Mathematics 21: Calculus One

Professor Linghai Zhang. Office: Christmas-Saucon Hall 236. Office telephone: 610-758-4116. Email address: liz5@lehigh.edu 
Office Hours: Monday, Wednesday, Friday, 1PM-2PM and by appointments.
Materials to be covered: Chapters 1, 2, 3, 4, 5: Functions and graphs, limits and continuity, derivative, differential and applications, indefinite and definite integrals, trigonometric functions, logarithmic functions, exponential functions and hyperbolic functions. See below for Chapters 1, 2, 3, 4, 5.
Calculators are not allowed in any test or any quizzes.
Homework will be collected every Monday before class begins. No late homework will be accepted. Quizzes will be given in recitation classes and no make up quizzes will be given.
There will be two midterm examinations (100 points each) in this semester. No make up exam will be given unless you have a good reason (such as sickness. Doctor’s note will be needed).
Homework and Quizzes: 100 points.
Midterm Examination One: 100 point.
Midterm Examination Two: 100 points.
Final Examination: 200 points.
Total score: 500 points.
Grading policy:
A, A-: 91%-100%;
B+, B, B-: 81%-90%;
C+, C, C-: 71%-80%;
D: 61%-70%;
F: 0%-60%.

Students: You are expected to spend at least six hours every week to get familiar with the materials and to do the homework. The homework is an important part of the course: no one can learn mathematics passively and as learning the material is your responsibility, you should try to solve all problems assigned. I strongly recommend doing the homework problems as soon as possible after the material has been covered in class - preferably on the same day. Students will benefit far more from the lectures if they familiarize themselves with the material to be covered before class and come to prepared to ask questions. It is very important that students keep up with the homework and the lectures as the material rapidly builds upon itself. If this becomes a problem, please ask the instructor for help as soon as possible.

Accommodations for students with disabilities: If somebody has a disability for which that student may be requesting accommodations, please contact both your mathematics professor and the Office of Academic Supporting Services, University Center 212 (Telephone: 610-758-4152) as early as possible in this semester. You must have documentation from the Academic Supporting Services Office before accommodations can be granted.

Lehigh University endorses “The Principles of Our Equitable Community” (http://www4.lehigh.edu/diversity/principles). We expect each member of this class to acknowledge and practice these Principles. Respect for each other and for differing viewpoints is a vital component of the learning environment inside and outside the classroom.
Mathematics 22: Calculus Two

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Office Hours: Monday, Wednesday, Friday, 1PM-2PM and by appointments.
Office Telephone: 610-758-4116. Email address: liz5@lehigh.edu.

Textbook: Calculus - Early Transcendentals, eighth Edition. Calculus two covers Chapters 6, 7, 8, 9, 10, 11 (only a few sections in those chapters will be skipped).

Author: James Stewart, published by Brooks/Cole. 2007.
Attendance is strongly required as the pace of the class is very fast. If your final grade is on the borderline, then your attendance may play a positive/negative role.
Midterm Examination One: 100 points.
Midterm Examination Two: 100 points.
Final Examination: 200 points.
Homework: 100 points.

Total Scores: 500 points. Grading policy: A: 471-500,
A-: 451-470,
B+: 435-450,
B: 417-434,
B-: 401-416,
C+: 385-400,
C: 367-384,
C-: 351-366,
D: 301-350,
F: 0-300.

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week to get familiar with the materials and to do the homework. The homework is an important part of the course: no one can learn mathematics passively and as learning the material is your responsibility, you should try to solve all problems assigned. I strongly recommend doing the homework problems as soon as possible after the material has been covered in class - preferably on the same day. Students will benefit far more from the lectures if they familiarize themselves with the material to be covered before class and come to prepared to ask questions. It is very important that students keep up with the homework and the lectures as the material rapidly builds upon itself. If this becomes a problem, please ask the instructor for help as soon as possible.

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Course Materials for Mathematics 21 and Mathematics 22

Chapter 1. Functions and Models
Section 1.1 Four ways to represent a function.
Section 1.2 Mathematical models: a catalog of essential functions.
Section 1.3 New functions from old functions.
Section 1.4 Graphing calculators and computers.
Section 1.5 Exponential functions.
Section 1.6 Inverse functions and logarithms.

Chapter 2. Limits and Derivatives
Section 2.1 The tangent and velocity problems.
Section 2.2 The limit of a function.
Section 2.3 Calculating limits using the limit laws.
Section 2.4 The precise definition of a limit.
Section 2.5 Continuity.
Section 2.6 Limits at infinity, horizontal asymptotes.
Section 2.7 Derivatives and rates of change.
Section 2.8 The derivative as a function.

Chapter 3. Differentiation Rules
Section 3.1 Derivatives of polynomial and exponential functions.
Section 3.2 The product and quotient rules.
Section 3.3 Derivatives of trigonometric functions.
Section 3.4 The chain rule.
Section 3.5 Implicit differentiation.
Section 3.6 Derivatives of logarithmic functions.
Section 3.7 Rates of change in the natural and social sciences.
Section 3.8 Exponential growth and decay.
Section 3.9 Related rates.
Section 3.10 Linear approximations and differentials.
Section 3.11 Hyperbolic functions.

Chapter 4. Applications of Differentiation
Section 4.1 Maximum and minimum values.
Section 4.2 The mean value theorem.
Section 4.3 How derivatives affect the shape of a graph.
Section 4.4 Intermediate forms and L’Hospital’s rule.
Section 4.5 Summary of curve sketching.
Section 4.6 Graphing with calculus and calculators.
(this section will be skipped) Section 4.7 Optimization problems.
Section 4.8 Newton’s method.
Section 4.9 Antiderivatives.

Chapter 5. Integrals
Section 5.1 Areas and distances.
Section 5.2 The definite integral.
Section 5.3 The fundamental theorem of calculus.
Section 5.4 Indefinite integrals and the net change theorem.
Section 5.5 The substitution rule.

Chapter 6. Applications of Integration
Section 6.1 Areas between curves.
Section 6.2 Volumes.
Section 6.3 Volumes by cylindrical shells.
Section 6.4 Work.
Section 6.5 Average value of a function.

Chapter 7. Techniques of Integration
Section 7.1 Integration by parts.
Section 7.2 Trigonometric integrals.
Section 7.3 Trigonometric substitution.
Section 7.4 Integration of rational functions by partial fractions.
Section 7.5 Strategy for integration.
Section 7.6 Integration using tables and computer algebraic sys-
tems.
Section 7.7 Approximate integration.
Section 7.8 Improper integrals.

Chapter 8. Further Applications of Integration
Section 8.1 Arc length.
Section 8.2 Area of a surface of revolution.
Section 8.3 Applications to physics and engineering.
Section 8.4 Applications to economics and biology.
Section 8.5 Probability.

Chapter 9. Differential Equations
Section 9.1 Modeling with differential equations.
Section 9.2 Direction fields ands Euler’s method.
Section 9.3 Separable equations.
Section 9.4 Models for population growth.
Section 9.5 Linear equations.
Section 9.6 Predator-prey systems.

Chapter 10. Parametric Equations and Polar Coordinates
Section 10.1 Curves defined by parametric equations.
Section 10.2 Calculus with parametric curves.
Section 10.3 Polar coordinates.
Section 10.4 Areas and lengths in polar coordinates.

Chapter 11. Infinite Sequences and Series
Section 11.1 Sequences.
Section 11.2 Series.
Section 11.3 The integral test and estimates of sums.
Section 11.4 The comparison tests.
Section 11.5 Alternating series.
Section 11.6 Absolute convergence and the ratio and root tests.
Section 11.7 Strategy for testing series.
Section 11.8 Power series.
Section 11.9 Representations of functions as power series.
Section 11.10 Taylor and Maclaurin series.

Mathematics 22 - Midterm Examination Three

Name:

This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit.

1. (40 points)
2. (28 points)
3. (32 points)

Total (100 points)
1. Use divergence test, integral test, comparison test, alternating series test, ratio test, root test or a combination of any two tests to determine if the series is absolutely convergent, conditionally convergent or divergent. Show all work to receive full credit.

(1) \[ \sum_{n=1}^{\infty} \left( \frac{5^n + 6^n}{7^n + 8^n} \right)^{\frac{1}{n}} \]

(2) \[ \sum_{n=1}^{\infty} \left\{ n^2 \ln \left( 1 + \frac{1}{n^2} \right) + n^4 \ln \left( 1 + \frac{1}{n^4} \right) \right\} \]

(3) \[ \sum_{n=1}^{\infty} (-1)^n \arccos \left( \frac{1}{n^2} \right) \]

(4) \[ \sum_{n=10}^{\infty} \frac{1}{n \ln n \ln(\ln n)} \]

(5) \[ \sum_{n=3}^{\infty} \sin \left\{ \frac{1}{n(\ln n)^2} \right\} \]

(6) \[ \sum_{n=2}^{\infty} \left( \frac{\ln n}{n} \right)^2 \]

(7) \[ \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} \cos \left( \frac{1}{n^2} \right) + \frac{1}{n^2} \sin \left( \frac{1}{n^2} \right) \right\} \]

(8) \[ \sum_{n=3}^{\infty} \left[ \frac{1 + (\sin n)^2}{n^8} + \frac{1 + (\cos n)^2}{n^8} \right] \]

(9) \[ \sum_{n=1}^{\infty} \frac{1}{1 + 2 \cosh n} \]

(10) \[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n^2} \right) \cos \left( \frac{1}{n^2} \right) \]
\[
\sum_{n=1}^{\infty} (-1)^n \arcsin \left( \frac{1}{n^2} \right)
\]

\[
\sum_{n=1}^{\infty} (-1)^n \sin \left( \frac{n}{1 + n^2} \right)
\]

\[
\sum_{n=1}^{\infty} \frac{e^n}{n!}
\]

\[
\sum_{n=1}^{\infty} \left( \frac{n!}{n^n + \frac{2^n}{n!}} \right)
\]

\[
\sum_{n=1}^{\infty} a_n, \quad a_1 = 1, \quad \frac{a_{n+1}}{a_n} = \frac{1 + \sin(n^2 + 2n + 1)}{n^2 + 2n + 1}
\]

\[
\sum_{n=3}^{\infty} \left\{ \arccos \left( \frac{1}{n^2} \right) \right\}^n
\]

\[
\sum_{n=1}^{\infty} \left[ \frac{1 + n^2 \arcsin \left( \frac{1}{n^2} \right)}{1 + n^2 \arccos \left( \frac{1}{n^2} \right)} \right]^n
\]

\[
\sum_{n=1}^{\infty} \left[ \left( \frac{5n^2 + 7n + 1}{6n^2 + 8n + 2} \right)^n + \left( \frac{7n^2 + 9n + 3}{8n^2 + 10n + 4} \right)^n \right]
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{(1 + \frac{1}{n})^n}
\]

\[
\sum_{n=1}^{\infty} \frac{(-n)^n}{\exp(n^2)}
\]
2. Find the radius of convergence, the interval of convergence and the sum of the following power series

\[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1}\]

\[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\]

\[\sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}\]

\[\sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1}\]
3. Find a power series representation for all of the following functions. Specify the interval of convergence for each power series.

\[
\begin{align*}
(1) \quad f(x) & = \frac{1}{1-x} \\
(2) \quad f(x) & = \frac{1}{1+x} \\
(3) \quad f(x) & = \frac{1}{(1-x)^2} \\
(4) \quad f(x) & = \frac{1}{(1+x)^2} \\
(5) \quad f(x) & = \ln(1-x) \\
(6) \quad f(x) & = \ln(1+x) \\
(7) \quad f(x) & = \ln(1-x^2) \\
(8) \quad f(x) & = \ln(1+x^2)
\end{align*}
\]
(9) \( f(x) = \ln \left( \frac{1 + x^2}{1 - x^2} \right) \)
(10) \( f(x) = x \arctan x \)
(11) \( f(x) = x \sin x \)
(12) \( f(x) = x \cos x \)
(13) \( f(x) = x \exp(x^2) \)
(14) \( f(x) = \frac{\exp(x) - \exp(-x)}{2} \)
(15) \( \int_0^x \frac{\ln(1 + t)}{t} \, dt \)
(16) \( f(x) = \int_0^x \frac{\arctan t - t}{t^3} \, dt \)
1. Solve the following integrals

(1) \[ \int (2x + 2) \arctan x \, dx \]

(2) \[ \int \frac{2}{(1 + x^2)^2} \, dx \]

(3) \[ \int \left( \frac{\ln x}{x} \right)^2 \, dx \]

(4) \[ \int \sqrt{1 + x^2} \, dx \]

(5) \[ \int 4x \sin x \cos x \, dx \]

(6) \[ \int \frac{4 \tan x (\sec x)^2}{(\tan x)^2[1 + (\tan x)^4]} \, dx \]

2. Solve the following integrals

(1) \[ \int 10(\sin x)^9 \cos x \, dx + \int 10(\cos x)^9 \sin x \, dx \]

(2) \[ \int 10(\tan x)^9(\sec x)^2 \, dx + \int 10(\sec x)^{10} \tan x \, dx \]

3. The region bounded by \( y = R \sin x \), \( y = 0 \) and \( 0 \leq x \leq \pi \) is rotated about the \( x \)-axis to generate a solid.
   (1) Find the volume of the solid.
   (2) Find the surface area of the solid.

4. Solve each of the following differential equations for solutions

(1) \[ 8y(1 + y^2) \ln(1 + y^2) \frac{dy}{dx} = 9x \cos(3x). \]

(2) \[ \exp(x^2) \frac{dy}{dx} + 2x \exp(x^2) y = 2 \sin x \cos x. \]

(3) \[ x \frac{dy}{dx} + y = \frac{2}{x(1 + x^2)} - \frac{1}{x^2(1 + x^2)}. \]
7. Let $R > 0$ be a positive constant. Given the polar curve $r = f(\theta) = R(1 + \cos \theta)$, where $0 \leq \theta \leq 2\pi$. Find the arclength of this polar curve.

