

**ON THE GESKE COMPOUND OPTION MODEL WHEN
INTEREST RATES CHANGE RANDOMLY – WITH AN
APPLICATION TO CREDIT RISK MODELING**

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ABSTRACT

To extend the Geske model (1979) to incorporate random interest rates is not trivial, even in the simplest case. This is because the “implied strike prices” in the Geske model must be numerically solved and can be solved with non-stochastic interest rates. This prevents the Geske model from evaluating credit derivatives like bonds and credit default swaps. In this paper, we derive a new valuation technique where no “implied strike prices” are needed. As a result, we can easily extend the Geske compound option model to incorporate random interest rates. We apply the new model to conduct various analyses.

Key words. compound option, binomial model, random interest rates, credit derivatives, credit default swaps, reduced-form models.

JEL Classifications. G, G1, G12, G13

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1 INTRODUCTION

The Geske model (1979) for compound options cannot be extended easily to incorporate random interest rates (see Frey and Sommer (1998) and Geman, Karoui, and Rochet, (1995)) in that the “implied strikes” are not easily solvable when there are multiple exercise dates in the compound option. In the Geske model, as the number of exercise dates of the compound option increases, the dimensionality of the implied strike also increases. Note that an n -exercise-date compound option (when $n > 2$) does not have a closed-form solution (as multivariate normal probabilities must be computed numerically) and as a result, solving for the implied strikes is extremely computationally expensive. With random interest rates, each of the implied strikes is a function of future interest rates and the problem becomes impossible to solve.

To overcome this difficulty, we propose a new approach that does not require solving for the implied strikes. This approach is similar to the valuation technique for collateralized debt obligations (CDO) that turns fixed strikes into random strikes. In doing so, we successfully avoid solving for the implied strikes. This is analogous to how the random strike option pricing model by Fisher (1981) is compared to the Black-Scholes model. This approach not only solves the dimensionality problem in the Geske model, but also provides a straightforward way to include random interest rates.

Random interest rates are critical when the Geske model is applied in the area of credit risk modeling. The Geske model for credit risk (1977) has not been

widely used mainly in that credit derivatives are interest rate sensitive and must be modeled with the interest rate risk. Our model, i.e. Geske model with random interest rates, therefore can be used to study credit and interest rate risks and how they interact in the prices of various credit derivatives. Furthermore, it will be shown in details later that our model provides a consistent framework for the endogenization of equity, interest rate, and credit risks. Such endogeneity allows for the right insight toward the interactions of these risks. As a result, our model can also provide regulatory suggestions as three major risk factors are endogenously related. For example, our model can generate the substitution effect between the interest rate risk and credit risk endogenously for any fixed income securities (as demonstrated explicitly in Section 4).

The implementation of the model makes use of a binomial lattice similar to Cox, Ross, and Rubinstein (1979). Given that there is no need to recursively solve for the implied strikes, the binomial model can compute accurate prices swiftly. The use of the binomial model has also other advantages. We can randomize the other parameters (such as interest rate and volatility) easily. We can include liquidity risk in the model.¹

The paper is organized as follows. Section 2 reviews the Geske compound option model (1979) and how Geske relates it to credit risk modeling (1977). In this section, an extension to the Geske-Johnson model (1984) which is a correction of the Geske credit risk model (1977) is provided. In other words, in this section, a correct, multi-period Geske model is presented. Section 3 is the main contribution of the paper. In this section, an alternative valuation approach of the Geske model is presented. Geske's fixed strike model is transformed into a random strike model. Then random interest rates (of the

¹ See, for example, Chen et. al. (2013).

Vasicek type) are introduced to the model. The binomial implementation of the model is explained in this section. In Section 4, various numerical examples are provided. Section 5 compares our model with the reduced-form models used in the industry. Consistency is shown and a unified framework for valuing fixed income securities is offered. Section 6 concludes the paper.

2 The Geske Model

The compound option model by Geske (1979) adopts the usual Black-Scholes assumptions that under the risk-neutral measure the underlying asset, V , follows the log normal process:

$$(1) \quad \frac{dV_t}{V_t} = r_t dt + \sigma_t dW_t$$

where r_t and σ_t represent the risk-free rate and the volatility respectively that can both be random. Define $\underline{T} = \langle T_1, \dots, T_n \rangle$ as a series of exercise times (note that these times need not be even intervals) and K_i where $i = 1, \dots, n$ as the strike price at time T_i .

