# A CHARACTERIZATION OF $(\mathbf{3}+\mathbf{1})$-FREE POSETS

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Abstract. Posets containing no subposet isomorphic to the disjoint sums of chains $\mathbf{3}+\mathbf{1}$ and/or $\mathbf{2}+\mathbf{2}$ are known to have many special properties [7], [10], [15], [17]. However, while posets free of $\mathbf{2}+\mathbf{2}$ and posets free of both $\mathbf{2}+\mathbf{2}$ and $\mathbf{3}+\mathbf{1}$ may be characterized as interval orders, no such characterization is known for posets free of only $\mathbf{3}+\mathbf{1}$. We give here a characterization of $(\mathbf{3}+\mathbf{1})$-free posets in terms of their antiadjacency matrices. Using results about totally positive matrices, we show that this characterization leads to a simple proof that the chain polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros.

Résumé. Les ensembles partiellement ordonnés qui ne contiennent pas un sous-ensemble partiellement ordonné isomorphique à $\mathbf{3}+\mathbf{1}$ et/ou $\mathbf{2}+\mathbf{2}$ possèdent propriétés interessantes [7], [10], [15], [17]. Cependant, alors que les ensembles sans $\mathbf{2}+\mathbf{2}$, et les ensembles sans $\mathbf{2}+\mathbf{2}$ et $\mathbf{3}+\mathbf{1}$ possèdent une caracterisation par ordres d'intervalle, aucune caracterisation analogue pour les ensembles seulement sans $\mathbf{3}+\mathbf{1}$ n'est connue. Nous présentons une caracterisation basée sur leurs matrices antiadjacentes. En utilisant les resultats sur les matrices totallement positives, nous montrons que cette caracterisation produit une preuve simple que le polynôme des chaînes d'un ensemble partiellement ordonné sans $\mathbf{3}+\mathbf{1}$ ne possède que des zéros réelles.

## 1. Introduction

For nonnegative integers $a$ and $b$, we denote by $\mathbf{a}+\mathbf{b}$ the poset which is the disjoint sum of an $a$-element chain and a $b$-element chain. A poset is called $(\mathbf{a}+\mathbf{b})$-*free* if it contains no induced subposet isomorphic to $\mathbf{a}+\mathbf{b}$. (See [13, ch. 3] for basic definitions.) For example, the first two posets in Figure 1.1 are $\mathbf{2}+\mathbf{2}$ and $\mathbf{3}+\mathbf{1}$. The third poset $P$ is $(\mathbf{2}+\mathbf{2})$-free but not $(\mathbf{3}+\mathbf{1})$-free, because the subposet induced by the elements $\{2,3,4,6\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$. For any poset $P$, we will denote the order relation in $P$ by the symbol $<_P$, reserving $<$ for comparisons of numbers and vectors.

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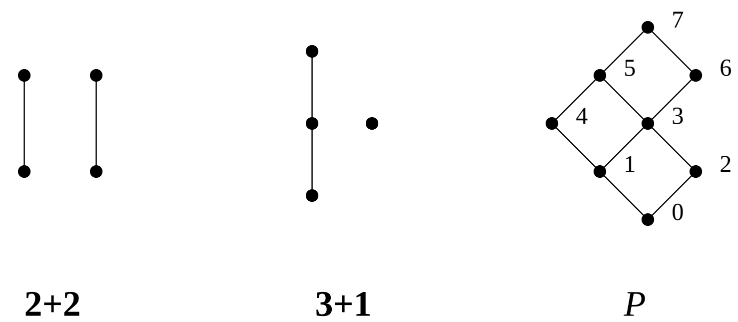


FIGURE 1.1

Fishburn [6] characterized $(\mathbf{2}+\mathbf{2})$-free posets by showing that any such poset $P$ may be represented as a set of closed intervals of real numbers $[c_i, d_i]$, ordered by

$$[c_i, d_i] <_P [c_j, d_j], \text{ if } d_i < c_j.$$

He also showed that for posets free of both $\mathbf{3}+\mathbf{1}$ and $\mathbf{2}+\mathbf{2}$, the intervals may be chosen to have unit length. In honor of these results, posets in the above classes are often called *interval orders* and *unit interval orders.*