8. Find the area of the region outside the curve $r = R(1 + \sin \theta)$ and inside the curve $r = 3R \sin \theta$.

Math 22 - Solutions to the Final Exam

1. Solutions:

(1) \[ \int (2x + 2) \arctan x \, dx \]

\[ = \int (x^2 + 2x + 1)' \arctan x \, dx \]

\[ = (x^2 + 2x + 1) \arctan x - \int \frac{x^2 + 2x + 1}{1 + x^2} \, dx \]

\[ = (x^2 + 2x + 1) \arctan x - x - \ln(1 + x^2) + C, \]

(2) \[ \int \frac{2}{(1 + x^2)^2} \, dx = \int \left[ \frac{2}{1 + x^2} + x \left( \frac{1}{1 + x^2} \right)' \right] \, dx \]

\[ = 2 \arctan x + \frac{x}{1 + x^2} - \int \frac{1}{1 + x^2} \, dx \]

\[ = \arctan x + \frac{x}{1 + x^2} + C, \]

(3) \[ \int \left( \frac{\ln x}{x} \right)^2 \, dx \]

\[ = \int \left( -\frac{1}{x} \right)' (\ln x)^2 \, dx \]

\[ = -\frac{1}{x} (\ln x)^2 + 2 \int \frac{1}{x^2} \ln x \, dx \]

\[ = -\frac{1}{x} (\ln x)^2 - 2 \frac{1}{x} \ln x + \int \frac{1}{x^2} \, dx \]

\[ = -\frac{1}{x} (\ln x)^2 - 2 \frac{1}{x} \ln x - \frac{1}{x} + C, \]
\[
\int \sqrt{1 + x^2} \, dx = x \sqrt{1 + x^2} - \int \frac{x^2}{\sqrt{1 + x^2}} \, dx
\]

\[
= x \sqrt{1 + x^2} - \int \sqrt{1 + x^2} \, dx + \int \frac{1}{\sqrt{1 + x^2}} \, dx
\]

\[
2 \int \sqrt{1 + x^2} \, dx = x \sqrt{1 + x^2} + \ln(x + \sqrt{1 + x^2}) + 2C,
\]

\[
\int \sqrt{1 + x^2} \, dx = \frac{1}{2} x \sqrt{1 + x^2} + \frac{1}{2} \ln(x + \sqrt{1 + x^2}) + \ln(2) + C,
\]

(5) \[
\int 4x \sin x \cos x \, dx = \int 2x [\sin^2 x] \, dx
\]

\[
= 2x (\sin x)^2 - \int 2 \sin^2 x \, dx
\]

\[
= 2x (\sin x)^2 - \int [1 - \cos(2x)] \, dx
\]

\[
= 2x (\sin x)^2 - x + \frac{1}{2} \sin(2x) + C,
\]

(6) \[u = (\tan x)^2, \quad du = 2 \tan x (\sec x)^2 \, dx\]

\[
\int \frac{4 \tan x (\sec x)^2}{(\tan x)^2[1 + (\tan x)^4]} \, dx
\]

\[
= \int \frac{2}{u(1 + u^2)} \, du = \int \left( \frac{2}{u} - \frac{2u}{1 + u^2} \right) \, du
\]

\[
= \ln(u^2) - \ln(1 + u^2) + C = \ln(\tan x)^4 - \ln[1 + (\tan x)^4] + C
\]

\[
= \ln \frac{(\tan x)^4}{1 + (\tan x)^4} + C.
\]

2. Solutions: (1) Let \( u = \sin x \) and \( v = \cos x \). Then \( du = \cos x \, dx \) and \( dv = -\sin x \, dx \). (2) Let \( u = \tan x \) and \( v = \sec x \).
Then $du = (\sec x)^2 \, dx$ and $dv = \sec x \tan x \, dx$.

(1) \[
\int 10(\sin x)^9 \cos x \, dx + \int 10(\cos x)^9 \sin x \, dx
= \int 10u^9 \, du - \int 10v^9 \, dv = u^{10} - v^{10} + C = (\sin x)^{10} - (\cos x)^{10} + C,
\]

(2) \[
\int 10(\tan x)^9 (\sec x)^2 \, dx + \int 10(\sec x)^{10} \tan x \, dx
= \int 10u^9 \, du + \int 10v^9 \, dv = u^{10} + v^{10} + C
= (\tan x)^{10} + (\sec x)^{10} + C.
\]

2. Solve the following integrals

\[
\int (\tan x) \, dx = -\ln |\cos x| + C,
\]

\[
\int (\tan x)^2 \, dx = \tan x - x + C,
\]

\[
\int (\tan x)^3 \, dx = \frac{1}{2}(\tan x)^2 + \ln |\cos x| + C,
\]

\[
\int (\tan x)^4 \, dx = \frac{1}{3}(\tan x)^3 - \tan x + x + C,
\]

\[
\int (\tan x)^5 \, dx = \frac{1}{4}(\tan x)^4 - \frac{1}{2}(\tan x)^2 - \ln |\cos x| + C,
\]

\[
\int (\tan x)^6 \, dx = \frac{1}{5}(\tan x)^5 - \frac{1}{3}(\tan x)^3 + \tan x - x + C.
\]
3. Solutions: The volume and the surface area are

\[
V = \int_0^\pi \pi (R \sin x)^2 \, dx = \frac{1}{2} \pi R^2 \int_0^\pi [1 - \cos(2x)] \, dx \\
= \frac{1}{4} \pi R^2 [2x - \sin(2x)]|_0^\pi = \frac{1}{2} \pi^2 R^2,
\]

\[
\int \sqrt{1 + u^2} \, du = \frac{1}{2} u \sqrt{1 + u^2} + \frac{1}{2} \ln(u + \sqrt{1 + u^2}),
\]

\[
A = \int_0^\pi 2\pi (R \sin x) \sqrt{1 + (R \cos x)^2} \, dx \\
= -2\pi \left\{ \frac{1}{2} R \cos x \sqrt{1 + (R \cos x)^2} + \frac{1}{2} \ln \left[ R \cos x + \sqrt{1 + (R \cos x)^2} \right] \right\} \bigg|_0^\pi \\
= 2\pi [R \sqrt{1 + R^2} + \ln(R + \sqrt{1 + R^2})].
\]

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(1) \quad \int_0^\pi \pi (\sin x)^2 \, dx = \frac{1}{2} \pi^2,

(2) \quad \int_0^\pi 2\pi \sin x \sqrt{1 + (\cos x)^2} \, dx \\
= -2\pi \left\{ \frac{1}{2} \cos x \sqrt{1 + (\cos x)^2} + \frac{1}{2} \ln \left[ \cos x + \sqrt{1 + (\cos x)^2} \right] \right\} \bigg|_0^\pi \\
= 2\pi [\sqrt{2} + \ln(1 + \sqrt{2})].
3. Solve the following integrals
\[ \int (\sec x)\,dx = \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C, \]
\[ \int (\sec x)^2\,dx = \tan x + C, \]
\[ \int (\sec x)^3\,dx = \frac{1}{2} \tan x \sec x + \frac{1}{4} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C, \]
\[ \int (\sec x)^4\,dx = \frac{1}{3} \tan x (\sec x)^2 + \frac{2}{3} \tan x + C, \]
\[ \int (\sec x)^5\,dx = \frac{1}{4} \tan x (\sec x)^3 + \frac{3}{8} \tan x \sec x + \frac{3}{16} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C, \]
\[ \int (\sec x)^6\,dx = \frac{1}{5} \tan x (\sec x)^4 + \frac{4}{15} \tan x (\sec x)^2 + \frac{8}{15} \tan x + C, \]

4. Solutions: (1) Let \( u = 1 + y^2 \). Then \( du = 2y\,dy \).
\[ \int 8y(1 + y^2) \ln(1 + y^2)\,dy = \int 4u \ln u\,du = \int (2u^2)' \ln u\,du \]
\[ = 2u^2 \ln u - \int 2udu = 2u^2 \ln u - u^2 = 2(1 + y^2)^2 \ln(1 + y^2) - (1 + y^2)^2. \]

Let \( v = 3x \). Then \( dv = 3\,dx \).
\[ \int 9x \cos(3x)\,dx = \int v \cos vd\!v = \int v(\sin v)'\,dv \]
\[ = v \sin v - \int \sin vd\!v = v \sin v + \cos v + C \]
\[ = 3x \sin(3x) + \cos(3x) + C, \]
\[ 2(1 + y^2)^2 \ln(1 + y^2) - (1 + y^2)^2 \]
\[ = 3x \sin(3x) + \cos(3x) + C \]
(2) 
\[
\frac{d}{dx} \left[ \exp(x^2)y \right] = 2 \sin x \cos x \\
\exp(x^2)y = (\sin x)^2 + C, \\
y = \frac{(\sin x)^2 + C}{\exp(x^2)}
\]

(3) 
\[
\frac{d}{dx} (xy) = \frac{2}{x} - \frac{2x}{1 + x^2} - \frac{1}{x} + \frac{1}{1 + x^2} \\
xy = 2 \ln |x| - \ln(1 + x^2) + C + \frac{1}{x} + \arctan x.
\]

5. Solutions:
(1), (2), (3), (4), (5), (7), (8) and (9) : They are absolutely convergent by using the comparison test.
(6): It is divergent by using the comparison test.
(11) and (12): They are absolutely convergent by using ratio test.
(10), (13) and (14): They are absolutely convergent by using the root test.
(15): It is divergent by using the divergence test.
6. Solutions:

(1) \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad R = 1, \quad I = (-1, 1), \)

(2) \( \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad R = 1, \quad I = (-1, 1), \)

(3) \( \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}, \quad R = 1, \quad I = (-1, 1), \)

(4) \( \frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^{n-1} nx^{n-1}, \quad R = 1, \quad I = (-1, 1), \)

(5) \( \ln(1-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \quad R = 1, \quad I = (-1, 1), \)

(6) \( \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \quad R = 1, \quad I = (-1, 1), \)

(7) \( \ln(1-x^2) = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{2n+2}, \quad R = 1, \quad I = (-1, 1), \)

(8) \( \ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2}, \quad R = 1, \quad I = (-1, 1), \)
\[
\begin{align*}
\text{(9)} & \quad \ln \frac{1 + x^2}{1 - x^2} = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n + 1} x^{2n+2}, \quad R = 1, \quad I = (-1, 1), \\
\text{(10)} & \quad \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+2}, \quad R = 1, \quad I = (-1, 1), \\
\text{(11)} & \quad x \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+2}, \quad R = \infty, \quad I = (-\infty, \infty), \\
\text{(12)} & \quad x \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1}, \quad R = \infty, \quad I = (-\infty, \infty), \\
\text{(13)} & \quad x \exp(x^2) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n+1}, \quad R = \infty, \quad I = (-\infty, \infty), \\
\text{(14)} & \quad \frac{\exp(x) - \exp(-x)}{2} = \sum_{n=0}^{\infty} \frac{1 - (-1)^n}{2(n!)} x^n, \quad R = \infty, I = (-\infty, \infty), \\
\text{(15)} & \quad \int_0^x \frac{\ln(1 + t)}{t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + 1)^2} x^{n+1}, \quad R = 1, \quad I = (-1, 1), \\
\text{(16)} & \quad \int_0^x \frac{\arctan t - t}{t^3} dt = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n + 1)(2n - 1)} x^{2n-1}, \quad R = 1, \quad I = (-1, 1).
\end{align*}
\]
7. Solution: The arclength of the polar curve is equal to

\[ L = \int_0^{2\pi} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta \]

\[ = 2 \int_0^{\pi} \sqrt{[R(1 + \cos \theta)]^2 + [-R \sin \theta]^2} \, d\theta \]

\[ = 2R \int_0^{\pi} \sqrt{1 + 2 \cos \theta + (\sin \theta)^2 + (\cos \theta)^2} \, d\theta \]

\[ = 2R \int_0^{\pi} \sqrt{2 + 2 \cos \theta} \, d\theta \]

\[ = 2R \int_0^{\pi} \sqrt{4 \left[ \cos \left( \frac{\theta}{2} \right) \right]^2} \, d\theta \]

\[ = 4R \int_0^{\pi} \cos \left( \frac{\theta}{2} \right) \, d\theta \]

\[ = 8R \sin \left( \frac{\theta}{2} \right) \bigg|_0^{\pi} = 8R, \]

where

\[ \left[ \cos \left( \frac{\theta}{2} \right) \right]^2 = \frac{1 + \cos \theta}{2}. \]

8. Solution: The intersection of the two curves are at \( \theta = \frac{\pi}{6} \) and \( \theta = \frac{5\pi}{6} \) and the area is

\[ \frac{1}{2} \int_{\pi/6}^{5\pi/6} \left[ (3R \sin \theta)^2 - (R + R \sin \theta)^2 \right] \, d\theta \]

\[ = \frac{1}{2} R^2 \int_{\pi/6}^{5\pi/6} \left[ 8(\sin \theta)^2 - 1 - 2 \sin \theta \right] \, d\theta \]

\[ = \frac{1}{2} R^2 \int_{\pi/6}^{5\pi/6} \left[ 4 - 4 \cos(2\theta) - 1 - 2 \sin \theta \right] \, d\theta \]

\[ = \frac{1}{2} R^2 \left[ 3\theta - 2 \sin(2\theta) + 2 \cos \theta \right] \bigg|_{\pi/6}^{5\pi/6} = \pi R^2. \]
8. Let $R > 0$ be a positive constant. Given the parametric equations of a cycloid

\[ x = f(\theta) = R(\theta - \sin \theta), \quad 0 \leq \theta \leq 2\pi, \]
\[ y = g(\theta) = R(1 - \cos \theta), \quad 0 \leq \theta \leq 2\pi. \]

Find the arclength of this curve.

Solution: \( f'(\theta) = R(1 - \cos \theta), \ g'(\theta) = R \sin \theta. \) The arclength of the parametric curve is equal to

\[
L = \int_0^{2\pi} \sqrt{[f'(\theta)]^2 + [g'(\theta)]^2} d\theta
\]
\[
= \int_0^{2\pi} \sqrt{R(1 - \cos \theta)^2 + (R \sin \theta)^2} \, d\theta
\]
\[
= R \int_0^{2\pi} \sqrt{1 - 2 \cos \theta + (\cos \theta)^2 + (\sin \theta)^2} \, d\theta
\]
\[
= R \int_0^{2\pi} \sqrt{2 - 2 \cos \theta} \, d\theta
\]
\[
= R \int_0^{2\pi} \sqrt{2 \sin \left( \frac{\theta}{2} \right)} \, d\theta
\]
\[
= R \int_0^{2\pi} 2 \sin \left( \frac{\theta}{2} \right) \, d\theta
\]
\[
= -4R \cos \left( \frac{\theta}{2} \right) \bigg|_0^{2\pi} = 8R.
\]
Solutions: Let \( R = 1 \) and \(|x| < 1\). Then

(1) \[ \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n, \]

(2) \[ \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n, \]

(3) \[ \frac{1}{(1 - x)^2} = \sum_{n=0}^{\infty} n x^{n-1}, \]

(4) \[ \frac{1}{(1 + x)^2} = \sum_{n=0}^{\infty} (-1)^{n-1} n x^{n-1}, \]

(5) \[ \ln(1 - x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \]

(6) \[ \ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \]

(7) \[ \ln(1 - x^2) = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{2n+2}, \]

(8) \[ \ln(1 + x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2}, \]

(9) \[ \ln \frac{1 + x^2}{1 - x^2} = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n+1} x^{2n+2}, \]

(10) \[ \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \]

(11) \[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \]

(12) \[ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \]

(13) \[ e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \]
Mathematics 22 - Midterm Examination Two

Name:

In this examination, \( R > 0 \) is a fixed positive constant. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit.
This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test.
Please do not start working before you are told to do so.