At each exercise time, T_i , the holder of the option decides if it is worthwhile to pay K_i to exercise the option. If the continuation value (i.e. expected present value of future uncertain cash-flows) is higher than K_i , the option is not exercised and kept alive; otherwise the option will be exercised.

Formally, at any time T_i , the comparison is made between $\mathbb{E}_{T_i} \left[\exp\left(-\int_{T_i}^{T_{i+1}} r_u du\right) C_{T_{i+1}} \right]$ (continuation value) and the strike price K_i . In other words, the current call option value C_{T_i} at time T_i must be:

$$(2) \quad C_{T_i} = \max \left\{ \mathbb{E}_{T_i} \left[\exp\left(-\int_{T_i}^{T_{i+1}} r_u du\right) C_{T_{i+1}} \right] - K_i, 0 \right\}$$

where $\mathbb{E}_{T_i}[\cdot]$ represents the conditional risk-neutral expectation taken at time T_i . This is a recursive equation, as in American option pricing, and can be easily solved by the binomial model.

In a separate paper, Geske (1977) demonstrates that the call compound option is identical to the equity value of the firm when the firm has multiple debts. In other words, the call option value in equation (2) can be replaced with the equity value E_{T_i} as follows:

$$(3) \quad E_{T_i} = \max \left\{ \mathbb{E}_{T_i} \left[\exp \left(- \int_{T_i}^{T_{i+1}} r_u du \right) E_{T_{i+1}} \right] - K_i, 0 \right\}$$

where K_i now represents the cash-flow due at time T_i necessary to be paid by the firm to avoid default. If the firm cannot make the K_i payment, then the firm must default (this is equivalent to the compound option decision not exercising the option and letting the debtholders take over the firm).

As one major purpose of the paper is to apply the Geske model on credit derivatives, we shall now present our model (with random interest rates) in the context of credit risk modeling. In doing so, the “implied strike” in the compound option model can be easily interpreted as the default boundary (a.k.a. default barrier) of the firm. Given that the implied strikes (default boundary) are internally solved, the Geske model, and hence our model, distinguishes itself from other reduced-form and barrier-option credit risk models where the default boundary is exogenously given. As a result, our model can be used for regulatory purposes.

In this section, we first present the model in a two-cash-flow setting in order to demonstrate the basic modeling structure and then we generalize it to any arbitrary number of cash flows. Finally, we carry out a number of analyses and show various numerical results such as default barrier, default probability

curve, and expected recovery. For the sake of easy exposition and no loss of generality, in the two-cash-flow model, we use continuous states while in the n -cash-flow model, we use discrete states.

2.1 The Two-Cash-flow Example – Geske (1977)

In a two-period model, the firm has two cash-flow payments K_1 and K_2 at times T_1 and T_2 . The firm liquidates at time T_2 . Let the value of debt be $D_{t,T}$ where the current time is t and the maturity time is T . It is apparent that $D_{T_2,T_2} = \min\{V_{T_2}, K_2\}$ and $E_{T_2} = \max\{V_{T_2} - K_2, 0\}$. At time T_1 , the equity value before checking for default is precisely the Black-Scholes value:

$$(4) \quad \begin{aligned} E_{T_1}^* &= e^{-r(T_2-T_1)} \mathbb{E}_{T_1}[\max\{V_{T_2} - K_2, 0\}] \\ &= V_{T_1} N(d^+) - e^{-r(T_2-T_1)} K_2 N(d^-) \end{aligned}$$

where $\mathbb{E}_\tau[\cdot]$ is the risk-neutral expectation conditional on (the information given at) time τ and

$$d^\pm = \frac{\ln V_{T_1} - \ln K_2 + (r \pm \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}$$