A second well known characterization of $(\mathbf{2}+\mathbf{2})$-free, $(\mathbf{3}+\mathbf{1})$-free posets involves natural labellings and antiadjacency matrices. Let $P$ be a poset with $n$ elements. Any bijective function $\phi : P \rightarrow \{1, \ldots, n\}$ is called a *labelling* of $P$, and is called *natural* if it satisfies $\phi(x) < \phi(y)$ whenever $x <_P y$. If the elements $x$ and $y$ are labelled as $i$ and $j$, we will often write $i <_P j$ to mean $x <_P y$. For a fixed labelling of $P$, we define an *antiadjacency matrix* $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} 0, & \text{if } i <_P j \\ 1, & \text{otherwise.} \end{cases}$$

**Theorem 1.1.** *Let $P$ be a poset on $n$ elements. $P$ is $(\mathbf{2}+\mathbf{2})$-free and $(\mathbf{3}+\mathbf{1})$-free if and only if it may be naturally labelled so that the corresponding antiadjacency matrix $A$ satisfies $a_{jk} \leq a_{i\ell}$, for all integers $1 \leq i \leq j \leq n$ and $1 \leq k \leq \ell \leq n$.*

That is, the positions $(i, j)$ such that $a_{ij} = 0$ form a Ferrers shape in the upper right corner of $A$.

Unfortunately, little is known about posets free *only* of $\mathbf{3}+\mathbf{1}$. Characterization of $(\mathbf{3}+\mathbf{1})$-free posets is desirable, because several results and conjectures about posets require avoidance only of $\mathbf{3}+\mathbf{1}$. For instance, Stanley's generalization of the chromatic polynomial [14] is known to be $s$-positive for the incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets [10], and is conjectured to be $e$-positive for these graphs as well [14, 17, 19]. Further, the chain polynomial of a $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros. (See [17] and Corollary 4.1). This implies that the $f$-vector of a $(\mathbf{3}+\mathbf{1})$-free poset is log-concave and unimodal.

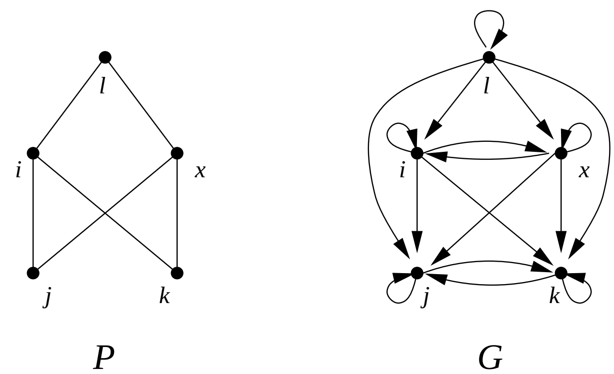


FIGURE 2.1

We will characterize $(\mathbf{3}+\mathbf{1})$-free posets with a result analogous to Theorem 1.1. Observing serveral properties of squared antiadjacency matrices in Section 2, we we use these properties to prove the main theorem in Section 3. In Section 4 we derive a result of Stanley [17, Cor. 2.9] as a corollary of the main theorem and discuss some open questions regarding the chain polynomials of posets.

## 2. The squared antiadjacency matrix

Let $P$ be a finite poset with elements labelled $\{1, \ldots, n\}$, and let $A$ be the corresponding antiadjacency matrix. The *squared antiadjacency matrix* $B = A^2$ has a simple combinatorial interpretation. Let $G = (P, E)$ be the graph whose *adjacency* matrix is $A$. The vertex set of $G$ consists of the elements $\{1, \ldots, n\}$ of $P$, and the edge set consists of the ordered pairs $(i, j) \in P \times P$ such that $i \not<_P j$. (See Figure 2.1.) Clearly, $B = [b_{ij}]$ counts paths of length two in $G$. That is, $b_{ij}$ is the number of ordered triples $(i, x, j)$, where $(i, x)$ and $(x, j)$ belong to $E$.

