A CHARACTERIZATION OF $(\mathbf{3}+\mathbf{1})$-FREE POSETS

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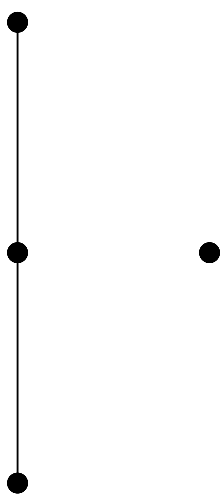
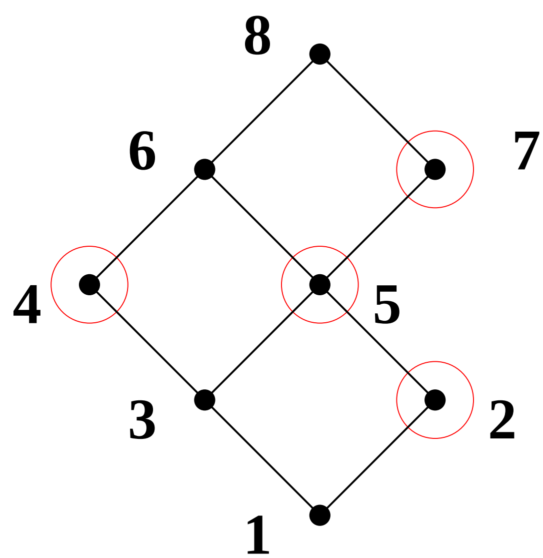
Outline

1. Definitions
2. Characterization Theorem
3. Related Problems

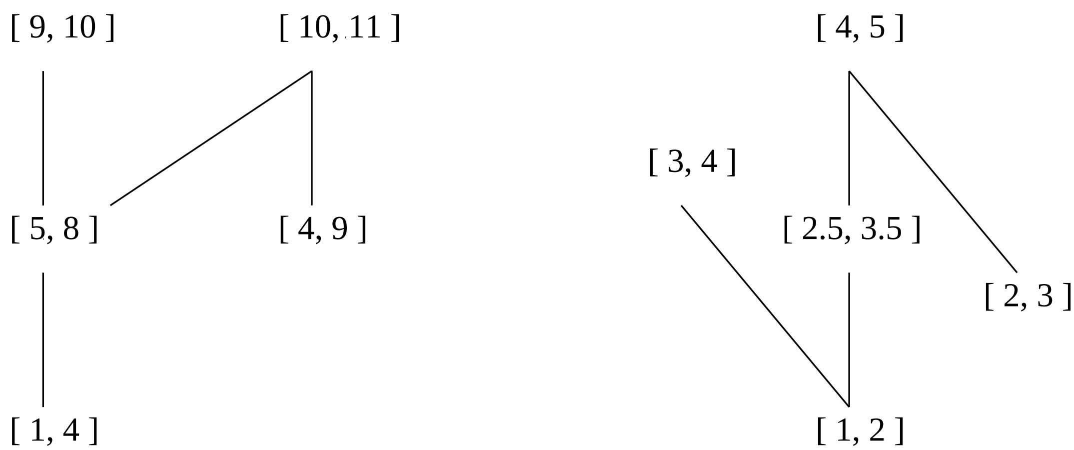


3 + 1



not (3 + 1)-free

Call a poset $\mathbf{(a+b)}$**-free** if it contains no induced subposet isomorphic to $\mathbf{a+b}$.

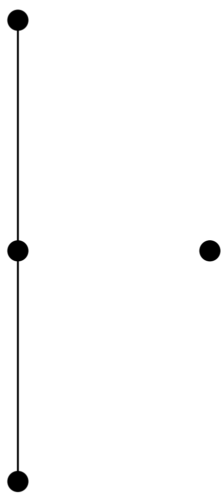


Interval orders are sets of closed intervals ordered by defining

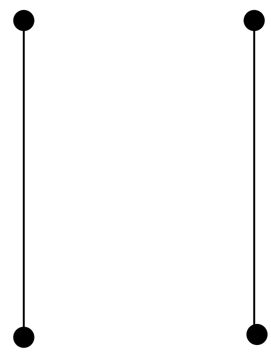
$$[a_i, b_i] < [a_j, b_j]$$

whenever $b_i < a_j$.

In a **unit interval order,** all intervals have the same length.



3 + 1



2 + 2

An **interval order** has no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$.