In the option literature this is called the “continuation value” which is the present value of future payoff. In the compound option model, this value needs to be compared against K_1 , the cash-flow amount (or strike price) at time T_1 . Hence the equity value at time T_1 is:

$$(5) \quad E_{T_1} = \max\{E_{T_1}^* - K_1, 0\}$$

The current value of the equity is

$E_t = e^{-r(T_1-t)} \mathbb{E}_t[E_{T_1}] = e^{-r(T_1-t)} \mathbb{E}_t[\max\{E_{T_1}^* - K_1, 0\}]$. Should we know the distribution of $E_{T_1}^*$, the solution can be easily derived. Unfortunately, the distribution of $E_{T_1}^*$ is unknown. As a result, the closed-form solution can only be derived if we perform the change of variable from $E_{T_1}^*$ to V_{T_1} using (4). In doing so, we must also translate the strike price of K_1 for the equity to the “implied strike price” of \bar{V}_1

for the firm value. This implied strike price for the firm value must be solved numerically as a value that makes the following equality hold:

$$(6) \quad E_{T_1}^* = K_1$$

Equations (5) and (6) are graphically depicted via dotted line and solid line respectively in Figure 1. In Figure 1, $E_{T_1}^*$ is the Black-Scholes call option. The situation where $E_{T_1}^* <> K_1$ is equivalent to $V_{T_1} <> \bar{V}_1$. As a result, we can solve for \bar{V}_1 which is a constant value today.

[Figure 1 Here]

The value \bar{V}_1 , which is the “implied strike price” in the context of compound option, is known as the default point at time T_1 . Once we know the value of \bar{V}_1 , we can then derive the closed-form solution for the equity today as:

$$(7) \quad \begin{aligned} E_t &= e^{-r(T_1-t)} \mathbb{E}_t[\max\{E_{T_1}^* - K_1, 0\}] \\ &= e^{-r(T_1-t)} \mathbb{E}_t[\max\{V_{T_1} - \bar{V}_1, 0\}] \\ &= V_t N_2\left[h_1^+, h_2^+; \sqrt{\frac{T_1-t}{T_2-t}}\right] - e^{-r(T_1-t)} K_1 N_1[h_1^-] - e^{-r(T_2-t)} K_2 N_2\left[h_1^-, h_2^-; \sqrt{\frac{T_1-t}{T_2-t}}\right] \end{aligned}$$

where $N_1[\cdot]$ and $N_2[\cdot, \cdot; \cdot]$ are uni-variate and bi-variate standard normal probabilities respectively with:

$$h_i^\pm = \frac{\ln V_t - \ln \bar{V}_1 + (r \pm \frac{1}{2}\sigma^2)(T_i - t)}{\sigma\sqrt{T_i - t}}$$

and $\sqrt{\frac{T_1-t}{T_2-t}}$ to be the correlation in the bi-variate normal probability function.

The total debt value is $V_t - E_t$ which is equal to:

$$(8) \quad D_t = V_t \left(1 - N_2\left[h_1^+, h_2^+; \sqrt{\frac{T_1-t}{T_2-t}}\right]\right) + e^{-r(T_1-t)} K_1 N_1[h_1^-] + e^{-r(T_2-t)} K_2 N_2\left[h_1^-, h_2^-; \sqrt{\frac{T_1-t}{T_2-t}}\right]$$

The first term is known as the expected recovery value. The second and third terms are the current values of the first and second cash-flows (K_1 and K_2) respectively. Note that $N_1[h_1^-]$ and $N_2\left[h_1^-, h_2^-; \sqrt{\frac{T_1-t}{T_2-t}}\right]$ are T_1 and T_2 risk-neutral survival probabilities respectively. The T_1 survival probability $N_1[h_1^-]$ is the

probability of $V_{T_1} > \bar{V}_1$ (or equivalently $E_{T_1}^* > K_1$). The T_2 survival probability $N_2\left[h_1^-, h_2^-; \sqrt{\frac{T_1-t}{T_2-t}}\right]$ is the joint probability of $V_{T_1} > \bar{V}_1$ and $V_{T_2} > K_2$.