**Observation 2.1.** *Assume that $P$ is $(\mathbf{3}+\mathbf{1})$-free, and let $i$, $j$, $k$, and $\ell$ be elements of $P$.*

1. *If $i <_P j <_P k$, then $b_{ik} = 0$.*
2. *If $b_{ik} > b_{i\ell}$, then there is an element $x <_P \ell$, such that $i \not<_P x \not<_P k$*
3. *If $b_{ik} > b_{jk}$, then there is an element $y >_P j$, such that $k \not>_P y \not>_P i$.*

*Proof.* (1) Assume $b_{ik} > 0$. Then, for some element $x$ of $P$, $(i, x)$ and $(x, k)$ belong to $E$, implying that $i \not<_P x \not<_P k$. In fact, $x$ must be incomparable to $i$ and $k$, for if $x <_P i$, then $x <_P i <_P j <_P k$, and if $k <_P x$, then $i <_P j <_P k <_P x$, both impossible. Similarly, $x$ cannot be comparable to $j$. Thus, the subposet of $P$ induced by $\{i, j, k, x\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$, contradicting our assumption that $P$ is $(\mathbf{3}+\mathbf{1})$-free.

(2) If $b_{ik} > b_{i\ell}$, then there are more paths of length two in $G$ from $i$ to $k$ than from $i$ to $\ell$. It follows that for some element $x$ of $P$, the pairs $(i, x)$ and $(x, k)$ belong to $E$, and the pair $(x, \ell)$ does not.

(3) Apply the argument of (2) to the dual poset of $P$. □

Elements such as $x$ in Observation 2.1 (2) are central to the proof of Lemma 2.2. To simplify notation, we introduce the following definition.

**Definition 2.1.** Let $i, k, \ell$, and $x$ be elements of $P$. Call $x$ a $(k, \ell)$-*advantage for* $i$ if $(i, x)$ and $(x, k)$ are edges in the graph $G$ and $(x, \ell)$ is not.

We use the word *advantage*, imagining that $x$ helps us to travel from $i$ to $k$, but not from $i$ to $\ell$. In the language of partially ordered sets, $x$ is a $(k, \ell)$-advantage for $i$ if $x <_P \ell$, and $i \not<_P x \not<_P k$. Note that in Figure 2.1, the vertex $x$ is a $(k, \ell)$-advantage for $i$, although it is not a $(k, \ell)$-advantage for $j$.

**Lemma 2.2.** *Assume that $P$ is $(\mathbf{3}+\mathbf{1})$-free, and let $i, j, k$, and $\ell$ be elements of $P$. If $b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell}$, then one of the following is true:*

1. *$P$ contains an element $x$ such that $j <_P x <_P \ell$ and $b_{j\ell} = 0$.*
2. *$P$ contains an element $y$ such that $i <_P y <_P k$ and $b_{ik} = 0$.*

*Proof.* Let us denote by $\alpha(k, \ell, i)$ the number of elements of $P$ which are $(k, \ell)$-advantages for $i$. Then,

$$b_{ik} - b_{i\ell} = \alpha(k, \ell, i) - \alpha(\ell, k, i).$$

Assuming that $b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell}$, we have

$$\alpha(k, \ell, i) + \alpha(\ell, k, j) > \alpha(k, \ell, j) + \alpha(\ell, k, i),$$

and at least one of the following two inequalities must be true.

$$\alpha(k, \ell, i) > \alpha(k, \ell, j),$$
$$\alpha(\ell, k, j) > \alpha(\ell, k, i).$$

Suppose that $\alpha(k, \ell, i) > \alpha(k, \ell, j)$. Then $P$ contains an element $x$ which is a $(k, \ell)$-advantage for $i$ and not a $(k, \ell)$-advantage for $j$. By Definition 2.1, the pairs $(i, x)$ and $(x, k)$ belong to $E$ and the pairs $(x, \ell)$ and $(j, x)$ do not. Thus, $j <_P x <_P \ell$ and by Observation 2.1 (1), $b_{j\ell} = 0$. Similarly, if $\alpha(\ell, k, j) > \alpha(\ell, k, i)$, then $P$ contains an element $y$ such that the pairs $(y, k)$ and $(i, y)$ do not belong to $E$. Thus, $i <_P y <_P k$ and $b_{ik} = 0$.