1. (10 points)
2. (10 points)
3. (10 points)
4. (10 points)
5. (10 points)
6. (10 points)
7. (10 points)
8. (10 points)
9. (10 points)
10. (10 points)
Total (100 points)
1. The parametric equations of a cycloid are given by

\[ x = R(\theta - \sin \theta), \quad y = R(1 - \cos \theta), \quad 0 \leq \theta \leq 2\pi. \]

(1) Find the area under one arch of the cycloid.
(2) Find the arclength of one arch of the cycloid.
(3) Find the equation of tangent line at the point at \( \theta = \pi/3. \)
2. (1) The curve

\[ y = \sqrt{R^2 - x^2}, \quad -R \leq x \leq R, \]

is the upper half of the circle \( x^2 + y^2 = R^2 \).

Find the area of the surface obtained by rotating this curve about the \( x \)-axis.

(2) Find the area of the surface generated by rotating the curve \( y = R \cos x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \) about the \( x \) axis.
3. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

(1) \( y = \sqrt{R^2 - x^2}, y = 0, x = -R, x = R, \) about the \( x \)-axis.

(2) \( y = -\cos(x), y = 0, x = \frac{\pi}{2}, x = \frac{3\pi}{2}, \) about the \( y \)-axis.
4. Let $R > 0$ be a given positive constant. A positive sequence is defined by $a_1 = \sqrt{R}$ and $a_{n+1} = \sqrt{R + a_n}$, for all integers $n \geq 1$.

(1) Show that the sequence $\{a_n\}$ is bounded.

(2) Show that the sequence $\{a_n\}$ is increasing.

(3) Show that the sequence $\{a_n\}$ has a limit.

(4) Find the limit $L = \lim_{n \to \infty} a_n$. 
5. Find all values of $R$ for which the following series

$$
\sum_{n=0}^{\infty} \frac{[\sin(1 + 4R + 4R^2)]^n + [\cos(1 + 4R + 4R^2)]^n}{(1 + 2|R|)^n}
$$

is convergent. Find the sum of the series for those values of $R$. 
6. Determine whether each series is convergent or divergent. Show all work to support your answer.

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \left( \frac{1}{n} \right) \]

\[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \]

\[ \sum_{n=1}^{\infty} \frac{1}{\left( \frac{4}{3} \right)^n + \left( \frac{7}{8} \right)^n} \]

\[ \sum_{n=1}^{\infty} \frac{1}{\left( \frac{6}{5} \right)^n + \left( \frac{8}{7} \right)^n} \]
7. Determine whether each series is convergent or divergent

(1) \[ \sum_{n=1}^{\infty} \left[ \frac{1}{n^2} \sin \left( \frac{1}{n^2} \right) + \frac{1}{n^2} \cos \left( \frac{1}{n^2} \right) \right] \]

(2) \[ \sum_{n=1}^{\infty} \left[ \frac{1}{n^4} \arcsin \left( \frac{1}{n^4} \right) + \frac{1}{n^4} \arccos \left( \frac{1}{n^4} \right) \right] \]

(3) \[ \sum_{n=1}^{\infty} \left[ \frac{1}{n^6} \sinh \left( \frac{1}{n^6} \right) + \frac{1}{n^6} \cosh \left( \frac{1}{n^6} \right) \right] \]

(4) \[ \sum_{n=1}^{\infty} \left[ n \ln \left( 1 + \frac{1}{n} \right) \right] \]

(5) \[ \sum_{n=1}^{\infty} \frac{6}{(n^2 + 11n + 24)(n^2 + 11n + 30)} \]
8. Determine whether each series is convergent or divergent. Show all work to support your answer.

(1) \[ \sum_{n=1}^{\infty} \frac{1}{n^2(1 + n^2)} \sin \left( \frac{1}{n^2(1 + n^2)} \right) \]

(2) \[ \sum_{n=1}^{\infty} \frac{1}{n^2(1 + n^2)} \cos \left( \frac{1}{n^2(1 + n^2)} \right) \]
9. Given two circles $r = 2R \cos \theta$ and $r = 2R \sin \theta$ in polar coordinate system. Find the area of the region that lies inside both circles.
10. A cardioid is the graph of a polar curve in polar coordinate system given by \( r = R(1 + \sin \theta) \), \( 0 \leq \theta \leq 2\pi \).

(1) Find the arclength of the cardioid.

(2) Find the area of the cardioid.

Math 22 - Solutions to Midterm Exam Two

1. Solutions: Let \( f(\theta) = R(\theta - \sin \theta) \) and \( g(\theta) = R(1 - \cos \theta) \).

Then

\[
   f'(\theta) = R(1 - \cos \theta), \quad g'(\theta) = R \sin \theta.
\]

The arclength is

\[
   L = \int_{0}^{2\pi} \sqrt{[R(1 - \cos \theta)]^2 + (R \sin \theta)^2} \, d\theta
   = \int_{0}^{2\pi} \sqrt{R^2[1 + (\sin \theta)^2 + (\cos \theta)^2 - 2 \cos \theta]} \, d\theta
   = \int_{0}^{2\pi} \sqrt{R^2(2 - 2 \cos \theta)} \, d\theta
   = \int_{0}^{2\pi} \sqrt{4R^2 \left[ \sin \left( \frac{\theta}{2} \right) \right]^2} \, d\theta
   = \int_{0}^{2\pi} 2R \sin \left( \frac{\theta}{2} \right) \, d\theta
   = -4R \cos \left( \frac{\theta}{2} \right) \bigg|_{0}^{2\pi} = 8R.
\]
The area under one arch is

\[ A = \int_{0}^{2\pi} [R(1 - \cos \theta)]^2 d\theta \]

\[ = \int_{0}^{2\pi} R^2[1 - 2 \cos \theta + (\cos \theta)^2] d\theta \]

\[ = \int_{0}^{2\pi} R^2 \left[ 1 - 2 \cos \theta + \frac{1 + \cos(2\theta)}{2} \right] d\theta \]

\[ = R^2 \left[ \theta - 2 \sin \theta + \frac{2\theta + \sin(2\theta)}{4} \right]_{0}^{2\pi} \]

\[ = 3\pi R^2. \]

The slope at \( \theta = \pi \) is

\[ \frac{dy}{dx} = \frac{g'(\theta)}{f'(\theta)} = \frac{\sin \theta}{1 - \cos \theta} = 0. \]

The equation of tangent line at \( \theta = \pi \) is \( y = 2R \).

2. Solution: (1) Let \( f(x) = \sqrt{R^2 - x^2} \). Then \( f'(x) = -\frac{x}{\sqrt{R^2 - x^2}} \) and

\[ 1 + [f'(x)]^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}. \]

The surface area

\[ \int_{-R}^{R} 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx \]

\[ = \int_{-R}^{R} 2\pi \sqrt{R^2 - x^2} \sqrt{\frac{R^2}{R^2 - x^2}} dx = \int_{-R}^{R} 2\pi R dx = 4\pi R^2. \]
(2) First of all, let us find the following integral.

\[
\int \sqrt{1+u^2}du = u\sqrt{1+u^2} - \int \frac{u^2}{\sqrt{1+u^2}}du
\]

\[
= u\sqrt{1+u^2} - \int \sqrt{1+u^2}du + \int \frac{1}{\sqrt{1+u^2}}du
\]

\[
= u\sqrt{1+u^2} - \int \sqrt{1+u^2}du + \ln(u + \sqrt{1+u^2}),
\]

\[
2\int \sqrt{1+u^2}du = u\sqrt{1+u^2} + \ln(u + \sqrt{1+u^2}),
\]

\[
\int \sqrt{1+u^2}du = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u + \sqrt{1+u^2}).
\]

Let \( u = R\sin x \). Then \( du = R\cos x\,dx \). The surface area

\[
\int_{-\pi/2}^{\pi/2} 2\pi f(x)\sqrt{1+[f'(x)]^2}dx
\]

\[
= \int_{-\pi/2}^{\pi/2} 2\pi R\cos x\sqrt{1+(R\sin x)^2}dx
\]

\[
= 2\pi \left\{ \frac{1}{2}R\sin x\sqrt{1+(R\sin x)^2} + \frac{1}{2}\ln \left[R\sin x + \sqrt{1+(R\sin x)^2}\right] \right\} \bigg|_{-\pi/2}^{\pi/2}
\]

\[
= 2\pi [R\sqrt{1+R^2} + \ln(R + \sqrt{1+R^2})].
\]

3.

(1) \( V = \pi \int_{-5}^{5} (25 - x^2)dx = \pi(25x - \frac{1}{3}x^3)|_{-5}^{5} = \frac{500\pi}{3} \).

(2) \( V = 2\pi \int_{\sqrt{\pi/2}}^{\sqrt{3\pi/2}} -x^3\cos(x^2)dx \)

\[
= -\pi [x^2\sin(x^2) + \cos(x^2)]\bigg|_{\sqrt{\pi/2}}^{\sqrt{3\pi/2}} = 2\pi^2.
\]

4. Solution: First of all, let us use mathematical induction method
to prove that the sequence is bounded and increasing. Clearly

\[ 0 < a_1 = \sqrt{R} < a_2 = \sqrt{R + \sqrt{R}} < 1 + R. \]

Suppose that for some integer \( k \geq 1 \), there holds the estimates

\[ 0 < a_k < a_{k+1} < 1 + R. \]

Adding \( R \) to all terms, then

\[ R < R + a_k < R + a_{k+1} < 1 + 2R < 1 + 2R + R^2. \]

Then taking the square root of every term, we have

\[ \sqrt{R} < \sqrt{R + a_k} < \sqrt{R + a_{k+1}} < \sqrt{1 + 2R + R^2}. \]

That is

\[ 0 < a_{k+1} < a_{k+2} < 1 + R. \]

Therefore, the sequence is an increasing and bounded sequence, so the limit of the sequence exists. Let

\[ L = \lim_{n \to \infty} a_n. \]

Since

\[ a_{n+1} = \sqrt{R + a_n}, \]

we have

\[ \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{R + a_n}. \]

That is

\[ L = \sqrt{R + L}. \]

Finally, the limit \( L = \frac{1 + \sqrt{1 + 4R}}{2} \).
5. Solution: If $R = 0$, then both geometric series $\sum_{n=0}^{\infty}(\sin 1)^n$ and $\sum_{n=0}^{\infty}(\cos 1)^n$ are convergent because $0 < \sin 1 < 1$ and $0 < \cos 1 < 1$. Therefore

$$\sum_{n=0}^{\infty}[(\sin 1)^n + (\cos 1)^n] = \sum_{n=0}^{\infty}(\sin 1)^n + \sum_{n=0}^{\infty}(\cos 1)^n = \frac{1}{1 - \sin 1} + \frac{1}{1 - \cos 1}.$$  

If $R \neq 0$, then $1 + 2|R| > 1$, both geometric series

$$\sum_{n=0}^{\infty}\left[\frac{\sin(1 + 4R + 4R^2)}{1 + 2|R|}\right]^n$$

and

$$\sum_{n=0}^{\infty}\left[\frac{\cos(1 + 4R + 4R^2)}{1 + 2|R|}\right]^n$$

are convergent. Therefore

$$\sum_{n=0}^{\infty}\left[\frac{\sin(1 + 4R + 4R^2)}{1 + 2|R|}\right]^n + \sum_{n=0}^{\infty}\left[\frac{\cos(1 + 4R + 4R^2)}{1 + 2|R|}\right]^n = \frac{1}{1 - \sin(1 + 4R + 4R^2)} + \frac{1}{1 - \cos(1 + 4R + 4R^2)}.$$ 

6. Solution: (1) We will use integral test.

$$\int_{1}^{\infty} \frac{1}{x^2} \sin \left(\frac{1}{x}\right) \, dx = \frac{\cos \left(\frac{1}{x}\right)}{1} \bigg|_{1}^{\infty} = 1 - \cos 1.$$ 

The series $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \left(\frac{1}{n}\right)$ is convergent.

(2) We will use integral test as well.

$$\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \ln(\ln x) \bigg|_{2}^{\infty} = \infty.$$
The series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent.

(3) Obviously, we have
\[
\left( \frac{4}{3} \right)^n + \left( \frac{7}{8} \right)^n > \left( \frac{4}{3} \right)^n.
\]

Hence
\[
\frac{1}{\left( \frac{4}{3} \right)^n + \left( \frac{7}{8} \right)^n} < \frac{1}{\left( \frac{4}{3} \right)^n} = \left( \frac{3}{4} \right)^n.
\]

Clearly
\[
\sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n
\]

is a convergent geometric series. Therefore,
\[
\sum_{n=0}^{\infty} \frac{1}{\left( \frac{4}{3} \right)^n + \left( \frac{7}{8} \right)^n}
\]

is also convergent.

(4) Obviously, we have
\[
\left( \frac{6}{5} \right)^n + \left( \frac{8}{7} \right)^n > 2 \left( \frac{8}{7} \right)^n.
\]

Hence
\[
\frac{1}{\left( \frac{6}{5} \right)^n + \left( \frac{8}{7} \right)^n} < \frac{1}{2 \left( \frac{8}{7} \right)^n} = \frac{1}{2} \left( \frac{7}{8} \right)^n.
\]

Obviously
\[
\sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{7}{8} \right)^n
\]
is a convergent geometric series. Therefore,
\[
\sum_{n=0}^{\infty} \frac{1}{\left(\frac{6}{5}\right)^n + \left(\frac{8}{7}\right)^n}
\]
is convergent as well.

7. Solutions: The first series is convergent if it is compared with
\[
\sum_{n=1}^{\infty} \frac{1}{n^2}.
\]
The second series is convergent if it is compared with
\[
\sum_{n=1}^{\infty} \frac{1}{n^4}.
\]
The third series is convergent if it is compared with
\[
\sum_{n=1}^{\infty} \frac{1}{n^6}.
\]
The fourth series is divergent because
\[
\lim_{n \to \infty} \left[n \ln \left(1 + \frac{1}{n}\right)\right] = 1 \neq 0.
\]
The last series is convergent if it is compared with
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ or } \sum_{n=1}^{\infty} \frac{1}{n^4}.
\]

8. Solution: (1) Let us compare the given series
\[
\sum_{n=1}^{\infty} \frac{1}{n^2(1 + n^2)} \sin \left(\frac{1}{n^2(1 + n^2)}\right)
\]
with
\[ \sum_{n=1}^{\infty} \frac{1}{n^2}. \]

For all positive integers \( n \geq 1 \), it is easy to see that
\[ \frac{1}{n^2(1 + n^2)} \sin \left( \frac{1}{n^2(1 + n^2)} \right) \leq \frac{1}{n^2}. \]

Therefore, the \( \sum_{n=1}^{\infty} \frac{1}{n^2(1+n^2)} \sin \left( \frac{1}{n^2(1+n^2)} \right) \) is convergent.