The total risk-neutral default probability, which is $1 - N_2\left[h_1^-, h_2^-; \sqrt{\frac{T_1-t}{T_2-t}}\right]$, represents either default at T_1 or T_2 . When the random recovery value (V_{T_1} or V_{T_2} paid at T_1 or T_2 should default occur) combines with the corresponding default probability, it gives rise to the expected recovery value of $V_t\left(1 - N_2\left[h_1^+, h_2^+; \sqrt{\frac{T_1-t}{T_2-t}}\right]\right)$ which is the current asset value multiplied by the default probability under the measure in which the random asset value is the numerarie.²

As one can see, the closed-form solution actually relies upon the numerical solution of the default point \bar{V}_1 at time T_1 . In a multi-period setting, although the solution can be easily extended, the solutions to the default points over time (i.e. $\bar{V}_1 \dots \bar{V}_n$ at $T_1 \dots T_n$) become increasingly complex. For example, in a three-period model, \bar{V}_2 is the internal solution to $E_{T_2}^* = K_2$ which is a uni-variate result. But $E_{T_1}^*$ is a bi-variate integration and hence \bar{V}_1 requires a bi-variate numerical search (such that $E_{T_1}^* = K_1$). Hence, as the number of dimensions increases (say n), the solution to \bar{V}_1 cannot be solved without an n -dimensional numerical algorithm.

2.2 The General (n -cash-flow) Case — Extension of Geske-Johnson (1984)

Although there is an n -cash-flow model in Geske (1977), a mistake was made with seniorities in debts, which was later corrected by Geske and Johnson (1984). However, Geske and Johnson only provide a two-cash-flow example.

² This expected value is performed under the following change of measure:

$\mathbb{E}_t[V_s X_s] = \mathbb{E}_t[V_s] \mathbb{E}_t^V[X_s]$ where $s > t$, V is the numerarie, and X is the random cash flow.

Here, we extend the Geske-Johnson result and present the general n -cash-flow model where seniorities of debts in Geske and Johnson are explicitly considered.

The notation we employ here is more complex than that in Geske-Johnson. Not only do we label firm default points, we also label the default points of each bond. In the previous section, the default points (n periods) are labeled as $\langle \bar{V}_1, \bar{V}_2, \dots, \bar{V}_n \rangle$ where in each period the default point is solved for by $E_{T_i}^* = K_i$ for $1 < i < n$ and $\bar{V}_n = K_n$. Here, these points are labeled as $\langle \bar{V}_{1,n}, \bar{V}_{2,n}, \dots, \bar{V}_{n,n} \rangle$ where $\bar{V}_{n,n} = K_n$. The reason for this more complex system of notation is due to the fact that not only must we capture firm default, we must also capture the default of each of the n bonds. In an n -period case, there are n cash-flows. For the sake of argument, we regard these cash flows as the face values of zero-coupon bonds whose current values are labeled as D_{t,T_i} for $i = 1, \dots, n$. Each bond has a set of default points labeled as $\bar{V}_{j,i}$ where $j < i$ (so there are $\frac{n(n-1)}{2}$ of them).³ Again, when $i = n$, it represents the firm default.

With this extension in notation, we can write the extension of the Geske-Johnson formula as follows:

$$(9) \quad D_{t,T_i} = V_t \{ N_{i-1} [h_1^+(\bar{V}_{1,i-1}) \cdots h_{i-1}^+(\bar{V}_{i-1,i-1}); \mathbf{C}_{i-1}] - N_i [h_1^+(\bar{V}_{1,i}) \cdots h_i^+(\bar{V}_{i,i}); \mathbf{C}_i] \} \\ + \sum_{j=1}^i e^{-r(T_j-t)} K_j \{ N_j [h_1^-(\bar{V}_{1,i}) \cdots h_1^-(\bar{V}_{j,i}); \mathbf{C}_j] - N_j [h_1^-(\bar{V}_{1,i-1}) \cdots h_1^-(\bar{V}_{j,i-1}); \mathbf{C}_j] \}$$

where

$$h_i^\pm(x) = \frac{\ln V_t - \ln x + (r \pm \frac{1}{2}\sigma^2)(T_i - t)}{\sigma\sqrt{T_i - t}}$$

$N_i(\cdot, \cdot; \mathbf{C}_i)$ is the i -dimensional normal probability function with the correlation matrix:

³ The notation system seems complex at the first glance, it can be easily understood with a simple example which we shall present and discuss in the next section with the help of Table 2.

