COMBINATORIAL EVALUATION OF HYPEROCTAHEDRAL GROUP CHARACTERS

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Outline

(1) Symmetric and hyperoctahedral groups
(2) Bipartitions
(3) Irreducible hyperoctahedral group characters
(4) The Kazhdan-Lusztig basis
(5) Decorated posets
(6) Combinatorial evaluation of irreducible characters

Irreducible character evaluation

Let χ^λ be an irreducible $\mathfrak{S}_n$-character.

For $w \in \mathfrak{S}_n$ we compute $\chi^\lambda(w) \in \mathbb{Z}$ by an alternating sum.
No simple formula $\chi^\lambda(w) = (-1)^{|S(\lambda,w)|}|T(\lambda, w)|$ is known.

Let $\{\widetilde{C}_w \,|\, w \in \mathfrak{S}_n\}$ be the Kazhdan-Lusztig basis of $\mathbb{C}[\mathfrak{S}_n]$ and suppose that w avoids the patterns 3412, 4231.

To compute $\chi^\lambda(\widetilde{C}_w) \in \mathbb{N}$ we

(1) draw a poset $P(w)$,
(2) place elements of $P(w)$ into a Young diagram of shape λ,
(3) count valid placements.

Goal: Do the same for the Kazhdan-Lusztig basis of $\mathbb{C}[\mathfrak{B}_n]$.

The Symmetric group $\mathfrak{S}_{[\overline{n},n]} = \mathfrak{S}_{\{\overline{n},\ldots,\overline{1},1,\ldots,n\}}$

Generated by $s_{\overline{n-1}}, \ldots, s_{\overline{1}}, s_0, s_1, \ldots, s_{n-1}$ subject to

$$\begin{aligned} s_i^2 &= e && \text{for all } i, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j| = 1, \\ s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2. \end{aligned}$$

Define action on words $a_{\overline{n}} \cdots a_{\overline{1}} a_1 \cdots a_n$ by

$$\begin{array}{lll} s_i & \text{swaps letters in positions } i, i+1, & i \geq 1, \\ s_{\overline{i}} & \text{swaps letters in positions } \overline{i}, \overline{i+1}, & i \geq 1, \\ s_0 & \text{swaps letters in positions } \overline{1}, 1. & \end{array}$$

One-line notation of w = action of w on $\overline{n} \cdots \overline{1} 1 \cdots n$.

Ex: $s_2 s_1 s_0(\overline{3}\overline{2}\overline{1}123) = s_2 s_1(\overline{3}\overline{2}1\overline{1}23) = s_2(\overline{3}\overline{2}12\overline{1}3) = \overline{3}\overline{2}123\overline{1}$.

The Hyperoctahedral group $\mathfrak{B}_n$

Subgroup of $\mathfrak{S}_{[\overline{n},n]}$ generated by $s'_0 := s_0$, $s'_i := s_i s_{\overline{i}}$.

Relations are

$$\begin{aligned} {s'_i}^2 &= e && \text{for } i = 0, \ldots, n-1, \\ s'_i s'_j s'_i &= s'_j s'_i s'_j && \text{for } |i-j| = 1, \quad i, j \geq 1, \\ s'_0 s'_1 s'_0 s'_1 &= s'_1 s'_0 s'_1 s'_0 && \\ s'_i s'_j &= s'_j s'_i && \text{for } |i-j| \geq 2. \end{aligned}$$

Inherited action on words $a_{\overline{n}} \cdots a_{\overline{1}} a_1 \cdots a_n$ is

s'_i swaps letters in positions $i, i+1$ and $\overline{i}, \overline{i+1}$,

s'_0 swaps letters in positions $\overline{1}, 1$.

Ex: $s'_2 s'_1 s'_0(\overline{3}\overline{2}\overline{1}123) = s'_2 s'_1(\overline{3}\overline{2}1\overline{1}23) = s'_2(\overline{3}1\overline{2}2\overline{1}3) = 1\overline{3}\overline{2}23\overline{1}$.