□

## 3. Main Result

**Theorem 3.1.** *Let $P$ be a poset on $n$ elements. $P$ is $(\mathbf{3}+\mathbf{1})$-free if and only if it may be naturally labelled so that the squared antiadjacency matrix $B$ satisfies the following two conditions for all integers $1 \leq i \leq j \leq n$ and $1 \leq k \leq \ell \leq n$.*

1. $b_{jk} \geq b_{i\ell}$
2. *If $b_{ik} - b_{i\ell} \neq b_{jk} - b_{j\ell}$, then $b_{i\ell} = 0$ and $b_{ik} < b_{jk} - b_{j\ell}$.*

**Example 3.1.** Corresponding to any natural labelling of the poset $P$ in Figure 2.1 is the squared antiadjacency matrix

$$B = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 4 & 4 & 2 & 2 & 0 \\ 4 & 4 & 2 & 2 & 0 \\ 5 & 5 & 3 & 3 & 1 \end{bmatrix}.$$

Condition (1) of the theorem says that entries of the squared antiadjacency matrix increase to the left in rows and downward in columns. To prove that a $(\mathbf{3}+\mathbf{1})$-free poset $P$ may be labelled to satisfy this condition, we will begin with an arbitrary labelling of $P$ and the squared antiadjacency matrix $B$. Note that relabelling $P$ corresponds to simultaneous row and column permutation of $B$. Let us denote the $i$th row and $i$th column of any matrix $M$ by $M_{i*}$ and $M_{*i}$.

**Observation 3.2.** *The following two conditions on any real matrix $M$ are equivalent.*

1. *It is possible to simlutaneously permute the columns and rows of $M$ so that its entries weakly increase to the left in rows and downward in columns.*
2. *The rows and columns of $M$ corresponding to any pair of indices $i$ and $j$ satisfy one of the following pairs of vector inequalities.*
   (a) $M_{i*} \geq M_{j*}$ *and* $M_{*i} \leq M_{*j}$.
   (b) $M_{i*} \leq M_{j*}$ *and* $M_{*i} \geq M_{*j}$.

The first statement simply says that we may sort the columns of $M$ in weakly decreasing order while simultaneously sorting the rows in weakly increasing order. With a moment's thought, we see that this is possible if and only if the conditions in the second statement are true.

**Proposition 3.3.** *Any $(\mathbf{3}+\mathbf{1})$-free poset may be naturally labelled so that the entries of its squared antiadjacency matrix weakly increase to the left in rows and downward in columns.*

*Proof.* Let $P$ be a $(\mathbf{3}+\mathbf{1})$-free poset and assume that for each labelling of $P$, the corresponding antiadjacency matrix fails to satisfy condition (2) of Observation 3.2.

(Trivially, condition (2) fails to hold for each non-natural labelling of $P$.) We consider two cases for a fixed labelling of $P$ and the corresponding squared antiadjacency matrix $B$.

**Case 1:** Two columns of $B$ are incomparable as vectors. That is,

$$B_{*i} \not\geq B_{*j} \text{ and } B_{*i} \not\leq B_{*j},$$

for some indices $i \neq j$. Then for some indices $k \neq \ell$ we have $b_{ik} > b_{i\ell}$ and $b_{jk} < b_{j\ell}$, implying that

$$b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell}.$$

Applying Lemma 2.2 to this inequality, we have $b_{j\ell} = 0$ or $b_{ik} = 0$, both contradictions. The argument for incomparable rows is identical.

**Case 2:** All columns of $B$ are pairwise comparable as vectors, as are all rows, but for some indices $i \neq j$, we have an incorrect pair of comparisons of the form

$$B_{i*} \gneqq B_{j*} \text{ and } B_{*i} \gneqq B_{*j}.$$

That is, there are elements $k$ and $\ell$ in $P$, not necessarily distinct, satisfying $b_{i\ell} > b_{j\ell}$ and $b_{ki} > b_{kj}$.

By Observation 2.1 (3), $P$ contains an element $x <_P j$, such that $k \not<_P x \not<_P i$. By Observation 2.1 (2), $P$ contains an element $y >_P j$, such that $i \not<_P y \not<_P \ell$. Thus, $x <_P j <_P y$ is a chain, and each of these three elements is incomparable to $i$. This contradicts our assumption that $P$ is $(\mathbf{3}+\mathbf{1})$-free. □

It is not hard to show that any labelling of a $(\mathbf{3}+\mathbf{1})$-free poset which satisfies the first condition of Theorem 3.1 also satisfies the second condition.