(2) Let us compare the given series
\[ \sum_{n=1}^{\infty} \frac{1}{n^2(1 + n^2)} \cos \left( \frac{1}{n^2(1 + n^2)} \right) \]

with
\[ \sum_{n=1}^{\infty} \frac{1}{n^2}. \]

For all positive integers \( n \geq 1 \), it is easy to see that
\[ \frac{1}{n^2(1 + n^2)} \cos \left( \frac{1}{n^2(1 + n^2)} \right) \leq \frac{1}{n^2}. \]

Therefore, the series \( \sum_{n=1}^{\infty} \frac{1}{n^2(1+n^2)} \cos \left( \frac{1}{n^2(1+n^2)} \right) \) is convergent.

\[ \lim_{n \to \infty} \cos \frac{1222}{2n} \geq 1 \frac{2n}{4}. \]
\[ = \lim_{n \to \infty} \frac{n}{4}. \]

. We finde limit1222nnn+2nBecause1+ddd1 2n1+nc , well. , 2n \leq 1+, 2n1+is that is \( \leq 1 \). Note that \cos x \ is decreasing function interval \[0, \frac{\pi}{2} \], s \cos \left( \frac{2n}{2} \right) \geq \cos 1 > \cos \frac{\pi}{2} = 0. \] Now we have
By using integral test, \( \int_1^\infty \frac{2x}{x} \, dx = \ln(1 + x)|_1^\infty = \infty \). Thus, \( 1 + 2n \) diverges. That means it diverges.

9. Solution. The two polar curves intersect at \( \theta = \frac{\pi}{4} \). The two circles are tangent to either the \( x \)-axis or the \( y \)-axis. The area inside both curves is

\[
A = \frac{1}{2} \int_0^{\pi/4} (R \sin \theta)^2 \, d\theta + \frac{1}{2} \int_{\pi/4}^{\pi/2} (R \cos \theta)^2 \, d\theta
\]
\[
= \frac{1}{2} R^2 \int_0^{\pi/4} \frac{1 - \cos(2\theta)}{2} \, d\theta + \frac{1}{2} R^2 \int_{\pi/4}^{\pi/2} \frac{1 + \cos(2\theta)}{2} \, d\theta
\]
\[
= \frac{1}{8} R^2 [2\theta - \sin(2\theta)]_0^{\pi/4} + \frac{1}{8} R^2 [2\theta + \sin(2\theta)]_{\pi/4}^{\pi/2}
\]

The arclength is

\[
L = \int_0^{\pi/4} \sqrt{(R \sin \theta)^2 + (R \cos \theta)^2} \, d\theta
\]
\[
+ \int_{\pi/4}^{\pi/2} \sqrt{(R \cos \theta)^2 + (-R \sin \theta)^2} \, d\theta
\]
\[
= \int_0^{\pi/4} Rd\theta + \int_{\pi/4}^{\pi/2} Rd\theta = \frac{1}{2} \pi R.
\]
10. Solutions: The area is

\[ A = \int_0^{2\pi} \frac{1}{2} [R(1 + \sin \theta)]^2 d\theta \]

\[ = \int_0^{2\pi} \frac{1}{2} R^2 [1 + 2 \sin \theta + (\sin \theta)^2] d\theta \]

\[ = \int_0^{2\pi} \frac{1}{2} R^2 \left[ 1 + 2 \sin \theta + \frac{1 - \cos(2\theta)}{2} \right] d\theta \]

\[ = \frac{1}{2} R^2 \left[ \theta - 2 \cos \theta + \frac{2\theta - \sin(2\theta)}{4} \right] \bigg|_0^{2\pi} = \frac{3}{2} \pi R^2 \]

The arclength is

\[ L = 2 \int_{-\pi/2}^{\pi/2} \sqrt{[R(1 + \sin \theta)]^2 + (R \cos \theta)^2} d\theta \]

\[ = 2 \int_{-\pi/2}^{\pi/2} \sqrt{R^2[1 + (\sin \theta)^2 + (\cos \theta)^2 + 2 \sin \theta]} d\theta \]

\[ = 2 \int_{-\pi/2}^{\pi/2} \sqrt{R^2(2 + 2 \sin \theta)} d\theta \]

\[ = 2 \int_{-\pi/2}^{\pi/2} \sqrt{2R^2} d\theta \]

\[ = 2 \int_{-\pi/2}^{\pi/2} \left[ \sin \left( \frac{\theta}{2} \right) + \cos \left( \frac{\theta}{2} \right) \right]^2 d\theta \]

\[ = 2 \int_{-\pi/2}^{\pi/2} R \]

45
The area is

\[ A = \frac{1}{2} \int_0^{2\pi} [R(1 + \sin \theta)]^2 d\theta \]

\[ = \frac{1}{2} R^2 \int_0^{2\pi} [1 + 2 \sin \theta + (\sin \theta)^2] d\theta \]

\[ = \frac{1}{2} R^2 \int_0^{2\pi} \left[ 1 + 2 \sin \theta + \frac{1 - \cos(2\theta)}{2} \right] d\theta \]

\[ = \frac{1}{2} R^2 \left[ \theta - 2 \cos \theta + \frac{2\theta - \sin(2\theta)}{4} \right]_0^{2\pi} = \frac{3}{2} \pi R^2. \]

\[ = 4R - \sin \theta + \cos \theta + \sin \theta + \cos \theta + \]

\[ \left[ \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) \right] \]

\[ \left[ \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) \right] \]

\[ \left[ \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) \right] \]

\[ \left[ \sin \theta + \cos \theta + \sin \theta + \cos \theta + \sin \theta + \cos \theta + \sin \theta + \cos \theta \right] \]

---

**Mathematics 22 - Midterm Examination One**

Name:

Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit. This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so.
1. (30 points)
2. (30 points)
3. (20 points)
4. (20 points)

Total (100 points)
1. Evaluate the following integrals by using integration by parts, substitution rule or partial fraction

(1) \[ \int \exp(4x) \cos(3x) \, dx \]
(2) \[ \int 3x (\sin x)^2 \cos x \, dx \]
(3) \[ \int \frac{2x \sin x}{(\cos x)^3} \, dx \]
(4) \[ \int (\arcsin x + 2x \arctan x) \, dx \]
(5) \[ \int \frac{-2 \arctan x}{x^3} \, dx \]
(6) \[ \int \frac{-2x \arctan x}{(1 + x^2)^2} \, dx \]
2. Evaluate the following integrals

(1) \[ \int (\sin x)^3(\cos x)^3 \, dx \]

(2) \[ \int (\cos x)^4 \, dx \]

(3) \[ \int (\tan x)^4(\sec x)^2 \, dx + \int (\sec x)^5 \tan x \, dx \]

(4) \[ \int \frac{\ln x}{x^2 + (\ln x)^2} \, dx \]

(5) \[ \int \frac{x^4 + x^2 + 1}{x^3 + x} \, dx \]

(6) \[ \int \frac{2x \exp(x^2)}{1 + \exp(2x^2)} \, dx \]
3. Determine if each of the following improper integrals is convergent or divergent. If it is convergent, then find the value of the improper integral.

(1) \( \int_{0}^{1} 4x(\ln x)^2 \, dx \)

(2) \( \int_{e}^{\infty} \frac{1}{x \ln x[1 + (\ln x)^2]} \, dx \)

(3) \( \int_{0}^{\infty} \frac{2x}{(1 + x^2)^2} \, dx \)

(4) \( \int_{0}^{\infty} \frac{30}{(x^2 + 11x + 24)(x^2 + 11x + 30)} \, dx \)
4. Solve the following differential equations for solutions

(1) \[ 4y(1 + y^2)\ln(1 + y^2)\frac{dy}{dx} = 2x^3 \cos(x^2) - 2x^3 \sin(x^2). \]

(2) \[ x^2 \frac{dy}{dx} + 2xy = \frac{1}{x^2(1 + x^2)} + \frac{1}{x(1 + x^2)}. \]

(3) \[ \exp(x)\frac{dy}{dx} + \exp(x)y = 2x \cos(x^2) - 2x \sin(x^2). \]
(Bonus problems) Evaluate the following integrals

(1) \[ \int 22(\sin x + \cos x)^{20} \cos(2x) \, dx \]

(2) \[ \int \frac{4 \sin x \cos x}{(\sin x)^4 + (\cos x)^4} (\tan x)^6 \, dx \]
1. Let us apply the integration by parts twice. We have

(1) \[ \int \exp(4x) \cos(3x)dx \]

\[ = \frac{1}{3} \int \exp(4x) \sin(3x) \] 

\[ = \frac{1}{3} \exp(4x) \sin(3x) - \frac{4}{3} \int \exp(4x) \sin(3x)dx \]

\[ = \frac{1}{3} \exp(4x) \sin(3x) + \frac{4}{9} \int \exp(4x) \cos(3x)dx \]

\[ = \frac{1}{3} \exp(4x) \sin(3x) + \frac{4}{9} \exp(4x) \cos(3x) - \frac{16}{9} \int \exp(4x) \cos(3x)dx, \]

\[ \frac{25}{9} \int \exp(4x) \cos(3x)dx = \frac{1}{3} \exp(4x) \sin(3x) + \frac{4}{9} \exp(4x) \cos(3x), \]

\[ \int \exp(4x) \cos(3x)dx = \frac{1}{25} \exp(4x)[4 \cos(3x) + 3 \sin(3x)] + C, \]

(2) \[ \int 3x \sin^2 x \cos x dx \]

\[ = \int x \sin^3 x \] 

\[ = x \sin^3 x - \int (\sin^2 x) \] 

\[ = x \sin^3 x - \int [1 - (\cos x)^2] \sin x dx \]

\[ = x \sin^3 x + \cos x - \frac{1}{3} (\cos x)^3 + C, \]

(3) \[ \int \frac{2x \sin x}{(\cos x)^3} dx \]

\[ = \int x \left[ \frac{1}{(\cos x)^2} \right]' \] 

\[ = \frac{x}{(\cos x)^2} - \int \frac{1}{(\cos x)^2} dx = \frac{x}{(\cos x)^2} - \tan x + C \]
\begin{align*}
\text{(4)} & \quad \int (\arcsin x + 2x \arctan x) \, dx \\
& = \int x' \arcsin x \, dx + \int (1 + x^2)' \arctan x \, dx \\
& = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx + (1 + x^2) \arctan x - \int \frac{1 + x^2}{1 + x^2} \, dx \\
& = x \arcsin x + \sqrt{1-x^2} + (1 + x^2) \arctan x - x + C, \\
\text{(5)} & \quad \int - \frac{2 \arctan x}{x^3} \, dx \\
& = \int \left( \frac{1}{x^2} \right)' \arctan x \, dx \\
& = \frac{1}{x^2} \arctan x - \int \frac{1}{x^2 (1 + x^2)} \, dx \\
& = \frac{1}{x^2} \arctan x - \int \left( \frac{1}{x^2} - \frac{1}{1 + x^2} \right) \, dx \\
& = \frac{1}{x^2} \arctan x + \frac{1}{x} + \arctan x + C, \\
\text{(6)} & \quad \int - \frac{2x \arctan x}{(1 + x^2)^2} \, dx \\
& = \int \left( \frac{1}{1 + x^2} \right)' \arctan x \, dx \\
& = \frac{1}{1 + x^2} \arctan x - \int \frac{1}{(1 + x^2)^2} \, dx \\
& = \frac{1}{1 + x^2} \arctan x - \int \frac{1 + x^2 - x^2}{(1 + x^2)^2} \, dx \\
& = \frac{1}{1 + x^2} \arctan x - \int \frac{1}{1 + x^2} \, dx + \int \frac{x^2}{(1 + x^2)^2} \, dx \\
& = \frac{1}{1 + x^2} \arctan x - \int \frac{1}{1 + x^2} \, dx - \frac{1}{2} \int x \left( \frac{1}{1 + x^2} \right)' \, dx \\
& = \frac{1}{1 + x^2} \arctan x - \arctan x - \frac{1}{2} \frac{x}{1 + x^2} + \frac{1}{2} \arctan x + C.
\end{align*}
2.

(1) \[ \int (\sin x)^3 (\cos x)^3 \, dx \]
\[ = \int (\sin x)^3 [1 - (\sin x)^2] \cos x \, dx \]
\[ = \int u^3(1 - u^2) \, du = \frac{1}{4} (\sin x)^4 - \frac{1}{6} (\sin x)^6 + C \]

(2) \[ \int (\cos x)^4 \, dx \]
\[ = \frac{1}{4} \int [1 + \cos(2x)]^2 \, dx \]
\[ = \frac{1}{4} \int \{1 + 2 \cos(2x) + [\cos(2x)]^2\} \, dx \]
\[ = \frac{1}{4} \int \left[ 1 + 2 \cos(2x) + \frac{1 + \cos(4x)}{2} \right] \, dx \]
\[ = \frac{3}{8} x + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C, \]

(3) \[ \int 5(\tan x)^4 (\sec x)^2 \, dx + \int 5(\sec x)^5 \tan x \, dx \]
\[ = \frac{1}{5} (\tan x)^5 + \frac{1}{5} (\sec x)^5 + C, \]

(4) \[ \int \frac{\ln x}{x \sqrt{1 + (\ln x)^2}} \, dx \]
\[ = \sqrt{1 + (\ln x)^2} + C \]

(5) \[ \int \frac{x^4 + x^2 + 1}{x^3 + x} \, dx \]
\[ = \int \left( x + \frac{1}{x} - \frac{x}{1 + x^2} \right) \, dx = \frac{1}{2} x^2 + \ln |x| - \frac{1}{2} \ln(1 + x^2) + C, \]

(6) \[ \int \frac{2x \exp(x^2)}{1 + \exp(2x^2)} \, dx \]
\[ = \arctan(\exp(x^2)) + C \]
3.