$$\mathbb{C}_i = \begin{bmatrix} 1 & \sqrt{\frac{T_1-t}{T_2-t}} & \sqrt{\frac{T_1-t}{T_3-t}} & \cdots & \sqrt{\frac{T_1-t}{T_i-t}} \\ & 1 & \sqrt{\frac{T_2-t}{T_3-t}} & \cdots & \sqrt{\frac{T_2-t}{T_i-t}} \\ & & \ddots & & \vdots \\ & & & 1 & \sqrt{\frac{T_{i-1}-t}{T_i-t}} \\ & & & & 1 \end{bmatrix}$$

and finally $\bar{V}_{i,i} = K_i$, $\bar{V}_{j,i} = -\infty$ for $j > i$ (which indicates that any probability involving this argument is 0), and $N_0[\cdot] = 1$. Summing all debt values yields:

$$(10) \quad \sum_{i=1}^n D_{t,T_i} = V_t [1 - N_n(h_1^+(\bar{V}_{1,n}) \cdots h_n^+(\bar{V}_{n,n}); \mathbb{C}_n)] \\ + \sum_{i=1}^n e^{-r(T_i-t)} K_i N_i(h_1^-(\bar{V}_{1,n}) \cdots h_n^-(\bar{V}_{n,n}); \mathbb{C}_n)$$

The equity value as a result is equal to:

$$(11) \quad E_t = V_t - \sum_{i=1}^n D_{t,T_i} \\ = V_t N_n(h_1^+(\bar{V}_{1,n}) \cdots h_n^+(\bar{V}_{n,n}); \mathbb{C}_n) \\ - \sum_{i=1}^n e^{-r(T_i-t)} K_i N_i(h_1^-(\bar{V}_{1,n}) \cdots h_n^-(\bar{V}_{n,n}); \mathbb{C}_n)$$

In (9), the first line represents the default value (via expected recovery) and the remaining terms of the equation represent the survival value (i.e. expected cash flows). Note that $N_i[h_1^-(\bar{V}_{1,i}) \cdots h_i^-(\bar{V}_{i,i})]$ and $N_i[h_1^+(\bar{V}_{1,i}) \cdots h_i^+(\bar{V}_{i,i})]$ represent the i -th survival probabilities under the risk-neutral measure and the measure with random asset value as the numeraire respectively. Hence in each period, the default probability is either (under the risk neutral measure) $N_{i-1}[h_1^-(\bar{V}_{1,i-1}) \cdots h_{i-1}^-(\bar{V}_{i-1,i-1})] - N_i[h_1^-(\bar{V}_{1,i}) \cdots h_i^-(\bar{V}_{i,i})]$ or (under the “random asset measure”) $N_{i-1}[h_1^+(\bar{V}_{1,i-1}) \cdots h_{i-1}^+(\bar{V}_{i-1,i-1})] - N_i[h_1^+(\bar{V}_{1,i}) \cdots h_i^+(\bar{V}_{i,i})]$. In (9), the second line represents the seniority structure assumed in the Geske model. In other words, the second line of (9) states that earlier maturing debts are more senior than later maturing debts.⁴ This assumption is quite reasonable if the firm issues only

⁴ This point was not clear in the original Geske model (1977) and made clear in Geske and Johnson (1984).

zero-coupon debts. When applying the model to coupon-bearing debts, one must bear in mind that within a coupon bond, earlier coupons have a higher priority than later coupons. With this in mind, it is quite easy to interpret (10) if one regards all cash-flows of the firm as a coupon bond. The first term of (10) is the expected recovery and the second term is the survival value which is equal to the sum of cash flows weighted by their corresponding survival probabilities.