**Proposition 3.4.** *Let $P$ be a $(\mathbf{3}+\mathbf{1})$-free poset, naturally labelled so that its squared antiadjacency matrix $B$ weakly increases to the left in rows and downward in columns. Let $i, j, k$, and $\ell$ be numbers satisfying $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. Then the $2 \times 2$ submatrix*

$$\begin{bmatrix} b_{ik} & b_{i\ell} \\ b_{jk} & b_{j\ell} \end{bmatrix}$$

*satisfies one of the following two conditions:*

1. $b_{ik} - b_{i\ell} = b_{jk} - b_{j\ell}$.
2. $b_{i\ell} = 0$ *and* $b_{ik} < b_{jk} - b_{j\ell}$.

*Proof.* Suppose that condition (1) is not satisfied.

**Case 1:** $(b_{ik} - b_{i\ell} > b_{jk} - b_{j\ell})$. We apply Lemma 2.2 to this inequality. If $b_{j\ell} = 0$, then $b_{i\ell} = 0$ and $b_{ik} > b_{jk}$, contraditing our assumptions about weakly increasing entries of $B$. If instead $b_{ik} = 0$, then $b_{i\ell} = 0$ and $b_{j\ell} > b_{jk}$, another contradiction.

**Case 2:** $(b_{ik} - b_{i\ell} < b_{jk} - b_{j\ell})$. Again we apply Lemma 2.2. If $b_{jk} = 0$, then all four numbers are zero, a contradiction. We conclude that $b_{i\ell} = 0$ and that condition (2) is satisfied. □

Finally, we show that the only posets satisfying the conditions of Theorem 3.1 are those which are $(\mathbf{3}+\mathbf{1})$-free.

**Proposition 3.5.** *Let $P$ be a labelled poset containing $\mathbf{3}+\mathbf{1}$ as an induced subposet, and let $B$ be its squared antiadjacency matrix. Then there are two distinct elements $i$ and $k$ such that*

$$b_{ik} \neq 0 \text{ and } b_{ii} - b_{ik} \neq b_{ki} - b_{kk}.$$

*Proof.* Let $1, 2, 3$, and 4 be four elements of $P$ such that $1 <_P 2 <_P 3$ is a chain, and 4 is incomparable to $1, 2$, and 3. Let $G = (P, E)$ be the graph defined in Section 2.

Clearly, $b_{13} \neq 0$, since $(1, 4)$ and $(4, 3)$ are edges in $G$. We claim that

$$b_{11} - b_{13} \neq b_{31} - b_{33}.$$

Define the sets

$$X = \{x \in P | (1, x) \in E, (x, 1) \in E, (x, 3) \notin E\},$$
$$Y = \{x \in P | (x, 1) \in E, (x, 3) \notin E\},$$

and note that

$$|X| = b_{11} - b_{13},$$
$$|Y| = b_{31} - b_{33}.$$

Certainly $X$ is a subset of $Y$. Moreover, it is a proper subset, since the element 2 belongs to $Y$ and not to $X$. Thus, $b_{11} - b_{13} < b_{31} - b_{33}$.

□

Having completed the proof of Theorem 3.1, we now reconsider the theorem in terms of totally positive matrices. A real matrix, finite or infinite, is called *totally positive* (or sometimes *totally nonnegative*) if each $k \times k$ minor is nonnegative. Totally positive matrices have many interesting properties [1] [2] and arise frequently in combinatorics. (See [5], [8], [9], [18], [19], [21].)

One important property of a finite square totally positive matrix is that all of its eigenvalues are nonnegative and real. (See [2, Thm 1.1]). It is well known that the antiadjacency matrix of any $(\mathbf{2}+\mathbf{2})$-free, $(\mathbf{3}+\mathbf{1})$-free poset is totally positive, provided the poset is labelled as in Theorem 1.1. An infinite matrix which is well known to be totally positive is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 2 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 3 & 2 & 1 & 0 & \cdot & \cdot & \cdot \\ 4 & 3 & 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \end{bmatrix}.$$

It is easy to see that the matrices satisfying the conditions of Theorem 3.1 are essentially submatrices of $C$: each is determined by a finite multiset of columns and a finite multiset of rows. (See Example 3.1.) Thus, the squared antiadjacency matrix $B = A^2$ of any $(\mathbf{3}+\mathbf{1})$-free poset is totally positive.