(1) \[ \int_0^1 4x(\ln x)^2 \, dx = \left[ 2x^2(\ln x)^2 - 2x^2 \ln x + x^2 \right] |_0^1 = 1, \]

(2) \[ \int_e^\infty \frac{1}{x \ln x[1 + (\ln x)^2]} \, dx = \left\{ \ln(\ln x) - \frac{1}{2} \ln[1 + (\ln x)^2] \right\} |_e^\infty = \frac{1}{2} \ln 2, \]

(3) \[ \int_0^\infty \frac{2x}{(1 + x^2)^2} \, dx = -\frac{1}{1 + x^2} |_0^\infty = 1, \]

(4) \[ \int_0^\infty \frac{30}{(x^2 + 11x + 30)(x^2 + 11x + 30)} \, dx = \int \left( \frac{5}{x^2 + 11x + 24} - \frac{5}{x^2 + 11x + 30} \right) \, dx = \left[ \ln \left( \frac{x + 3}{x + 8} \right) + 5 \ln \left( \frac{x + 6}{x + 5} \right) \right] |_0^\infty = 5 \ln \frac{6}{5} - \ln \frac{8}{3}. \]
4. Solutions: (1) Let \( u = 1 + y^2 \). Then \( du = 2ydy \).

\[
\int 4y(1 + y^2) \ln(1 + y^2)dy
\]

\[
= \int 2u \ln u du
\]

\[
= \int (u^2)' \ln u du
\]

\[
= u^2 \ln u - \int u du
\]

\[
= u^2 \ln u - \frac{1}{2}u^2
\]

\[
= (1 + y^2)^2 \ln(1 + y^2) - \frac{1}{2}(1 + y^2)^2.
\]

Let \( v = x^2 \). Then \( dv = 2xdx \).

\[
\int [2x^3 \cos(x^2) - 2x^3 \sin(x^2)]dx
\]

\[
= \int (v \cos v - v \sin v)dv
\]

\[
= \int v(\sin v + \cos v)'dv
\]

\[
= v(\sin v + \cos v) - \int (\sin v + \cos v)dv
\]

\[
= v(\sin v + \cos v) + \cos v - \sin v + C
\]

\[
= x^2[\sin(x^2) + \cos(x^2)] + \cos(x^2) - \sin(x^2) + C.
\]

\[
(1 + y^2)^2 \ln(1 + y^2) - \frac{1}{2}(1 + y^2)^2
\]

\[
= x^2[\sin(x^2) + \cos(x^2)] + \cos(x^2) - \sin(x^2) + C.
\]

(2)

\[
\frac{d}{dx}(x^2y) = \frac{1}{x^2} - \frac{1}{1 + x^2} + \frac{1}{x} - \frac{x}{1 + x^2}.
\]
\[ x^2 y = -\frac{1}{x} - \arctan x + \ln |x| - \frac{1}{2} \ln(1 + x^2) + C. \]

(3)

\[
\frac{d}{dx} \left[ \exp(x) y \right] = \frac{d}{dx} \left[ \sin(x^2) + \cos(x^2) + C \right].
\]

\[
y = \frac{\sin(x^2) + \cos(x^2) + C}{\exp(x)}.
\]

**Bonus**

(1) \[
\int 22(\sin x + \cos x)^{20} \cos(2x)dx = \int 22[1 + \sin(2x)]^{10} \cos(2x)dx = [1 + \sin(2x)]^{11} + C,
\]

(2) \[
\int \frac{4 \sin x \cos x}{(\sin x)^4 + (\cos x)^4} (\tan x)^6 dx = \int \frac{4 \tan x (\sec x)^2}{1 + (\tan x)^4} (\tan x)^6 dx = \int \frac{2u^3}{1 + u^2} du = u^2 - \ln(1 + u^2) + C = (\tan x)^4 - \ln[1 + (\tan x)^4] + C \text{ where } u = (\tan x)^2, du = 2 \tan x (\sec x)^2 dx
\]

**Mathematics 21 - Midterm Examination One**

Name:

This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not
start working before you are told to do so. Show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit.

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1. Find the following limit
\[
\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}
\]

2. Use squeeze theorem to find the following limit
\[
\lim_{x \to 0} x^2 \cos \left( \frac{1}{x^2} \right)
\]

3. Use the \( \varepsilon - \delta \) definition to prove that
\[
\lim_{x \to 3} (4x - 5) = 7.
\]

4. Find the following limit
\[
\lim_{x \to \infty} \left( \sqrt{x^2 + 4x} - \sqrt{x^2 + 3x} \right).
\]

5. Define
\[
f(x) = \begin{cases} 
\frac{x^2 + x - 6}{x^2} + 2x & \text{for all } x \leq 2, \\
\frac{x^2 + 2x - 8}{x^2 - 2} + c & \text{for all } x > 2.
\end{cases}
\]

Find the value of the real constant \( c \) so that the function \( f \) is continuous everywhere on \( \mathbb{R} \).

6. Show that the equation \( e^x + x + \sin x = 10 \) has a solution in the open interval \((0, 11)\).

7. Find an equation of the tangent line to the parabola \( y = x^2 \) at the point \((a, a^2)\).

8. Find the derivative of \( f(x) = e^x(\sin x + \cos x) \).

9. Find the derivative of \( f(x) = \frac{\sin x + \cos x}{e^x} \).

10. Find the derivative of \( f(x) = \sin(e^x) + \exp(\cos x) \).
This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. Show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit.

Problem 1 /10 points
Problem 2 /10 points
Problem 3 /10 points
Problem 4 /10 points
Problem 5 /10 points
Problem 6 /10 points
Problem 7 /10 points
Problem 8 /10 points
Problem 9 /10 points
Problem 10 /10 points
Total /100 points
1. Let
\[ f(x) = \ln \frac{(1 + x^2)^6(1 + x^4)^7}{(x + e^x)^8(x + \ln x)^9} \]
Find the derivative of \( f(x) \).

2. Use the implicit differentiation rule to find the derivative \( y' \) if
\[ \sin x + \cos y = \arcsin x + \arccos y. \]

3. Prove that
\[ \arctan x + \arccot x = \frac{\pi}{2}, \quad \text{on} \ (-\infty, \infty). \]

4. Let
\[ f(x) = x \exp(-x^2). \]
Find the intervals on which the function is increasing or decreasing.

5. Let
\[ f(x) = x^3 + 6x^2 + 9x. \]
Find the intervals on which the function is concave upward or concave downward.

6. Let
\[ f(x) = x^2 \exp(-x^2). \]
Find the absolute maximum and absolute minimum of \( f \).
Hint:
\[ f'(x) = 2x \exp(-x^2) - 2x^3 \exp(-x^2) = 2x(1 - x^2) \exp(-x^2). \]

7. Use L’Hospital’s rule to calculate the limit
\[ \lim_{x \to \infty} \frac{(\ln x)^2}{x} = \lim_{x \to \infty} \frac{2 \ln x}{x} = \lim_{x \to \infty} \frac{2}{x} = 0. \]

8. Find the area of the largest rectangle that can be inscribed in a semicircle of radius \( r = 10 \).
9. Draw the graph of

\[ f(x) = \ln(5 - x^2). \]

Hint:

\[ f'(x) = \frac{-2x}{5 - x^2}, \quad f''(x) = -2 \frac{5 + x^2}{(5 - x^2)^2}. \]

Is the graph of the function symmetric about the y-axis or about the origin?

Find the intervals where the function \( f \) is increasing or decreasing.

Find the intervals where the function \( f \) is concave upward or downward.

Find the horizontal asymptote and the vertical asymptote if it exists.

Find the absolute maximum and absolute minimum of \( f \).

Draw the graph of the function.

10. Find the antiderivative of

\[
\begin{align*}
f(x) &= \sin x + 2 \cos x + 3(\sec x)^2 + 10x^9 + 4e^x \\
&\quad + \frac{5}{x} + \frac{6}{2\sqrt{x}} + \frac{7}{1 + x^2} + \frac{8}{\sqrt{1 - x^2}}.
\end{align*}
\]
This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. Show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit.

Problem 1 /20 points
Problem 2 /20 points
Problem 3 /10 points
Problem 4 /10 points
Problem 5 /10 points
Problem 6 /10 points
Problem 7 /10 points
Problem 8 /10 points
Problem 9 /20 points
Problem 10 /20 points
Problem 11 /10 points
Problem 12 /10 points
Problem 13 /10 points
Problem 14 /10 points
Problem 15 /20 points
Total /200 points
1. Find the following limits

   (1) \( \lim_{x \to 0} \left\{ x^2 \sin \left( \frac{1}{x^2} \right) + x^4 \cos \left( \frac{1}{x^4} \right) \right\} = 0 \)

   (2) \( \lim_{x \to 5} \frac{\sqrt{x + 4} - 3}{x - 5} \)

   (3) \( \lim_{x \to 0} \left( x^2 + \frac{|x|}{x} \right) \)

   (4) \( \lim_{x \to 1} \arcsin \left( \frac{x - 1}{x^2 - 1} \right) \)

2. Let

   \[ f(x) = x^x (\sin x)^{\sin x}. \]

   Find the derivative \( f'(x) \).

3. Find the derivative \( \frac{dy}{dx} \) if

   \[ x^2 (\arcsin y + y) = y (\arccos x + x). \]

4. Prove that there exists a unique solution to the equation

   \[ e^x + x + \sin x = 10. \]

5. Prove that

   (1) \( | \sin a - \sin b | \leq |a - b|, \)
   (2) \( | \cos a - \cos b | \leq |a - b|, \)

for all real numbers \( a \) and \( b \).

6. Let

   \[ f(x) = x^2 (\ln x)^2. \]

   Find the intervals where the function \( f \) is increasing or decreasing.
7. Let

\[ f(x) = x \exp(-2x^2). \]

Find the local maximum and local minimum of the function \( f(x) \).

8. Let

\[ f(x) = x^2 e^x. \]

Find the intervals where the graph of the function \( f(x) \) is concave upward or concave downward.

9. Express each limit as a definite integral and then calculate that integral

\[
(1) \quad \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\sqrt{1 - (-1 + 2k/n)^2}} \frac{2}{n} \\
(2) \quad \lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}
\]

10. Use the properties of integrals to prove the estimates

\[
(1) \quad \int_{0}^{\pi/2} x \sin x \, dx \leq \frac{\pi^2}{8} \quad \text{Hint: } x \sin x \leq x \\
(2) \quad \int_{0}^{\pi/6} \cos(x^2) \, dx \geq \frac{1}{2} \quad \text{Hint: } \cos(x^2) \geq \cos x
\]

11. Use the Fundamental Theorem of Calculus and the chain rule to find the derivatives of all functions

\[
(1) \quad \int_{0}^{\sin x + \cos x} t^3 \, dt \\
(2) \quad \int_{\arcsin x}^{\arccos x} t^2 \ln t \, dt
\]
12. Evaluate the definite integrals by using the substitution rule

\[(1) \int_{-\pi/2}^{\pi/2} 2 \sin x \cos x \, dx\]
\[(2) \int_{0}^{1} \left[ \frac{12(\text{arcsin} \, x)^{11}}{\sqrt{1 - x^2}} + \frac{12(\text{arctan} \, x)^{11}}{1 + x^2} \right] \, dx\]

13. Find the area of the region bounded by the curves \( f(x) = 12x \sin(x^2) \) and \( g(x) = -12x \sin(x^2) \) on the interval \([0, \sqrt{\pi}]\).

14. If \( f \) is a continuous function such that
\[
\int_{0}^{x} f(t) \, dt = x \exp(2x) + x \ln x - \int_{0}^{x} \exp(-t) f(t) \, dt
\]
for all \( x \), find an explicit formula for \( f(x) \).

15. Use the L’Hospital’s rule to find the following limits

\[(1) \lim_{x \to 0} \frac{\sin(x^2)}{\sin(x^2)},\]
\[(2) \lim_{x \to 3} \left[ \frac{1}{x - 3} \int_{3}^{x} \frac{\sin((t - 3)^2)}{t - 3} \, dt \right].\]
Mathematics 52: Survey of Calculus II

Professor Linghai Zhang. Office: Christmas-Saucon Hall 236. Office Hours: Mondays, Tuesdays, Wednesdays, Thursdays, 1PM-2PM and by appointments. Office Telephone: 610-758-4116. Email: liz5@lehigh.edu.


Attendance is strongly required as the pace of the class is very fast. If your final grade is on the borderline, then your attendance may play a positive/negative role.

Homework will be assigned every meeting day and will be collected the next meeting day. Quizzes will be given everyday unless there is a midterm exam on that day.

Quizzes: 100 points.

Midterm Examination One: 100 points.

Midterm Examination Two: 100 points.

Final Examination: 200 points.

Sample examinations will be given.

Total Scores: 500 points.

Grading Policy:
91% - 100%: A, A-;
81% - 90%: B+, B, B-;
71% - 80%: C+, C, C-;
61% - 70%: D;
0% - 60%: F.
Other parts of the syllabus as the same as that of Math 21.
Mathematics 52 - Midterm Exam I

1. (30 points) Solve the following indefinite integrals

   (1) \[ \int \frac{10[1 + \ln(\ln x)]^9}{x \ln x} \, dx \]

   (2) \[ \int 2x^3 \cos(x^2) \, dx \]

   (3) \[ \int 4x(\ln x)^2 \, dx \]

2. (30 points) Evaluate the following definite integrals

   (1) \[ \int_{0}^{\pi} 4x \cos(2x) \, dx \]

   (2) \[ \int_{-1}^{1} 2x^3 \exp(x^2) \, dx \]

   (3) \[ \int_{\ln(\pi/2)}^{\ln \pi} e^x \cos(e^x) \, dx \]

3. (10 points) Solve the following separable differential equation

   \[ 4y(1 + y^2) \ln(1 + y^2) \frac{dy}{dx} = 2x \cos(1 + x^2) + 2x \sin(1 + x^2). \]

4. (10 points) Solve the following separable differential equation

   \[ (2y + \exp(y)) \cos(y^2 + \exp(y)) \frac{dy}{dx} = [2x \cos(x^2) - 2x \sin(x^2)] \exp[\sin(x^2) + \cos(x^2)]. \]

5. (10 points) Solve the differential equation by using an integrating factor

   \[ (\sin x) \frac{dy}{dx} + (\cos x)y = -\sin x + 2 \sin x \cos x. \]
6. (10 points) Solve the first order linear differential equation

\[ \frac{dy}{dx} + 2xy = 2x + 2x \exp(-x^2). \]
Problem 1 (5 points)
Problem 2 (25 points)
Problem 3 (20 points)
Problem 4 (20 points)
Problem 5 (10 points)
Problem 6 (20 points)
Total: (100 points)
1. Let $R > 1$ be a positive constant. Find the sum of the geometric series if it is convergent

$$\sum_{n=0}^{\infty} \left( \frac{2R}{1+R^2} \right)^n = 1 + 2\frac{R}{1+R^2} + \left( 2\frac{R}{1+R^2} \right)^2 + 2\left( \frac{R}{1+R^2} \right)^3 + \left( 2\frac{R}{1+R^2} \right)^4 + \left( 2\frac{R}{1+R^2} \right)^5 + \cdots$$

2. Find the Taylor series for each of the following functions. Show work to support your solutions.

(1A) $\frac{1}{1-x}$ \hspace{1cm} (1B) $\frac{1}{1+x}$

(2A) $\frac{1}{(1-x)^2}$ \hspace{1cm} (2B) $\frac{1}{(1+x)^2}$

(3A) $\ln(1-x)$ \hspace{1cm} (3B) $\ln(1+x)$

(4A) $\ln(1-x^2)$ \hspace{1cm} (4B) $\ln(1+x^2)$

(5A) $\frac{1}{1+x^2}$ \hspace{1cm} (5B) $\arctan(x)$

3. Use the integral test to determine whether each series is convergent or divergent.

(1) $\sum_{n=1}^{\infty} \frac{2n}{(1+n^2)^2}$

(2) $\sum_{n=3}^{\infty} \frac{1}{n(n\ln n)^2}$

(3) $\sum_{n=1}^{\infty} \frac{2}{n^3} \sin \left( \frac{1}{n^2} \right)$

(4) $\sum_{n=1}^{\infty} 2n[1 + \cos(n^2)]$
4. Use the comparison test to determine whether each series is convergent or divergent. Show all work to receive credit.

\[(1) \sum_{n=1}^{\infty} \frac{5n + 6n^2}{7n^5 + 8n^6}\]
\[(2) \sum_{n=1}^{\infty} \frac{5^n + 6^n}{7^n + 8^n}\]
\[(3) \sum_{n=1}^{\infty} \frac{1}{\left(\frac{6}{5}\right)^n + \left(\frac{7}{8}\right)^n}\]
\[(4) \sum_{n=1}^{\infty} \frac{1}{\left(\frac{5}{6}\right)^n + \left(\frac{7}{8}\right)^n}\]

5. Define the following functions

\[(1) f(x, y, z) = \frac{\exp(x^2) - \exp(-x^2)}{\exp(x^2) + \exp(-x^2)} + \frac{\exp(y^2) - \exp(-y^2)}{\exp(y^2) + \exp(-y^2)} + \frac{\exp(z^2) - \exp(-z^2)}{\exp(z^2) + \exp(-z^2)},\]
\[(2) g(x, y, z) = \exp[(\sin x)^2 + (\sin y)^2 + (\sin z)^2],\]
\[(3) h(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.\]

Find the partial derivatives \(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\) and \(\frac{\partial f}{\partial z}\) for each function.