The major difficulty that prevents the Geske model from being widely used is the complexity in calculating the default points $\langle \bar{V}_{1,n} \dots \bar{V}_{n,n} \rangle$. The last default point is the last strike price $\bar{V}_{n,n} = K_n$. The next-to-last default point $\bar{V}_{n-1,n}$ is the solution to $E_{T_{n-1}}^* = K_{n-1}$ which is a uni-variate numerical algorithm.⁵ The default point for the firm at T_{n-2} , $\bar{V}_{n-2,n}$, is the solution to $E_{T_{n-2}}^* = K_{n-2}$ which is a bi-variate numerical algorithm. Also at T_{n-2} , we need to solve for the default point of the bond that expires at time T_{n-1} as $\bar{V}_{n-2,n-1}$ which is the solution to $E_{T_{n-2}}^* + D_{T_{n-2},T_{n-1}}^* = V_{T_{n-2}}$ where $D_{T_{n-2},T_{n-1}}^* = \mathbb{E}_{T_{n-2}}[D_{T_{n-1},T_{n-1}}]$.⁶ As we proceed backwards in time, we have a total of $\frac{n(n-1)}{2}$ numerical solutions to identify, and the values to solve at time T_1 are n -dimensional numerical solutions.

Furthermore, the Geske model cannot be extended to include random interest rates, as Frey and Sommer (1998) demonstrate. This is easily observable from (6) in which adding another dimension makes (6) unsolvable. Although Eom, Helwedge, and Huang (2002) and Chen, Chidambaran, Imerman, and Soprazetti (2010) propose a more straightforward way to compute the equity value without solving for the default points, their methods are not extendable to other debts of the firm. The main reason is that the way they avoid solving the

⁵ See equation (6).

⁶ Note that, similar to $E_{T_i}^*$, D_{T_i,T_j}^* is the continuation value of K_j at time T_i .

default points is by using the equity value at each time. This method is not able to solve for other debt values. We shall discuss this in details in the next section.

3 A New Approach toward the Compound Option Problem

In this section, we introduce a new approach toward the compound option problem. This new approach improves the Geske model in the following two significant ways. First, it avoids solving for the default points (i.e., “implied strike prices”) and hence avoids the formidable work of high-dimensional numerical algorithms. To achieve this, we transform the model from a fixed strike problem of the Black-Scholes to a random strike problem of Fisher (1978).⁷ The solution is certainly not closed-form but it remains one-dimensional, which we can implement easily with the Cox-Ross-Rubinstein binomial model.

Secondly, due to the new structure, we can easily extend the model to include random interest rates which is a crucial necessity for the model to be used in pricing various financial assets that are exposed to interest rate risk. We note in general that financial assets face three common types of risk: market (equity), interest rate, and credit. The Geske model, using the corporate finance approach, properly prices market (equity) and credit risks. Our model is to complete the Geske model by including the interest rate risk.

[Figure 2 Here]

Before we present the general case with n cash-flows, it is most intuitive to see a two-cash-flow case. Figure 2 presents the basic idea with a two-cash-flow example. Compared with Figure 1, it is clear that the equity value is the same. However, the equity value is not obtained by solving the default point, but rather using the concept of random strike, i.e. $E_{T_1} = \max\{V_{T_1} - V_{12}^*, 0\}$ where

⁷ Or equivalently the exchange option model by Margrabe (1978).

$V_{12}^* = D_{T_1, T_2}^* + K_1$. V_{12}^* is random because D_{T_1, T_2}^* is random as a function of V_{T_1} .⁸ Note that Figure 1 plots the payoff using $E_{T_1}^* - K_1$ and Figure 2 plots the payoff using $V_{T_1} - V_{12}^*$. These two payoffs differ when $V_{T_1} < V_{12}^*$ (or identically $E_{T_1}^* < K_1$) but equal each other when $V_{T_1} \geq V_{12}^*$ (or identically $E_{T_1}^* \geq K_1$). As a result, both produce the same payoff which takes only positive values. Algebraically, we present the result in the following Theorem.

[Theorem]

In Geske's two-period compound call option model, the implied strike price (or default point for the firm) \bar{V}_{12} which is constant can be replaced by V_{12}^ which is random.*

[Proof]