## 4. Chain Polynomials and Open Questions

Many open problems in algebraic combinatorics concern the *$f$-vectors* of posets and simplicial complexes. (See for example [16, Ch. 2-3].) We define the $f$-vector of a finite poset $P$ to be the integer sequence

$$f_P = (f_0, \dots, f_{d-1}),$$

where $f_{i-1}$ is the number of $i$-element chains in $P$, and $d$ is the maximum cardinality of a chain in $P$. While it would be desirable to characterize the $f$-vectors corresponding to particular classes of posets, few results of this type are known. Typical results relate the integers $f_0, \dots, f_{d-1}$ by linear and quadratic inequalities. (See [3].) The $f$-vector is called *unimodal* if

$$f_0 \leq \cdots \leq f_j \geq \cdots f_{d-1},$$

for some index $j$, and *log concave* if

$$f_i^2 \geq f_{i-1} f_{i+1},$$

for all $i = 1, \dots, d-2$. To prove that $f_P$ is unimodal and log concave, it suffices to show that the related *chain polynomial*,

$$f_P(x) = 1 + \sum_{i=1}^{d} f_{i-1} x^i,$$

has only real zeros. Further, the following identity relates the chain polynomial $f_P(x)$ to the antiadjacency matrix $A$ [15].

$$f_P(x) = \det(I + xA). \tag{4.1}$$

Thus, $f_P(x)$ has only real zeros if and only if $A$ has only real eigenvalues.

By Theorem 1.1, the chain polynomial of a $(\mathbf{2}+\mathbf{2})$-free, $(\mathbf{3}+\mathbf{1})$-free poset has only real zeros. (See discussion following Proposition 3.5.) Similarly, by Theorem 3.1, the same holds for $(\mathbf{3}+\mathbf{1})$-free posets. This result was originally proved in [17, Cor. 2.9], using facts about symmetric functions [10] [17, Thm. 2.8].

**Corollary 4.1.** *Let $P$ be a $(\mathbf{3}+\mathbf{1})$-free poset. Then the chain polynomial $f_P(x)$ has only real zeros.*

*Proof.* Label $P$ as in Theorem 3.1, and let $A$ be the corresponding antiadjacency matrix. By the discussion following Proposition 3.5, the matrix $B = A^2$ is totally positive and therefore has only nonnegative real eigenvalues. It follows that $A$ has only real eigenvalues, and that $f_P(x) = \det(I + xA)$ has only real zeros. □

The converse of Corollary 4.1 is not true, for there are many posets containing $\mathbf{3}+\mathbf{1}$ as an induced subposet, whose chain polynomials have only real zeros. An important open problem is to determine which posets have this property. In particular, we have the following conjecture due to Stanley [18] and Neggers [11].

**Conjecture 4.2.** *Let $J(Q)$ be a finite distributive lattice. Then the chain polynomial $f_{J(Q)}(x)$ has only real zeros.*

Various proofs show that the conjecture holds for the special cases in which $Q$ is a disjoint sum of chains [12], a Ferrers poset [4], and a series-parallel poset [22]. In addition, Stembridge has verified the conjecture for all posets $Q$ having eight or fewer elements [20]. A more general open problem is to determine whether the conjecture holds for the larger class of *modular* lattices. No counterexamples are known. It would be interesting to apply the identity (4.1) to either open question or to the special cases.

Another open problem is to count the $(\mathbf{3}+\mathbf{1})$-free posets of cardinality $n$. By the comment following Theorem 1.1, the Catalan numbers count $(\mathbf{2}+\mathbf{2})$-free, $(\mathbf{3}+\mathbf{1})$-free posets. It would be interesting to apply the discussion following Proposition 3.5 to obtain a simple formula for $(\mathbf{3}+\mathbf{1})$-free posets.

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