6. Let
\(\phi(x, y, z) = (\sin x)^2 + (\sin y)^2 + (\sin z)^2.\)

(1) Find the first order partial derivatives.
(2) Find the second order partial derivatives of \(\phi = \phi(x, y, z)\).
(3) Is $\phi(0, 0, 0)$ a local maximum or a local minimum or neither?
Justify your answer.
Mathematics 52 - Final Examination

Name:

This is a closed book test. Please close your textbook, notebook, cell phone. No calculators are allowed in this test. Please do not start working before you are told to do so. Please show all work to support your solutions. Our policy is that if you show no supporting work, you will receive no credit.

1. (20 points)
2. (20 points)
3. (10 points)
4. (20 points)
5. (20 points)
6. (20 points)
7. (20 points)
8. (10 points)
9. (20 points)
10. (10 points)
11. (20 points)
12. (10 points)

Total (200 points)
1. Use substitution rule and integration by parts to evaluate the following integrals

\( \int \frac{(\ln x) \cos(\ln x)}{x} \, dx \)

\( \int \frac{2}{x^5} \sin \left( \frac{1}{x^2} \right) \, dx \)

2. Use integration by parts and substitution rule to evaluate the following integrals

\( \int \arcsin x \, dx \)

\( \int 2x \arctan x \, dx \)

3. Determine if the following improper integrals are convergent or divergent. If it is convergent, then find its value

\( \int_0^{\infty} 5x^4 \exp(-x^5) \, dx \),

\( \int_1^{\infty} \frac{1}{x^2} \ln x \, dx \).

4. Solve each of the following differential equations for solutions

\( \frac{dy}{dx} - \frac{y}{2(1 + x)} = \frac{1}{2}, \quad x > -1, \)

\( 8y(1 + y^2) \ln(1 + y^2) \frac{dy}{dx} = (2x + e^x) \cos(x^2 + e^x) - (2x + e^x) \sin(x^2 + e^x). \)
5. Find the Taylor series of the following functions

(1) \( f(x) = \frac{1}{1 - x} \)

(2) \( f(x) = \frac{1}{1 + x} \)

(3) \( f(x) = \ln(1 - x) \)

(4) \( f(x) = \ln(1 + x) \)

(5) \( f(x) = \ln \frac{1 + x}{1 - x} \)

(6) \( f(x) = \ln \frac{1 + x^2}{1 - x^2} \)

(7) \( f(x) = \frac{1}{1 + x^2} \)

(8) \( f(x) = \text{arctan} \ x \)

6. Use the integral test to determine if the following series is convergent or divergent

(1) \[ \sum_{n=1}^{\infty} \left[ \frac{1}{n^2} \sin \left( \frac{1}{n} \right) + \frac{1}{n^3} \cos \left( \frac{1}{n} \right) \right] \]

(2) \[ \sum_{n=1}^{\infty} (2n e^{-n} - n^2 e^{-n}) \]

Hints: Compute the derivatives of \( f(x) = -\frac{1}{x} \sin(\frac{1}{x}) \) and \( g(x) = x^2 \exp(-x) \).

7. Use comparison test to determine if the following series is
convergent or divergent

(1) \[ \sum_{n=1}^{\infty} \frac{1}{n^4 + 2n^2 + 1} \]

(2) \[ \sum_{n=1}^{\infty} \frac{1}{5^n} \left[ \arccos \left( \frac{1}{n^2} \right) \right]^n \]

(3) \[ \sum_{n=1}^{\infty} \left[ \sin \left( \frac{1}{n^2} \right) \right]^2 \left[ \cos \left( \frac{1}{n^2} \right) \right]^2 \]

8. Let \( X \) represent a continuous random variable on the interval \((0, \infty)\) with the probability density function \( f(x) = 5x^9 \exp(-x^5) \).

(1) Find the cumulative distribution function \( F(x) \).

(2) Compute the probability \( P(1, \infty) \).

9. Let \( X \) denote a continuous random variable on the interval \((0, \infty)\) with the probability density function \( f(x) = k \exp(-kx) \), where \( k > 0 \) is a positive constant.

(1) Compute the expected value \( E(X) \).

(2) Compute the variance \( \text{Var}(X) \).

10. Let the function \( f = f(x, y) \) be defined by

\[ f(x, y) = \exp(-x^2 - y^2), \]

inside the circle \( x^2 + y^2 \leq 1 \).

(1) Compute the first order partial derivatives of the function \( f \).

(2) Compute the second order partial derivatives of the function \( f \).

(3) Find the maximum of the function \( f \).

11. Use the method of Lagrange multipliers to find the values of \( x \) and \( y \) that maximize the objective function \( f(x, y, z) = 2xy + 2yz + 2zx \) subject to the constraint \( g(x, y, z) = x^2 + y^2 + z^2 = 243 \).
12. Compute the double integral

\[ \int_{-1}^{1} \left[ \int_{-1}^{1} 2y(x^2 + y^2) \cos(x^2 + y^2) \, dy \right] \, dx \]
Math 21 - Solutions to Midterm Exam One

1. Solution: First of all, for all $x \neq 0$, we have

\[
\frac{\sqrt{x^2 + 9} - 3}{x^2} = \frac{(\sqrt{x^2 + 9} - 3)(\sqrt{x^2 + 9} + 3)}{x^2(\sqrt{x^2 + 9} + 3)}
\]

\[
= \frac{x^2}{x^2(\sqrt{x^2 + 9} + 3)} = \frac{1}{\sqrt{x^2 + 9} + 3}.
\]

Therefore

\[
\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 9} + 3} = \frac{1}{6}.
\]

2. Solution: For all real number $x \neq 0$, we know that $x^2 > 0$ and $-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$. Hence

\[-x^2 \leq x^2 \cos\left(\frac{1}{x^2}\right) \leq x^2.\]

It is easy to see that

\[
\lim_{x \to 0} x^2 = 0, \quad \lim_{x \to 0} -x^2 = 0.
\]

Based on the squeeze theorem, we find the limit

\[
\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0.
\]

3. Solution: For any positive number $\varepsilon > 0$, there exists another positive number $\delta = \frac{\xi}{4} > 0$, such that if $0 < |x - 3| < \delta = \frac{\xi}{4}$, then

\[|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < \varepsilon.\]

Therefore, we have

\[
\lim_{x \to 3} (4x - 5) = 7.
\]
4. Solution: First of all, we have

\[
(\sqrt{x^2 + 4x} - \sqrt{x^2 + 3x}) \\
= \frac{(\sqrt{x^2 + 4x} - \sqrt{x^2 + 3x})(\sqrt{x^2 + 4x} + \sqrt{x^2 + 3x})}{\sqrt{x^2 + 4x} + \sqrt{x^2 + 3x}} \\
= \frac{x}{\sqrt{x^2 + 4x} + \sqrt{x^2 + 3x}} = \frac{1}{\sqrt{1 + \frac{4}{x} + \sqrt{1 + \frac{3}{x}}}}.
\]

Therefore, we get

\[
\lim_{x \to \infty} (\sqrt{x^2 + 4x} - \sqrt{x^2 + 3x}) = \frac{1}{2}.
\]

5. Solution: Let us find the left hand limit and the right hand limit. We have

\[
\lim_{x \to 2^{-}} \frac{x^2 + x - 6}{x - 2} + 2x = \lim_{x \to 2^{-}} \frac{(x - 2)(x + 3)}{x - 2} + 2x = \lim_{x \to 2^{-}} x + 3 + 2x = 9,
\]

\[
\lim_{x \to 2^{+}} \frac{x^2 + 2x - 8}{x - 2} + c = \lim_{x \to 2^{+}} \frac{(x - 2)(x + 4)}{x - 2} + c = \lim_{x \to 2^{+}} x + 4 + c = 6 + c.
\]

To be continuous, the condition \( \lim_{x \to 2^{-}} f(x) = \lim_{x \to 2^{+}} f(x) \), namely \( 9 = 6 + c \), must be satisfied. Hence \( c = 3 \).

6. Solution: Define a function

\[ f(x) = e^x + x + \sin x. \]

This function is continuous on \([0, 11]\). It is easy to find that

\[ f(0) = 1 < 10, \quad f(11) = e^{11} + 11 + \sin(11) > 10. \]

Based on the intermediate value theorem, there exists a number \( c \), \( 0 < c < 11 \), such that \( f(c) = 10 \). That is

\[ e^c + c + \sin c = 10. \]
7. Solution: The slope of the tangent line is $2a$. The equation of the tangent line is

$$y - a^2 = 2a(x - a).$$

8. Solution: By using the product rule, we find the derivative

$$[e^x(\sin x + \cos x)]' = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x.$$ 

9. Solution: By using the quotient rule, we find the derivative

$$\left(\frac{\sin x + \cos x}{e^x}\right)' = \frac{e^x(\cos x - \sin x) - e^x(\sin x + \cos x)}{\exp(2x)} = -\frac{2\sin x}{e^x}.$$ 

10. Solution: By using the chain rule, we find the derivative

$$[\sin(e^x) + \exp(\cos x)]' = e^x \cos(e^x) - \sin x \exp(\cos x).$$

Math 21 - Solutions to Midterm Exam Two

1. Solution: First of all, by using the properties of the natural log function, we have

$$f(x) = 6\ln(1 + x^2) + 7\ln(1 + x^4) - 8\ln(x + e^x) - 9\ln(x + \ln x).$$ 

Then we find the derivative

$$f'(x) = \frac{12x}{1 + x^2} + \frac{28x^3}{1 + x^4} - \frac{8(1 + e^x)}{x + e^x} - \frac{9(1 + x)}{x(x + \ln x)}.$$ 

2. Solution: First of all, we have

$$\cos x - \frac{dy}{dx} \sin y = 1 \sqrt{1 - x^2} - \frac{1}{\sqrt{1 - y^2}} \frac{dy}{dx}.$$ 

Then

$$\frac{dy}{dx} \left(\frac{1}{\sqrt{1 - y^2}} - \sin y\right) = \frac{1}{\sqrt{1 - x^2}} - \cos x.$$
Now we obtain the derivative
\[
\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \cos x
\]
\[
\frac{1}{\sqrt{1-y^2}} - \sin y
\]

3. Proof. Define the following function
\[
f(x) = \arctan x + \text{arccot } x.
\]
This function is continuous and differentiable everywhere on \((-\infty, \infty)\). The derivative
\[
f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0.
\]
Therefore, \(f\) is a constant function on \((-\infty, \infty)\). Let \(x = 0\), we find
\[
f(0) = \arctan 0 + \text{arccot } 0 = \frac{\pi}{2}.
\]
Hence
\[
f(x) = \arctan x + \text{arccot } x = f(0) = \frac{\pi}{2}, \quad \text{on } (-\infty, \infty).
\]
The proof is finished.

4. Solution.
\[
f'(x) = (1 - 2x^2) \exp(-x^2) = 0 \text{ if } x = \pm \sqrt{\frac{1}{2}}.
\]
\[
f'(x) = (1 - 2x^2) \exp(-x^2) > 0 \text{ on } \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right).
\]
\[
f'(x) = (1 - 2x^2) \exp(-x^2) < 0 \text{ on } \left(-\infty, \sqrt{\frac{1}{2}}\right) \cup \left(\sqrt{\frac{1}{2}}, \infty\right).
\]
The function is increasing on $\left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$ and the function is decreasing on $\left(-\infty, \sqrt{\frac{1}{2}}\right) \cup \left(\sqrt{\frac{1}{2}}, \infty\right)$.

5. Solutions

$$f'(x) = 3x^2 + 12x + 9, \quad f''(x) = 6x + 12.$$

$$f''(x) = 0, \quad \text{if} \quad x = -2.$$

The function is concave upward on $(-2, \infty)$. The function is concave downward on $(-\infty, -2)$.

6. Solution

$$f'(x) > 0, \quad \text{on} \quad (-\infty, -1) \cup (0, 1),$$

$$f'(x) < 0, \quad \text{on} \quad (-1, 0) \cup (1, \infty).$$

The absolute maximum is $f(\pm 1) = \frac{1}{e}$ and the absolute minimum is $f(0) = 0$.

7.

8. The area of the rectangle is given by

$$A = 2x_0y_0 = 2r^2 \cos \theta \sin \theta = r^2 \sin(2\theta) = f(\theta).$$

The derivative is

$$f'(\theta) = 2r^2 \cos(2\theta) = 0 \quad \text{if} \quad \theta = \frac{\pi}{4}.$$

Now we find

$$x_0 = r \cos \theta = 10 \cos \frac{\pi}{4} = 5\sqrt{2}, \quad y_0 = r \sin \theta = 10 \sin \frac{\pi}{4} = 5\sqrt{2}.$$
asymptote. The absolute maximum is attained at $x = 0$, that is $f(0) = \ln 5$.

10. Solution. The antiderivative is

$$- \cos x + 2 \sin x + 3 \tan x + x^{10} + 4e^x + 5 \ln |x| + 6\sqrt{x} + 7 \arctan x + 8 \arcsin x + C.$$
Math 21 - Solutions to Final Exam 1. Solutions. (1) For all 
\( x \neq 0 \), we have

\[-1 \leq \sin \left( \frac{1}{x^2} \right) \leq 1,\]

\[-1 \leq \cos \left( \frac{1}{x^4} \right) \leq 1.\]

Thus

\[-x^2 \leq x^2 \sin \left( \frac{1}{x^2} \right) \leq x^2,\]

\[-x^4 \leq x^4 \cos \left( \frac{1}{x^4} \right) \leq x^4.\]

By using the Squeeze Theorem, the first limit is equal to zero.

(2) The second limit is equal to \( \frac{1}{6} \) because

\[
\lim_{x \to 5} \frac{\sqrt{x+4} - 3}{x-5} = \lim_{x \to 5} \frac{\sqrt{x+4} - 3}{(x-5)(\sqrt{x+4} + 3)}\]

\[= \frac{x-5}{(x-5)(\sqrt{x+4} + 3)} = \frac{1}{\sqrt{x+4} + 3},\]

\[= \lim_{x \to 5} \frac{1}{\sqrt{x+4} + 3} = \frac{1}{6}.\]

(3) The third limit does not exist because

\[
\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1, \quad \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1.
\]

(4) The last limit is equal to

\[
\arcsin \left( \lim_{x \to 1} \frac{x - 1}{x^2 - 1} \right) = \arcsin \left( \lim_{x \to 1} \frac{1}{x + 1} \right) = \arcsin \left( \frac{1}{2} \right) = \frac{\pi}{6}.
\]

2.