A **unit interval order** has no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$ or to $\mathbf{3}+\mathbf{1}$.

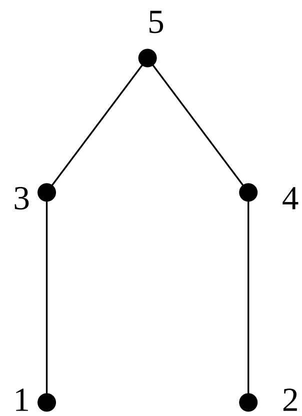
$(\mathbf{3}+\mathbf{1})$-free posets

1. Stanley's generalization of the chromatic polynomial $X_G(x)$ is conjectured to be **e-positive** for the incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets. (Stanley, Stembridge 1993)

2. $X_G(x)$ is known to be **s-positive** for the same graphs. (Gasharov 1996)

3. The **chain polynomial** of a $(\mathbf{3}+\mathbf{1})$-free poset **has only real zeros**. (Stanley 1995, Gasharov 1996)

How can we characterize posets free *only* of $\mathbf{3}+\mathbf{1}$?



Anti-adjacency matrices

Given any labelling of P, we define its **anti-adjacency matrix** $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} 0 & i <_P j \\ 1 & \text{otherwise.} \end{cases}$$

Example

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Theorem. *A poset P is $(\mathbf{3}+\mathbf{1})$-free if and only if there is a natural labelling of P such that its squared anti-adjacency matrix is a submatrix of the infinite Toeplitz matrix C,*

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & \cdots \\ 4 & 3 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix},$$

with row and column repetition allowed.

These are the non-negative integer matrices which **weakly increase** toward the southwest corner, in which each 2×2 submatrix

$$\begin{bmatrix} b_{ik} & b_{i\ell} \\ b_{jk} & b_{j\ell} \end{bmatrix}$$

satisfies

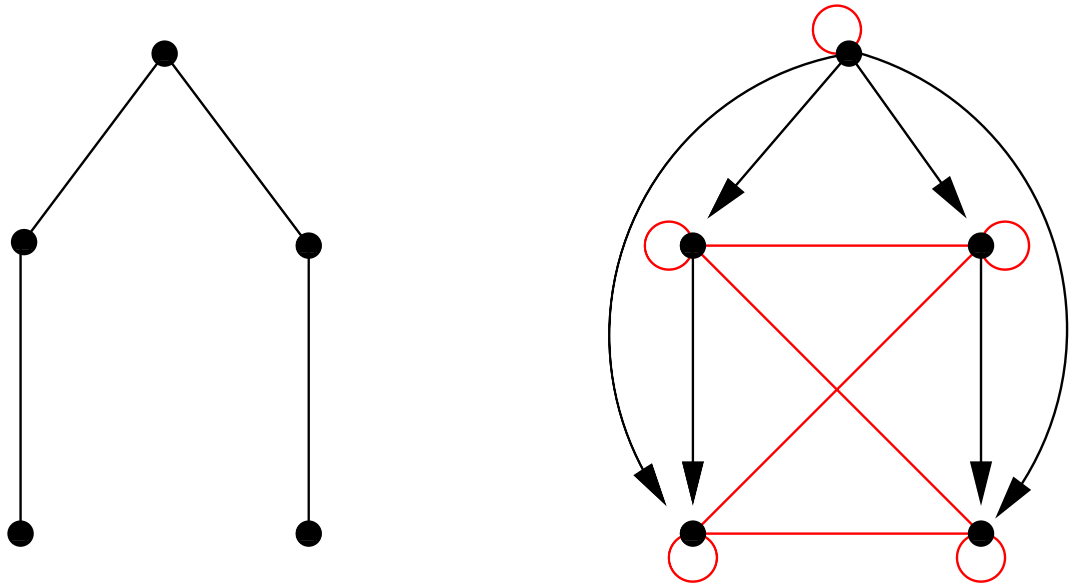
(i) $b_{ik} - b_{i\ell} = b_{jk} - b_{j\ell}$

or

(ii) $b_{i\ell} = 0$ and $b_{jk} - b_{j\ell} > b_{ik}$

Example

$$B = A^2 = \begin{bmatrix} 3 & 3 & 2 & 2 & 0 \\ 3 & 3 & 2 & 2 & 0 \\ 4 & 4 & 3 & 3 & 0 \\ 4 & 4 & 3 & 3 & 0 \\ 5 & 5 & 4 & 4 & 1 \end{bmatrix}.$$



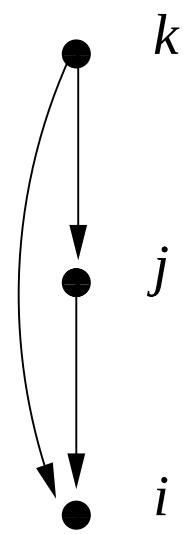
P G

The squared anti-adjacency matrix

Suppose A is the anti-adjacency matrix of a poset P. Let G be the graph whose **adjacency** matrix is A.