Short one-line notation = right-half of one-line notation.

The Symmetric group $\mathfrak{S}_n$

Subgroup of $\mathfrak{B}_n$ generated by $s'_1, \ldots, s'_{n-1}$. Relations are

$$\begin{aligned} {s'_i}^2 &= e && \text{for } i = 1, \ldots, n-1, \\ s'_i s'_j s'_i &= s'_j s'_i s'_j && \text{for } |i-j| = 1, \\ s'_i s'_j &= s'_j s'_i && \text{for } |i-j| \geq 2. \end{aligned}$$

Given a representation $\mathfrak{S}_n \to \mathrm{GL}_k(\mathbb{C})$ with character θ, extend this to a representation $\mathfrak{B}_n \to \mathrm{GL}_k(\mathbb{C})$ by mapping

$$s'_0 \mapsto I \quad \text{or} \quad s'_0 \mapsto -I.$$

Call the $\mathfrak{B}_n$-characters 1θ, $\delta\theta$, respectively.

Bipartitions

$\mathfrak{B}_n$ has one irreducible character for each *bipartition* (λ, μ),

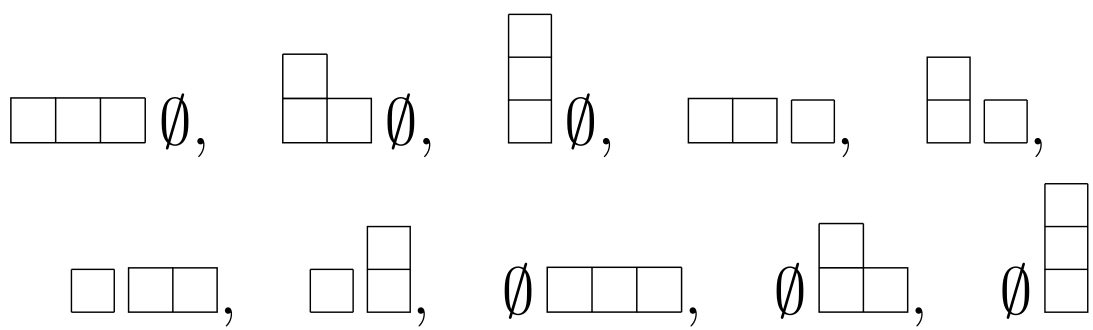
$$\lambda = (\lambda_1, \ldots, \lambda_r) \vdash k, \qquad \mu = (\mu_1, \ldots, \mu_s) \vdash n - k,$$

two weakly decreasing $\mathbb{Z}_{>0}$-sequences summing to n.
Write $(\lambda, \mu) \vdash n$.

Ex: Bipartitions of 3 are

$$(3, \emptyset), \quad (21, \emptyset), \quad (111, \emptyset), \quad (2, 1), \quad (11, 1),$$
$$(1, 2), \quad (1, 11), \quad (\emptyset, 3), \quad (\emptyset, 21), \quad (\emptyset, 111).$$

Corresponding Young diagrams have λ_i, μ_i boxes in row i,



Irreducible $\mathfrak{B}_n$-characters

From irreducible characters χ^λ of $\mathfrak{S}_k$ and χ^μ of $\mathfrak{S}_{n-k}$, create character $1\chi^\lambda \otimes \delta\chi^\mu$ of $\mathfrak{B}_{[1,k]} \times \mathfrak{B}_{[k+1,n]}$.

Induce to obtain $(\chi\chi)^{\lambda,\mu} := (1\chi^\lambda \otimes \delta\chi^\mu)\uparrow_{\mathfrak{B}_{[1,k]}\times\mathfrak{B}_{[k+1,n]}}^{\mathfrak{B}_n}$.