Solution. Let

\[ y = g(x)^{g(x)}. \]
Then

\[ \ln y = g(x) \ln g(x). \]

Compute the derivatives of both sides of the last equation. We have

\[ \frac{1}{y} \frac{dy}{dx} = g'(x) \ln g(x) + g'(x) = g'(x)[1 + \ln g(x)]. \]

Now

\[ \frac{dy}{dx} = g'(x)[1 + \ln g(x)]g(x)^{g(x)}. \]

By using this formula, we find

\[
\frac{d}{dx} (x^x) = (1 + \ln x)x^x, \\
\frac{d}{dx} [(\sin x)^{\sin x}] = \cos x[1 + \ln(\sin x)](\sin x)^{\sin x}.
\]

Now by using the product rule and the above results, we obtain the derivative

\[
\frac{d}{dx} [x^x(\sin x)^{\sin x}] = (1 + \ln x)x^x(\sin x)^{\sin x} + \cos x[1 + \ln(\sin x)]x^x(\sin x)^{\sin x}.
\]

3. Solution: Computing the derivative of both sides of the equation, we get

\[
2x(\arcsin y + y) + x^2 \left( \frac{1}{\sqrt{1 - y^2}} + 1 \right) \frac{dy}{dx} = (\arctan x + x) \frac{dy}{dx} + y \left( -\frac{1}{\sqrt{1 - x^2}} + 1 \right).
\]

That is

\[
\left[ x^2 \left( 1 + \frac{1}{\sqrt{1 - y^2}} \right) - x - \arctan x \right] \frac{dy}{dx} = y \left( 1 - \frac{1}{\sqrt{1 - x^2}} \right) - 2x(\arcsin y + y).
\]
Finally, we get the derivative
\[
\frac{dy}{dx} = \frac{y \left( 1 - \frac{1}{\sqrt{1-x^2}} \right) - 2x(\arcsin y + y)}{x^2 \left( 1 + \frac{1}{\sqrt{1-y^2}} \right) - x - \arctan x}.
\]

4. Solution. Define the function
\[
f(x) = e^x + x + \sin x.
\]
It is easy to see that the function \( f \) is continuous and differentiable everywhere. Moreover
\[
f(0) = e^0 + 0 + \sin 0 = 1 < 10. \quad f(10) = e^{10} + 10 + \sin(10) > 10.
\]
By using the intermediate value theorem, there exists a real number \( c, \ 0 < c < 10 \), such that \( f(c) = 10 \), that is
\[
e^c + c + \sin c = 10.
\]
Suppose that there are two distinct solutions \( c_1 < c_2 \) to the equation \( f(x) = 10 \), that is \( f(c_1) = 10 \) and \( f(c_2) = 10 \). By using the mean value theorem, we know that
\[
\frac{f(c_1) - f(c_2)}{c_1 - c_2} = f'(c_0),
\]
where \( c_0 \) is a real number, \( c_1 < c_0 < c_2 \). However, the left hand side of the above equation is equal to zero and the right hand side
\[
f'(c_0) = e^{c_0} + 1 + \cos c_0 > 0.
\]
This is a contradiction. Therefore, there exists a unique solution to the equation \( e^x + x + \sin x = 10 \).

5. Solutions. (1) By using Lagrange mean value theorem, for all real numbers \( a \) and \( b \), if \( a \neq b \), then we have
\[
\frac{\sin a - \sin b}{a - b} = \cos c,
\]
where \( c \) is a real number and \( a < c < b \). Therefore
\[
\left| \frac{\sin a - \sin b}{a - b} \right| = |\cos c| \leq 1.
\]
That is
\[
|\sin a - \sin b| \leq |a - b|.
\]
(2) By using Lagrange mean value theorem, for all real numbers \( a \) and \( b \), if \( a \neq b \), then we have
\[
\frac{\cos a - \cos b}{a - b} = -\sin c,
\]
where \( c \) is a real number and \( a < c < b \). Therefore
\[
\left| \frac{\cos a - \cos b}{a - b} \right| = |\sin c| \leq 1.
\]
That is
\[
|\cos a - \cos b| \leq |a - b|.
\]
\[
f'(x) = 2x(\ln x)^2 + 2x \ln x = 2x(1 + \ln x) \ln x.
\]
Hence
\[
f'(c) = 0, \quad \text{if } c = e^{-1}, c = 1,
\]
\[
f'(x) > 0, \quad \text{on } (0, e^{-1}) \cup (1, \infty),
\]
\[
f'(x) < 0, \quad \text{on } (e^{-1}, 1).
\]
Therefore, the function is increasing on the interval \((0, e^{-1})\cup(1, \infty)\).
The function is decreasing on the interval \((e^{-1}, 1)\). 7. Solutions.
\[
f'(x) = \exp(-2x^2) - 4x^2 \exp(-2x^2) = (1 - 4x^2) \exp(-2x^2).
\]
Now
\[ f'(x) > 0, \quad \text{on } \left( -\frac{1}{2}, \frac{1}{2} \right), \]
\[ f'(x) < 0, \quad \text{on } \left( -\infty, -\frac{1}{2} \right) \cup \left( \frac{1}{2}, \infty \right). \]

Therefore, the local maximum is \( f \left( \frac{1}{2} \right) = \frac{1}{2} \exp \left( -\frac{1}{2} \right) \) and the local minimum is \( f \left( -\frac{1}{2} \right) = -\frac{1}{2} \exp \left( -\frac{1}{2} \right) \).

8. Solutions.
\[ f'(x) = 2xe^x + x^2e^x = (x^2 + 2x)e^x, \]
\[ f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = (2 + 4x + x^2)e^x. \]

It is easy to see that
\[ f''(x) = 0, \quad \text{if } x = -2 \pm \sqrt{2}, \]
\[ f''(x) > 0, \quad \text{on } (-\infty, -2 - \sqrt{2}) \cup (-2 + \sqrt{2}, \infty), \]
\[ f''(x) < 0, \quad \text{on } (-2 - \sqrt{2}, -2 + \sqrt{2}). \]

The graph of the function is concave upward on \((-\infty, -2 - \sqrt{2}) \cup (-2 + \sqrt{2}, \infty)\) and is concave downward on \((-2 - \sqrt{2}, -2 + \sqrt{2})\).

9. Solutions:
\[ \int_{1}^{1} \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin x|_1^1 = \pi, \]
\[ \int_{0}^{1} \frac{1}{1 + x^2} \, dx = \arctan x|_0^1 = \frac{\pi}{2}. \]

10. Solutions.
\[ \int_{0}^{\pi/2} x \sin x \, dx \leq \int_{0}^{\pi/2} x \, dx = \frac{1}{2} x^2|_0^{\pi/2} = \frac{\pi^2}{8}, \]
\[ \int_{0}^{\pi/6} \cos(x^2) \, dx \geq \int_{0}^{\pi/6} \cos x \, dx = \sin x|_0^{\pi/6} = \frac{1}{2}. \]
11. Solutions. The derivatives are equal to

\[ \begin{align*}
(1) & \quad (\sin x + \cos x)^3 (\cos x - \sin x) \\
(2) & \quad \frac{(\arcsin x)^2 \ln(\arcsin x)}{\sqrt{1 - x^2}} + \frac{(\arccos x)^2 \ln(\arccos x)}{\sqrt{1 - x^2}}
\end{align*} \]

12. Solutions. The integrals are equal to

\[ \begin{align*}
(1) & \quad \int_{-\pi/2}^{\pi/2} 2 \sin x \cos x \, dx = (\sin x)^2 \bigg|_{-\pi/2}^{\pi/2} = 0, \\
(2) & \quad \int_{0}^{1} \left[ \frac{12(\arcsin x)^{11}}{\sqrt{1 - x^2}} + \frac{12(\arctan x)^{11}}{1 + x^2} \right] \, dx \\
& \quad = [ (\arcsin x)^{12} + (\arctan x)^{12} ] \bigg|_{0}^{1} = \left( \frac{\pi}{2} \right)^{12} + \left( \frac{\pi}{4} \right)^{12}.
\end{align*} \]

13. Solution. The area is equal to the definite integral

\[ \int_{0}^{\sqrt{\pi}} 24x \sin(x^2) \, dx = -12 \cos(x^2) \bigg|_{0}^{\sqrt{\pi}} = 24. \]

14. Solution. By using the Fundamental Theorem of Calculus, we see

\[ f(x) = \exp(2x) + 2x \exp(2x) + 1 + \ln x - \exp(-x)f(x). \]

Thus

\[ f(x) = \frac{\exp(2x) + 2x \exp(2x) + 1 + \ln x}{1 + \exp(-x)}. \]

15. Solutions. In the first problem, let \( t = \sin(x^2) \). As \( x \to 0 \), we
have \( t \to 0^+ \).

(1) \[ [\sin(x^2)]^{\sin(x^2)} = \exp\{\sin(x^2) \ln[\sin(x^2)]\} = \exp(t \ln t), \]

\[
\lim_{x \to 0} \{\sin(x^2) \ln[\sin(x^2)]\} = \lim_{t \to 0^+} \frac{\ln t}{t} = \lim_{t \to 0^+} \frac{t}{t^2} = \lim_{t \to 0^+} (-t) = 0,
\]

\[
\lim_{x \to 0^+} [\sin(x^2)]^{\sin(x^2)} = e^{\lim_{x \to 0^+} \{\sin(x^2) \ln[\sin(x^2)]\}} = e^{\lim_{t \to 0^+} (t \ln t)} = 1,
\]

(2) \[
\lim_{x \to 3} \left[ \frac{1}{x - 3} \int_3^x \frac{\sin((t - 3)^2)}{t - 3} \, dt \right] = \lim_{x \to 3} \frac{\sin((x - 3)^2)}{x - 3}
\]

\[
= \lim_{x \to 3} \frac{2(x - 3) \cos((x - 3)^2)}{1} = 0.
\]
Math 52 - Solutions to Midterm Exam I

1. \( \int \frac{10[1 + \ln(\ln x)]^9}{x \ln x} \, dx \)

\[ = [1 + \ln(\ln x)]^{10} + C, \]

\[ u = 1 + \ln(\ln x), \quad du = \frac{1}{x \ln x} \, dx, \]

(2) \( \int 2x^3 \cos(x^2) \, dx \)

\[ = x^2 \sin(x^2) + \cos(x^2) + C \]

(3) \( \int 4x(\ln x)^2 \, dx \)

\[ = \int (2x^2)'(\ln x)^2 \, dx = 2x^2(\ln x)^2 - \int 2x^2 \frac{2\ln x}{x} \, dx \]

\[ = 2x^2(\ln x)^2 - \int (2x^2)' \ln x \, dx \]

\[ = 2x^2(\ln x)^2 - 2x^2 \ln x + x^2 + C \]

2. 

(1) \( \int_0^\pi 4x \cos(2x) \, dx \)

\[ = \int u \cos u \, du = u \sin u + \cos u \]

\[ = [2x \sin(2x) + \cos(2x)]|_0^\pi = 0 \]

(2) \( \int_{-1}^1 2x^3 \exp(x^2) \, dx \)

\[ = (x^2 \exp(x^2) - \exp(x^2))|_{-1}^1 = 0 \]

(3) \( \int_{\ln(\pi/2)}^{\ln \pi} e^x \cos(e^x) \, dx \)

\[ = \sin(e^x)|_{\ln(\pi/2)}^{\ln \pi} = -1. \]
3.

\[ u = 1 + y^2, \quad du = 2ydy, \]
\[ \int 4y(1 + y^2) \ln(1 + y^2)dy \]
\[ = \int 2u \ln u du = \int (u^2)' \ln u du \]
\[ = u^2 \ln u - \frac{1}{2}u^2 \]
\[ = (1 + y^2)^2 \ln(1 + y^2) - \frac{1}{2}(1 + y^2)^2, \]

\[ v = 1 + x^2, \quad dv = 2xdx, \]
\[ \int [2x \cos(1 + x^2) + 2x \sin(1 + x^2)]dx \]
\[ = \int (\cos v + \sin v)dv \]
\[ = \sin v - \cos v + C = \sin(1 + x^2) - \cos(1 + x^2) + C, \]
\[ (1 + y^2)^2 \ln(1 + y^2) - \frac{1}{2}(1 + y^2)^2 = \sin(1 + x^2) - \cos(1 + x^2) + C \]

4.

\[ u = y^2 + \exp(y), \quad du = (2y + \exp(y))dy, \]
\[ \int (2y + \exp(y)) \cos(y^2 + \exp(y))dy \]
\[ = \int \cos u du = \sin u = \sin(y^2 + \exp(y)), \]
\[ v = \sin(x^2) + \cos(x^2), \quad dv = [2x \cos(x^2) - 2x \sin(x^2)]dx, \]
\[ \int [2x \cos(x^2) - 2x \sin(x^2)] \exp[\sin(x^2) + \cos(x^2)]dx \]
\[ = \int e^v dv = e^v + C = \exp[\sin(x^2) + \cos(x^2)] + C, \]
\[ \sin(y^2 + \exp(y)) = \exp[\sin(x^2) + \cos(x^2)] + C \]
5. \[
\frac{d}{dx}[(\sin x)y] = -\sin x + 2 \sin x \cos x.
\]
\[(\sin x)y = \cos x + (\sin x)^2 + C.
\]
\[y = \frac{\cos x + (\sin x)^2 + C}{\sin x}
\]

6. The integrating factor is \(\exp(x^2)\).

\[
\exp(x^2) \frac{dy}{dx} + 2x \exp(x^2)y = 2x \exp(x^2) + 2x.
\]
\[
\frac{d}{dx}(\exp(x^2)y) = 2x \exp(x^2) + 2x.
\]
\[\exp(x^2)y = \exp(x^2) + x^2 + C.
\]
\[y = 1 + (x^2 + C) \exp(-x^2).
\]
Math 52 - Solutions to Midterm Exam II

1. Solution: Because $R > 1$ so $2R < 1 + R^2$ and therefore the geometric series is convergent. The sum is

$$\frac{1}{1 - \frac{2R}{1 + R^2}} = \frac{1 + R^2}{(R - 1)^2}.$$ 

Find the sum of each of the following geometric series if it is convergent

(1) $\sum_{n=0}^{\infty} \frac{m^n}{(1 + m^2)^n}$

(2) $\sum_{n=0}^{\infty} \frac{7^n + 8^n}{10^n}$

where $m > 0$ is a positive constant.