(i,j) is a **directed edge** if $i >_P j$, and an **undirected edge** if i and j are incomparable or identical.

The matrix $B = A^2$ counts paths of length 2 in G.



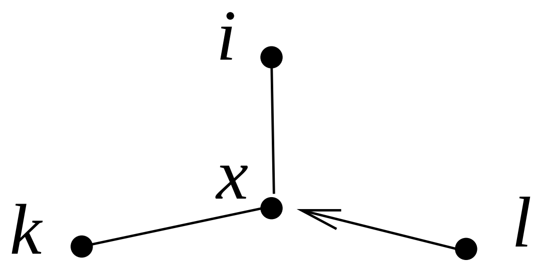
Let P be a $(\mathbf{3}+\mathbf{1})$-free poset.

Observation. If there is a chain of three elements $i <_P j <_P k$ in P, then $b_{ik} = 0$.

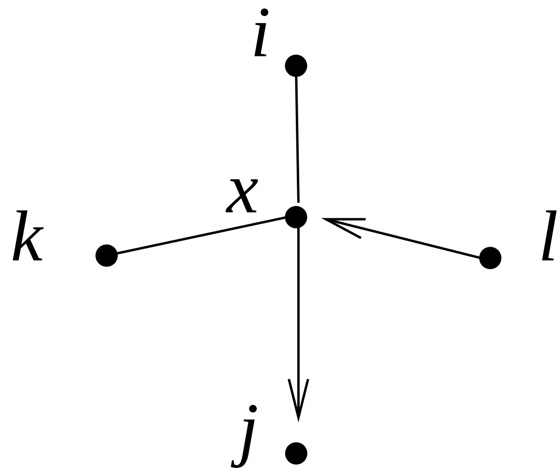
Proof. If $b_{ik} > 0$, then there is an element x, incomparable both to i and to k. $\square$

Observation. Let i, k, and ℓ be elements of P. If $b_{ik} > b_{i\ell}$, then there is an element $x <_P \ell$, such that $x \not<_P k$, and $x \not>_P i$.

Proof. (1) If $b_{ik} > b_{i\ell}$, then there are more paths of length two in G from i to k than from i to ℓ. There must be a vertex x such that (i, x, k) **is a path** in G and (i, x, ℓ) **is not**. In particular, the edge (ℓ, x) is directed. $\square$



Definition. Call such a vertex x a **(k,ℓ)-advantage** for i, imagining that it "helps" i get to k, but not to ℓ.



Advantage Lemma

Let P be a $(\mathbf{3}+\mathbf{1})$-free poset.

Lemma. If $b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell}$, then either i has a (k, ℓ)-advantage that j doesn't have, or j has an (ℓ, k)-advantage that i doesn't have. In particular, one of the following is true:

1. There is an element x such that $j <_P x <_P \ell$, and $b_{j\ell} = 0$.
2. There is an element y such that $i <_P y <_P k$, and $b_{ik} = 0$.

Proof.

$$\begin{aligned} b_{ik} - b_{i\ell} &= \#\{(k,\ell)\text{-advantages for } i\} \\ &\quad - \#\{(\ell,k)\text{-advantages for } i\}. \end{aligned}$$

Thus, if $b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell}$, then

$$\begin{aligned} &\#\{(k,\ell)\text{-adv. for } i\} + \#\{(\ell,k)\text{-adv. for } j\} \\ &\quad > \quad \#\{(k,\ell)\text{-adv. for } j\} + \#\{(\ell,k)\text{-adv. for } i\}. \end{aligned}$$

If an element x is a (k,ℓ)-advantage for i, and is not a (k,ℓ)-advantage for j, then $j <_P x$, implying that $j <_P x <_P \ell$. By our first observation, $b_{j\ell} = 0$.