Irreducible characters of $\mathfrak{B}_n$ are $\{(\chi\chi)^{\lambda,\mu} \mid (\lambda,\mu) \vdash n\}$.

Ex: Irreducible characters of $\mathfrak{B}_3$ are

$$(\chi\chi)^{3,\emptyset}, \quad (\chi\chi)^{21,\emptyset}, \quad (\chi\chi)^{111,\emptyset}, \quad (\chi\chi)^{2,1}, \quad (\chi\chi)^{11,1},$$
$$(\chi\chi)^{1,2}, \quad (\chi\chi)^{1,11}, \quad (\chi\chi)^{\emptyset,3}, \quad (\chi\chi)^{\emptyset,21}, \quad (\chi\chi)^{\emptyset,111}.$$

Kazhdan-Lusztig basis of $\mathbb{C}[\mathfrak{B}_n]$

Define the Kazhdan-Lusztig basis $\{\widetilde{C}_w \,|\, w \in \mathfrak{B}_n\}$ in terms of Kazhdan-Lusztig polynomials $\{P_{v,w}(q) \,|\, v, w \in \mathfrak{B}_n\}$ by

$$\widetilde{C}_w = \sum_{v \leq w} P_{v,w}(1) v.$$

If $w = w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n$ avoids patterns 3412, 4231, then

$$\widetilde{C}_w = \sum_{v \leq w} v.$$

For $a < b < c < d$, w does not contain $c \cdots d \cdots a \cdots b$ or $d \cdots b \cdots c \cdots a$.

$v \leq w$ in the Bruhat order if some (equivalently, every) reduced word for w contains a reduced word for v.

Decorated posets for the Kazhdan-Lusztig basis

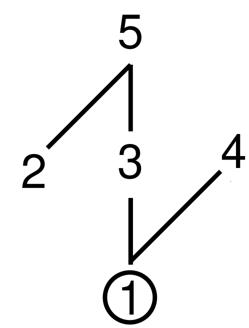
For $w \in \mathfrak{B}_n$ avoiding 3412, 4231, create $P = P(w)$ as follows:

(1) Define $m_{\overline{n}} \cdots m_{\overline{1}} m_1 \cdots m_n$ by $m_i = \max\{w_{\overline{n}}, \ldots, w_i\}$.
(2) For $i = 1, \ldots, n$, define $i <_P m_{i+1}, \ldots, n$.
(3) For $i = 1, \ldots, n$, circle $|w_i|$ if $w_i < 0$.

Example:

$$w = 1\,\overline{5}\,\overline{3}\,\overline{4}\,\overline{2}\,2\,4\,3\,5\,\overline{1}$$
$$m = 1\,1\,1\,1\,1\,2\,4\,4\,5\,5$$
$$(1\,2\,3\,4\,5)$$