2. Solution: Let $|x| < 1$ for all of the following series.

(1A) $\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$

(1B) $\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n,$

(2A) $\frac{1}{(1 - x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$

(2B) $\frac{1}{(1 + x)^2} = \sum_{n=0}^{\infty} (-1)^{n-1} nx^{n-1},$

(3A) $\ln(1 - x) = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1}$

(3B) $\ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1},$

(4A) $\ln(1 - x^2) = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{2n+2}$

(4B) $\ln(1 + x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2},$

(5A) $\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

(5B) $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$
3. Solution: We will use integral test in this problem.

\[
\int_1^\infty \frac{2x}{(1 + x^2)^2} \, dx = -\frac{1}{1 + x^2}\bigg|_1^\infty = \frac{1}{2},
\]

\[
\sum_{n=1}^{\infty} \frac{2n}{(1+n^2)^2} \text{ is convergent.}
\]

\[
\int_3^\infty \frac{1}{x(\ln x)^2} \, dx = -\frac{1}{\ln x}\bigg|_3^\infty = \frac{1}{\ln 3},
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2} \text{ is convergent.}
\]

\[
\int_1^\infty \frac{2}{x^3} \sin \left(\frac{1}{x^2}\right) \, dx = \cos \left(\frac{1}{x^2}\right)\bigg|_1^\infty = 1 - \cos 1,
\]

\[
\sum_{n=1}^{\infty} \frac{2}{n^2} \sin \left(\frac{1}{n^2}\right) \text{ is convergent.}
\]

\[
\int_1^\infty 2x[1 + \cos(x^2)] \, dx = [x^2 + \sin(x^2)]\bigg|_1^\infty \text{does not exist,}
\]

\[
\sum_{n=1}^{\infty} 2n[1 + \cos(n^2)] \text{ is divergent.}
\]

4. Solution: We will use comparison test in this problem.

\[
\frac{5n + 6n^2}{7n^5 + 8n^6} < \frac{6n^2 + 6n^2}{7n^5 + 7n^5} < \frac{1}{n^3}
\]

The series

\[
\sum_{n=1}^{\infty} \frac{5n + 6n^2}{7n^5 + 8n^6}
\]

is convergent.

\[
\frac{5^n + 6^n}{7^n + 8^n} < \frac{6^n + 6^n}{7^n + 7^n} = \left(\frac{6}{7}\right)^n.
\]

The series

\[
\sum_{n=1}^{\infty} \frac{5^n + 6^n}{7^n + 8^n}
\]
is convergent.

\[
\left( \frac{6}{5} \right)^n + \left( \frac{7}{8} \right)^n > \left( \frac{6}{5} \right)^n,
\]
\[
\frac{1}{\left( \frac{6}{5} \right)^n} + \left( \frac{7}{8} \right)^n < \frac{1}{\left( \frac{6}{5} \right)^n} = \left( \frac{5}{6} \right)^n.
\]

The series

\[
\sum_{n=1}^{\infty} \frac{1}{\left( \frac{6}{5} \right)^n + \left( \frac{7}{8} \right)^n}
\]

is convergent.

\[
\left( \frac{5}{6} \right)^n + \left( \frac{7}{8} \right)^n < 2 \left( \frac{7}{8} \right)^n.
\]
\[
\frac{1}{\left( \frac{5}{6} \right)^n} + \left( \frac{7}{8} \right)^n > \frac{1}{2 \left( \frac{7}{8} \right)^n} = \frac{1}{2} \left( \frac{8}{7} \right)^n.
\]

The series

\[
\sum_{n=1}^{\infty} \frac{1}{\left( \frac{5}{6} \right)^n + \left( \frac{7}{8} \right)^n}
\]

is divergent.

5. Solution: (1)

\[
\frac{\partial f}{\partial x} = \frac{8x}{(\exp(x^2) + \exp(-x^2))^2}, \quad \frac{\partial f}{\partial y} = \frac{8y}{(\exp(y^2) + \exp(-y^2))^2}, \quad \frac{\partial f}{\partial z} = \frac{8z}{(\exp(z^2) + \exp(-z^2))^2}.
\]
\begin{align*}
\frac{\partial g}{\partial x} &= 2 \sin x \cos x \exp[(\sin x)^2 + (\sin y)^2 + (\sin z)^2], \\
\frac{\partial g}{\partial y} &= 2 \sin y \cos y \exp[(\sin x)^2 + (\sin y)^2 + (\sin z)^2], \\
\frac{\partial g}{\partial z} &= 2 \sin z \cos z \exp[(\sin x)^2 + (\sin y)^2 + (\sin z)^2].
\end{align*}

\begin{align*}
\frac{\partial h}{\partial x}(x, y, z) &= -\frac{x}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}}, \\
\frac{\partial h}{\partial y}(x, y, z) &= -\frac{y}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}}, \\
\frac{\partial h}{\partial z}(x, y, z) &= -\frac{z}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}}.
\end{align*}

6. Solution: The partial derivatives are
\begin{align*}
\frac{\partial \phi}{\partial x} &= 2 \sin x \cos x, \quad \frac{\partial \phi}{\partial y} = 2 \sin y \cos y, \quad \frac{\partial \phi}{\partial z} = 2 \sin z \cos z, \\
\frac{\partial^2 \phi}{\partial x^2} &= 2(\cos x)^2 - 2(\sin x)^2, \\
\frac{\partial^2 \phi}{\partial y^2} &= 2(\cos y)^2 - 2(\sin y)^2, \\
\frac{\partial^2 \phi}{\partial z^2} &= 2(\cos z)^2 - 2(\sin z)^2.
\end{align*}

Clearly \( \phi(0, 0, 0) \) is a local minimum because in any small neighbourhood of \((0, 0, 0)\), \( \phi(x, y, z) > 0 \).
1. Solutions:

(1) \( u = \ln x, \, du = \frac{1}{x}dx. \)

\[
\int \frac{(\ln x) \cos(\ln x)}{x} \, dx = \int u \cos u \, du
\]

\[
= \int u (\sin u)' \, du = u \sin u - \int \sin u \, du
\]

\[
= u \sin u + \cos u + C = (\ln x) \sin(\ln x) + \cos(\ln x) + C.
\]

(2) \( u = \frac{1}{x^2}, \, du = -\frac{2}{x^3}dx, \)

\[
\int \frac{2}{x^5} \sin \left( \frac{1}{x^2} \right) \, dx = -\int u \sin u \, du
\]

\[
= \int u (\cos u)' \, du = u \cos u - \int \cos u \, du
\]

\[
= u \cos u - \sin u + C = \frac{1}{x^2} \cos \left( \frac{1}{x^2} \right) - \sin \left( \frac{1}{x^2} \right) + C
\]

2. Solutions:

(1) \[ \int \arcsin x \, dx = \int x' \arcsin x \, dx \]

\[
= x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} \, dx
\]

\[
= x \arcsin x + \sqrt{1 - x^2} + C,
\]

(2) \[ \int 2x \arctan x \, dx = \int (1 + x^2)' \arctan x \, dx \]

\[
= (1 + x^2) \arctan x - \int \frac{1 + x^2}{1 + x^2} \, dx
\]

\[
= (1 + x^2) \arctan x - x + C
\]
3. Solutions: Let $u = x^5$. Then $du = 5x^4 dx$.

$$(1) \quad \int 5x^4 \exp(-x^5) dx = \int e^{-u} du = -e^{-u} = -\exp(-x^5),$$

$$\int_0^\infty 5x^4 \exp(-x^5) dx = -\exp(-x^5)|_0^\infty = 1,$$

$$\int_1^\infty \frac{1}{x^2} \ln x dx = \int_1^\infty \left(-\frac{1}{x}\right) \ln x dx$$

$$= \left(-\frac{1}{x} \ln x\right)|_1^\infty + \int_1^\infty \frac{1}{x^2} dx = \left(-\frac{1}{x} \ln x - \frac{1}{x}\right)|_1^\infty = 1.$$

4. Solutions: Let $u = 1 + y^2$. Then $du = 2y dy$. Let $v = x^2 + e^x$. 
Then $dv = (2x + e^x)dx$.

(1) \[ \mu(x) = \exp \left[ - \int \frac{1}{2(1 + x)} \, dx \right] \]
\[ = \exp \left[ - \frac{1}{2} \ln(1 + x) \right] = \frac{1}{\sqrt{1 + x}}, \]
\[ \frac{1}{\sqrt{1 + x}} \frac{dy}{dx} - \frac{y}{2(1 + x)\sqrt{1 + x}} = \frac{1}{2\sqrt{1 + x}}, \]
\[ \frac{d}{dx} \left[ \frac{y}{\sqrt{1 + x}} \right] = \frac{1}{2\sqrt{1 + x}}, \]
\[ \frac{y}{\sqrt{1 + x}} = \sqrt{1 + x + C}, \quad y = 1 + x + C\sqrt{1 + x}, \]
\[ \int 8y(1 + y^2) \ln(1 + y^2) \, dy = \int 4u \ln u \, du \]
\[ = 2u^2 \ln u - \int u \, du = 2u^2 \ln u - u^2 \]
\[ = 2(1 + y^2)^2 \ln(1 + y^2) - (1 + y^2)^2, \]
\[ \int [(2x + e^x) \cos(x^2 + e^x) - (2x + e^x) \sin(x^2 + e^x)] \, dx \]
\[ = \int (\cos u - \sin u) \, du = \sin u + \cos u + C \]
\[ = \sin(x^2 + e^x) + \cos(x^2 + e^x) + C, \]
\[ (2) \quad 2(1 + y^2)^2 \ln(1 + y^2) - (1 + y^2)^2 \]
\[ = \sin(x^2 + e^x) + \cos(x^2 + e^x) + C. \]
5. Solutions:

(1) \( \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \)

(2) \( \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n \)

(3) \( \ln(1 - x) = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1} \)

(4) \( \ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \)

(5) \( \ln \frac{1 + x}{1 - x} = \ln(1 + x) - \ln(1 - x) = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n+1} x^{n+1} \)

(6) \( \ln(1 - x^2) = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{2n+2} \)

\( \ln(1 + x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2} \)

\( \ln \frac{1 + x^2}{1 - x^2} = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n+1} x^{2n+2} \)

(7) \( \frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \)

(8) \( \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1} \).

6. Solutions: The first series is convergent and the second series
is convergent, because

\[ (1) \quad \int_1^\infty \left[ \frac{1}{x^2} \sin \left( \frac{1}{x} \right) + \frac{1}{x^3} \cos \left( \frac{1}{x} \right) \right] \, dx \]

\[ = -\frac{1}{x} \sin \left( \frac{1}{x} \right) \bigg|_1^\infty = \sin 1, \]

\[ (2) \quad \int_1^\infty \left( 2x \exp(-x) - x^2 \exp(-x) \right) \, dx \]

\[ = (x^2 \exp(-x)) \bigg|_1^\infty = e^{-1}. \]

7. Solutions: All series are convergent because we may compare

\[ (1) \quad \sum_{n=1}^{\infty} \frac{1}{n^4 + 2n^2 + 1} \text{ with } \sum_{n=1}^{\infty} \frac{1}{n^4}, \]

\[ (2) \quad \sum_{n=1}^{\infty} \frac{1}{5^n} \left[ \arcsin \left( \frac{1}{n^2} \right) \right]^n \text{ with } \sum_{n=1}^{\infty} \left( \frac{\pi}{5} \right)^n, \]

\[ (3) \quad \sum_{n=1}^{\infty} \left[ \sin \left( \frac{1}{n^2} \right) \right]^2 \left[ \cos \left( \frac{1}{n^2} \right) \right]^2 \text{ with } \sum_{n=1}^{\infty} \frac{1}{n^4}. \]

8. Solutions: Let \( u = -x^5 \). Then \( du = -5x^4 \, dx \).

\[ (1) \quad F(x) = \int 5x^9 \exp(-x^5) \, dx = \int u e^u \, du \]

\[ = 1 + ue^u - e^u = 1 - x^5 \exp(-x^5) - \exp(-x^5), \]

\[ (2) \quad P(1, \infty) = \int_1^{\infty} 5x^9 \exp(-x^5) \, dx = 2e^{-1}. \]

9. Solutions:

\[ (1) \quad E(X) = \int_0^{\infty} xf(x) \, dx = \int_0^{\infty} k x \exp(-kx) \, dx = \frac{1}{k}, \]

\[ \int_0^{\infty} x^2 f(x) \, dx = \int_0^{\infty} k x^2 \exp(-kx) \, dx = \frac{2}{k^2}, \]

\[ Var(X) = \int_0^{\infty} x^2 f(x) \, dx - [E(X)]^2 = \frac{1}{k^2}. \]
10. Solutions:
\[
\frac{\partial f}{\partial x} = -2x \exp(-x^2 - y^2),
\]
\[
\frac{\partial f}{\partial y} = -2y \exp(-x^2 - y^2),
\]
\[
\frac{\partial^2 f}{\partial x^2} = -2 \exp(-x^2 - y^2) + 4x^2 \exp(-x^2 - y^2),
\]
\[
\frac{\partial^2 f}{\partial y^2} = -2 \exp(-x^2 - y^2) + 4y^2 \exp(-x^2 - y^2),
\]
\[
\frac{\partial^2 f}{\partial x \partial y} = 4xy \exp(-x^2 - y^2).
\]

The maximum is attained at \((x, y) = (0, 0)\), that is, \(f(0, 0) = 1\).

11. Solutions: Define the auxiliary function
\[
F(x, y, z, \lambda) = (2xy + 2yz + 2zx) - \lambda(x^2 + y^2 + z^2 - 243).
\]

Then
\[
\frac{\partial F}{\partial x} = 2y + 2z - 2\lambda x,
\]
\[
\frac{\partial F}{\partial y} = 2x + 2z - 2\lambda y,
\]
\[
\frac{\partial F}{\partial z} = 2y + 2x - 2\lambda z,
\]
\[
\frac{\partial F}{\partial \lambda} = -(x^2 + y^2 + z^2 - 243).
\]

Let these derivatives be equal to zero, we find
\[
2\lambda x = 2y + 2z, \quad 2\lambda y = 2x + 2z, \quad 2\lambda z = 2y + 2x, \quad x^2 + y^2 + z^2 = 243.
\]

Solving this system, we find that
\[
x^2 = 81, \quad y^2 = 81, \quad z^2 = 81, 2\lambda \neq -1.
\]
The maximum is

\[ f(9, 9, 9) = 486. \]

12. Solutions: Let \( u = x^2 + y^2 \). Then \( du = 2ydy \). First of all, we have

\[
\int 2y(x^2 + y^2) \cos(x^2 + y^2)dy = \int u \cos u du
\]

\[ = u \sin u + \cos u = (x^2 + y^2) \sin(x^2 + y^2) + \cos(x^2 + y^2). \]

Second, we have

\[
\int_{-1}^{1} 2y(x^2 + y^2) \cos(x^2 + y^2)dy
\]

\[ = [(x^2 + y^2) \sin(x^2 + y^2) + \cos(x^2 + y^2)]_{-1}^{1} = 0,
\]

\[
\int_{-1}^{1} \left[ \int_{-1}^{1} 2y(x^2 + y^2) \cos(x^2 + y^2)dy \right] dx = 0.
\]