□

Observation. Let B be **any** real matrix. The following two conditions on B are equivalent:

1. It is possible to simlutaneously permute the columns and rows of B so that it **weakly increases to the southwest,** i.e.

$$\begin{bmatrix} b_{i,j} & \geq & b_{i+1,j} \\ |\wedge & & |\wedge \\ b_{i+1,j} & \geq & b_{i+1,j+1} \end{bmatrix},$$

2. The **rows and columns** of B corresponding to any pair of indices i and j **satisfy** one of the following pairs of **vector inequalities**.

$$\mathrm{row}(i) \geq \mathrm{row}(j) \text{ and } \mathrm{column}(i) \leq \mathrm{column}(j)$$

$$\mathrm{row}(i) \leq \mathrm{row}(j) \text{ and } \mathrm{column}(i) \geq \mathrm{column}(j)$$

Proposition. If P is a $(\mathbf{3}+\mathbf{1})$-free poset then there is a natural labelling of P such that the entries of its **squared anti-adjacency matrix** B **weakly increase** toward the southwest.

Proof. Assume that B fails to increase toward the southwest.

Case 1: Some pair of columns (or rows) is **not comparable** as a pair of vectors.

Then there is a 2×2 submatrix

$$\begin{bmatrix} b_{ik} & > & b_{i\ell} \\ b_{jk} & < & b_{j\ell} \end{bmatrix},$$

and

$$b_{ik} - b_{i\ell} > 0 > b_{jk} - b_{j\ell}.$$

Applying the Advantage Lemma to this inequality, we have either $b_{j\ell} = 0$ or $b_{ik} = 0$, both contradictions.

Case 2: For some i, j,

$$\text{row}(i) \geq \text{row}(j) \text{ and column}(i) \geq \text{column}(j).$$

Then for some elements k, ℓ, of P we have the **incorrect pair** of inequalities

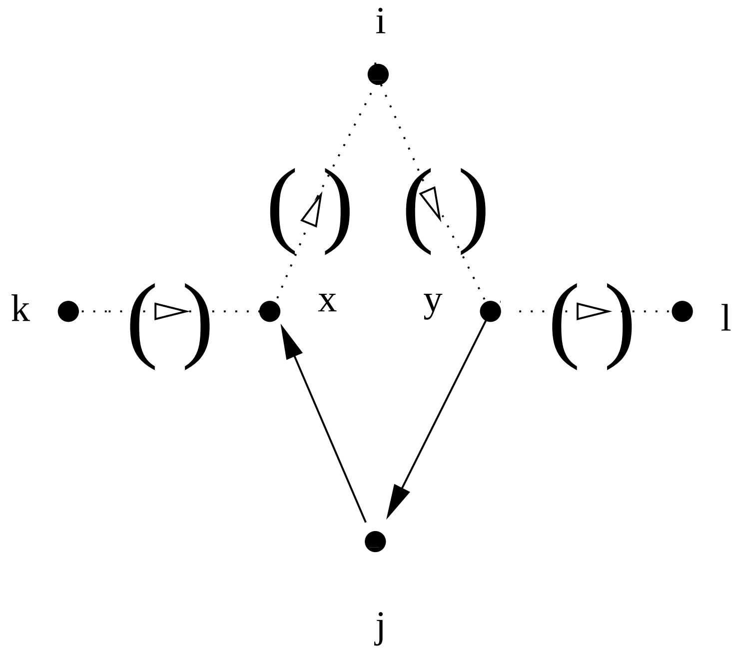
$$b_{ki} > b_{kj} \text{ and } b_{i\ell} > b_{j\ell}.$$

There must be an element $x \neq j$ such that

$$x <_P j,\ x \not<_P i, \text{ and } x \not>_P k,$$

and an element $y \neq j$ such that

$$j <_P y,\ y \not>_P i, \text{ and } y \not<_P \ell.$$



By the graph above, both x and y must be incomparable to i, contradicting our assumption that P is $(\mathbf{3}+\mathbf{1})$-free. $\square$

Proposition. Let P be a $(\mathbf{3}+\mathbf{1})$-free poset, naturally labelled so that its squared anti-adjacency matrix B weakly increases toward the southwest.

Then each 2×2 submatrix

$$\begin{bmatrix} b_{ik} & b_{i\ell} \\ b_{jk} & b_{j\ell} \end{bmatrix}$$

satisfies one of the following two conditions:

1. $b_{ik} - b_{i\ell} = b_{jk} - b_{j\ell}$
2. $b_{i\ell} = 0$ and $b_{jk} - b_{j\ell} > b_{ik}$

$$\begin{bmatrix} b_{ik} & b_{i\ell} \\ b_{jk} & b_{j\ell} \end{bmatrix}$$

Proof. Suppose that condition (1) is not satisfied, and apply the Advantage Lemma.

