \end{cases} \tag{4.10}$$

Combining (4.6), (4.8), and (4.10) we obtain

$$a_i = b_i + c_{i-1} \leq b_{i+1} + c_i = a_{i+1}, \quad \text{for } i = 0, \ldots, \lfloor \tfrac{m-1}{2} \rfloor,$$

which proves the inequality (4.4).

Applying the inductive hypothesis to (4.5), we see that $b(t)$ satisfies

$$b_i \leq b_{m-i}, \quad \text{for } i = 0, \ldots, \lfloor \tfrac{m-1}{2} \rfloor, \tag{4.11}$$

while $c(t)$ satisfies

$$c_i \leq c_{m-2-i}, \quad \text{for } i = -1, \ldots, \lfloor \tfrac{m-3}{2} \rfloor, \quad \text{if } c_{m-1} \text{ is zero,} \tag{4.12}$$

$$c_i \leq c_{m-1-i}, \quad \text{for } i = 0, \ldots, \lfloor \tfrac{m-2}{2} \rfloor, \quad \text{if } c_{m-1} \text{ is positive.} \tag{4.13}$$

First suppose c_{m-1} is zero. Then we have the inequality

$$c_{i-1} \leq c_{m-2-(i-1)} = c_{m-1-i}, \quad \text{for } i = 0, \ldots, \lfloor \tfrac{m-1}{2} \rfloor.$$

Combining this inequality with (4.7) and (4.11) we obtain

$$a_i = b_i + c_{i-1} \leq b_{m-i} + c_{m-1-i} = a_{m-i},$$

which proves the inequality (4.5).

Now suppose that c_{m-1} is positive. Combining (4.6), (4.10), (4.11), and (4.13) we obtain

$$a_i = b_i + c_{i-1} \leq b_i + c_i \leq b_{m-i} + c_{m-1-i} = a_{m-i}, \tag{4.14}$$

for $i = 0, \ldots, \lfloor \frac{m-2}{2} \rfloor$. Indeed (4.14) also holds for $i = \lfloor \frac{m-1}{2} \rfloor$, for when m is even we have

$$\lfloor \tfrac{m-2}{2} \rfloor = \lfloor \tfrac{m-1}{2} \rfloor,$$

and when m is odd we have

$$c_{\lfloor \frac{m-1}{2} \rfloor} = c_{\frac{m-1}{2}} = c_{m-1-\lfloor \frac{m-1}{2} \rfloor}.$$

This proves the inequality (4.5). □

5. Open problems

A more thorough understanding of the f-vectors of $(\mathbf{3}+\mathbf{1})$-free posets (equivalently, of unit interval orders) would be interesting because this might help to prove conjectures that certain combinatorially defined polynomials have only real zeros. (See for example [13, p. 114] or [24, Prob. 20].)

Problem 5.1. Characterize the f-vectors of unit interval orders.

On the other hand, a better understanding of the factorization in Theorem 3.1 might help to obtain results for $(\mathbf{3}+\mathbf{1})$-free posets analogous to those already known for unit interval orders. For instance the number of nonisomorphic unit interval orders on n elements is well-known to be the nth Catalan number [5], [27], but no such formula is known for $(\mathbf{3}+\mathbf{1})$-free posets.

Problem 5.2. Find a formula for the number of nonisomorphic $(\mathbf{3}+\mathbf{1})$-free posets on n elements.

Perhaps one could obtain a lower bound for the formula in Problem 5.2 by counting appropriate pairs of unit interval orders.

Problem 5.3. Characterize the pairs of unit interval orders which result from the factorization in Corollary 3.2. (Equivalently, characterize the corresponding pairs of antiadjacency matrices, or the corresponding planar networks.)

A second consequence of an answer to Problem 5.3 might be a new interpretation of the elements of a $(\mathbf{3}+\mathbf{1})$-free poset.

Problem 5.4. Find a representation of $(\mathbf{3}+\mathbf{1})$-free posets analogous to the interval representation of unit interval orders.

It is not hard to show that an n-element unit interval order is uniquely determined by its sequence $(\alpha(1), \ldots, \alpha(n))$ of altitudes. The analogous statement is not true for $(\mathbf{3}+\mathbf{1})$-free posets. Nevertheless, one might use altitude sequences to obtain a lower bound for the number of $(\mathbf{3}+\mathbf{1})$-free posets on n elements.

Question 5.5. Is there a simple characterization of the altitude sequences arising from $(\mathbf{3}+\mathbf{1})$-free posets on n elements, or a simple formula counting such sequences?

An interesting related problem, stated by Postnikov [14], is based upon a conjecture of Kostant [11].

Problem 5.6. Let P be a labeled poset with altitude sequence $(\alpha(1),\ldots,\alpha(n))$. Show that if P is a unit interval order, then there are two permutations $\pi=\pi_1,\ldots,\pi_n$ and $\sigma_1,\ldots,\sigma_n$ in S_n which satisfy

$$\pi_i-\sigma_i=\alpha(i),$$

for $i=1,\ldots,n$.

The statement is trivially true if we assume P has dimension at most two, and thus is true for many unit interval orders, which have dimension at most three [15]. (See also [6, Sect. 5.4], [25, Sect. 8.3].) The analogous statement involving $(\mathbf{3}+\mathbf{1})$-free posets is false however.

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