$1 <_P 3, 4, 5,$
$2 <_P 5,$
$3 <_P 5,$

$P =$ 

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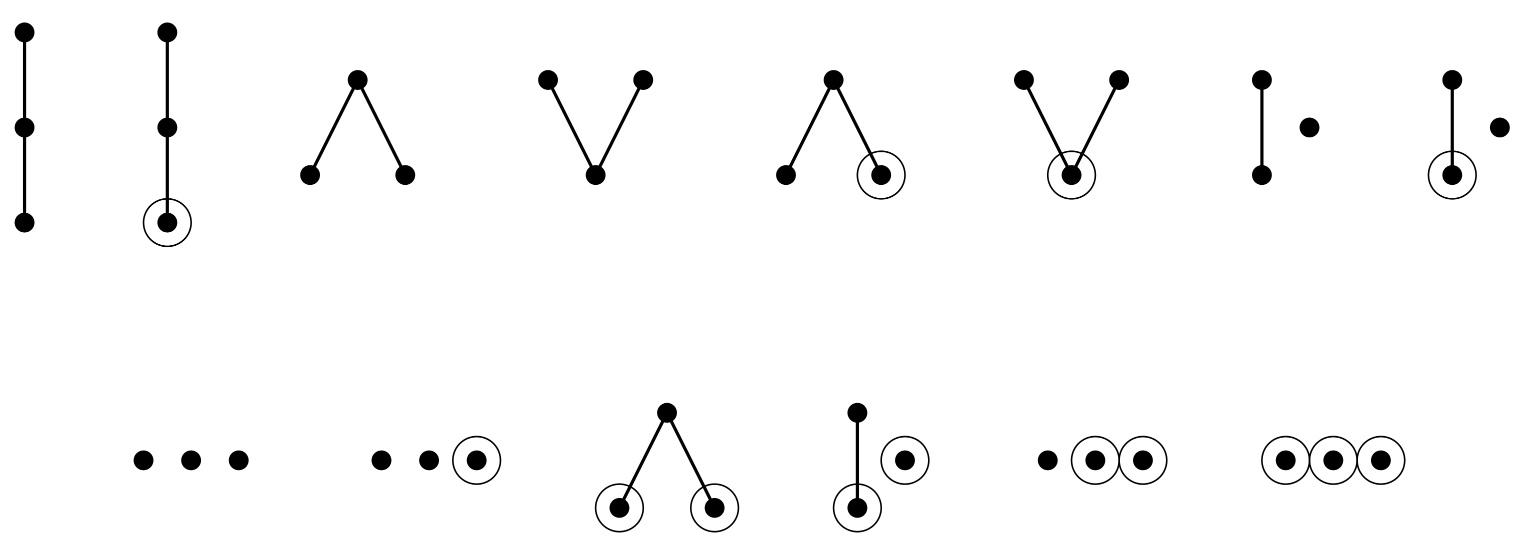
Decorated posets for the Kazhdan-Lusztig basis

The $\frac{1}{n+2}\binom{2n+2}{n+1}$ resulting decorated posets are the n-element unit interval orders with some minimal elements circled.

Example: For $n = 3$ these are



Standard marked P-bitableaux

A P-*tableau* is a Young diagram filled with elements of P. Call it *column-strict* or *row-semistrict* if

$$\begin{array}{|c|}\hline b \\ \hline a \\ \hline \end{array} \Rightarrow a <_P b \qquad \text{or} \qquad \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \Rightarrow a \not>_P b,$$

respectively. Call it *standard* if it has both properties.

Call a pair (U, V) of standard P-tableaux a *standard* P-*bitableau of shape* (λ, μ) if all elements of P appear once, and $\text{shape}(U) = \lambda$, $\text{shape}(V) = \mu$.

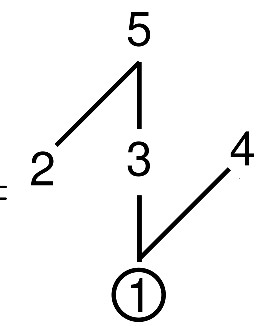
For P a decorated poset, call P-bitableau (U, V) *marked* if all circled elements appear in U, and some subset of these is marked with asterisks.

Main Theorem

Thm: For $(\lambda, \mu) \vdash n$ and $w \in \mathfrak{B}_n$ avoiding 3412, 4231, $(\chi\chi)^{\lambda,\mu}(\widetilde{C}_w) = \#$ std. marked P-tableaux of shape (λ, μ).

Example: Computations for $(\chi\chi)^{21,2}(\widetilde{C}_{2435\overline{1}}) = 48$ are

$$w = 1\,\overline{5}\,\overline{3}\,\overline{4}\,\overline{2}\,2\,4\,3\,5\,\overline{1} \qquad 1 <_P 3, 4, 5,$$
$$m = 1\,1\,1\,1\,1\,2\,4\,4\,5\,5 \qquad 2 <_P 5,$$
$$(1\,2\,3\,4\,5) \qquad 3 <_P 5,$$

$P =$  ,