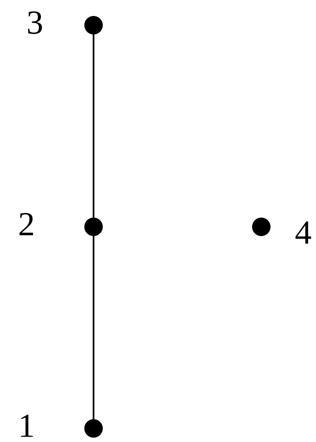
Case 1: $(b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell})$.
If $b_{j\ell} = 0$, then $b_{i\ell} = 0$ and $b_{ik} > b_{jk}$. $\Rightarrow\Leftarrow$.
If $b_{ik} = 0$, then $b_{i\ell} = 0$ and $b_{j\ell} > b_{jk}$. $\Rightarrow\Leftarrow$.

Case 2: $(b_{ik} - b_{i\ell} < b_{jk} - b_{j\ell})$.
If $b_{jk} = 0$, then $b_{ik} = b_{i\ell} = b_{j\ell} = 0$. $\Rightarrow\Leftarrow$.
Conclude: $b_{i\ell} = 0$, and $b_{jk} - b_{j\ell} > b_{ik}$. □

Converse

Proposition. If P is naturally lablelled poset **containing** $\mathbf{3}+\mathbf{1}$ as an induced subposet, then its squared anti-adjacency matrix is **not** a submatrix of the infinite Toeplitz matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & \cdots \\ 4 & 3 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$



Proof. Let $Q = \mathbf{3} + \mathbf{1}$ and label Q so that

$$B = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 3 & 2 & 1 & 3 \\ 4 & 3 & 2 & 4 \\ 4 & 3 & 2 & 4 \end{bmatrix}.$$

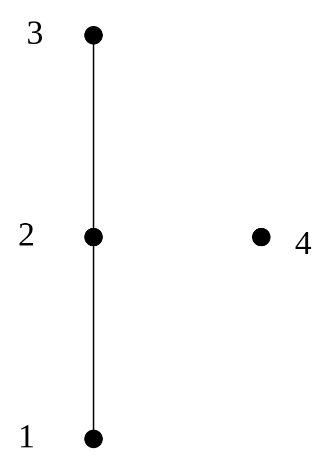
Then, $b_{11} - b_{13} < b_{31} - b_{33}$.

To increase $b_{11} - b_{13}$ without increasing $b_{31} - b_{33}$, we need a new element x to be a $(1,3)$-advantage for 1 and **not** a $(1,3)$-advantage for 3. This is clearly impossible. Similarly, we cannot decrease the second difference without decreasing the first. □

Definition. We define the **chain polynomial** of a finite poset P by

$$f_P(x) = \sum_{i=0}^{r} c_i x^i,$$

where c_i is the number of i-element chains in P, and r is the maximum cardinality of a chain in P. We define $c_0 = 1$.



$$f_P(x) = 1 + 4x + 3x^2 + x^3.$$

Formula for the chain polynomial

If A is the anti-adjacency matrix of P then the chain polynomial is given by

$$f_P(x) = \det(I + xA).$$

(Stanley, 1996). From this formula we see that $f_P(x)$ has only **real zeros** if and only if A has only **real eigenvalues**.

Corollary. Let P be a $(\mathbf{3}+\mathbf{1})$-free poset. Then the chain polynomial $f_P(x)$ has only real zeros.

Proof. Since $B = A^2$ is a submatrix of a totally positive matrix, it is totally positive, and therefore has only **nonnegative real** eigenvalues.

Thus, A has only **real** eigenvalues, and

$$f_P(x) = \det(I + xA)$$

has only real zeros. $\square$

Conjecture: (Stanley-Neggers) Let $J(Q)$ be a finite **distributive lattice**. Then the chain polynomial $f_{J(Q)}(x)$ has only real zeros.

By Simion (1984), the conjecture holds for the special case of **products of chains**.

By Wagner (1990), if the conjecture holds for **two distributive lattices**, it holds for **their product**.

The question of whether the conjecture holds for the larger class of **modular lattices** is open as well.