Note that:

$$\begin{aligned}
 E_t &= e^{-r(T_1-t)} \mathbb{E}_t[E_{T_1}] \\
 (12) \quad &= e^{-r(T_1-t)} \mathbb{E}_t[\max\{E_{T_1}^* - K_1, 0\}] \\
 &= e^{-r(T_1-t)} \int_{K_1}^{\infty} (E_{T_1}^* - K_1) f(E_{T_1}^*) dE_{T_1}^*
 \end{aligned}$$

where $f(\cdot)$ is the density for $E_{T_1}^*$. Geske recommends the change of variable from $E_{T_1}^*$ to V_{T_1} , we can re-write (12) as follows:

$$(13) \quad E_t = e^{-r(T_1-t)} \int_{\bar{V}_{12}}^{\infty} (E_{T_1}^* - K_1) g(V_{T_1}) dV_{T_1}$$

where $g(\cdot)$ is the density for V_{T_1} . This is the fixed strike case where \bar{V}_{12} is the solution to $E_{T_1}^* = K_1$.⁹

By accounting identity, $E_{T_1}^* - K_1 = (V_{T_1} - D_{T_1, T_2}^*) - K_1 = V_{T_1} - V_{12}^*$. Hence, $E_{T_1} = \max\{V_{T_1} - V_{12}^*, 0\} = \max\{E_{T_1}^* - K_1, 0\}$. As a result (12) becomes:

⁸ See footnote 6.

⁹ Note that the Jacobian is embedded in the change of variable.

$$\begin{aligned}
(14) \quad E_t &= e^{-r(T_1-t)} \mathbb{E}_t[E_{T_1}] \\
&= e^{-r(T_1-t)} \mathbb{E}_t[\max\{V_{T_1} - V_{12}^*, 0\}]
\end{aligned}$$

The strike price V_{12}^* is random because D_{T_1, T_2}^* is random as $V_{12}^* = K_1 + D_{T_1, T_2}^*$ where $D_{T_1, T_2}^* = e^{-r(T_2-T_1)} \mathbb{E}_{T_1}[\min\{V_{T_2}, K_2\}]$ in the two-period case.

[Q.E.D.]

To present the result more intuitively, we can transform (14) into the following:

$$\begin{aligned}
(15) \quad E_t &= e^{-r\Delta t} \mathbb{E}_t \left[V_{T_1} \max \left\{ 1 - \frac{V_{12}^*}{V_{T_1}}, 0 \right\} \right] \\
&= e^{-r\Delta t} \mathbb{E}_t[V_{T_1}] \mathbb{E}_t^V \left[\max \left\{ 1 - \frac{V_{12}^*}{V_{T_1}}, 0 \right\} \right] \\
&= V_t \mathbb{E}_t^V \left[\max \left\{ 1 - \frac{V_{12}^*}{V_{T_1}}, 0 \right\} \right] \\
&= V_t \left\{ \Pr^- \left(1 > \frac{V_{12}^*}{V_{T_1}} \right) - \mathbb{E}_t^V \left[\frac{V_{12}^*}{V_{T_1}} \right] \Pr^+ \left(1 > \frac{V_{12}^*}{V_{T_1}} \right) \right\}
\end{aligned}$$

where \mathbb{E}_t^V is the expected value taken under the measure where the asset value V_{T_1} serves as the numeraire, \Pr^- is the probability where V_{T_1} serves as the numeraire, and \Pr^+ is the probability where $\frac{V_{12}^*}{V_{T_1}}$ serves as the numeraire. While the closed-form solution with this new approach is not possible, it can be quickly calculated via a binomial lattice.

To help understand the mechanics of the Theorem, we provide a three-period example to demonstrate the details of the mechanics, which is summarized in Table 1.

[Table 1 Here]

This three-period example includes the above theorem as a special case and also paves the way to the understanding of the n -period model. The cash-flows are K_1 , K_2 , and K_3 which are paid at T_1 , T_2 , and T_3 . The current time is t . At time T_3 , the firm liquidates and the equity and bond holders receive

$\max\{V_{T_3} - K_3, 0\}$ and $\min\{V_{T_3}, K_3\}$ respectively. Clearly the default point at time T_3 is $V_{33}^* = K_3$ which is a constant.

At time T_2 , there are two default points, one where the firm defaults, and the other point where the firm cannot even pay K_2 . At time T_2 , the firm defaults when the asset value of the firm falls short of its current cash liability K_2 and the current value of the future liability which is $D_{T_2, T_3}^* = e^{-r(T_3 - T_2)} \mathbb{E}_{T_2}[\min\{V_{T_3}, K_3\}]$. In other words, the firm survives when $V_{T_2} > K_2 + D_{T_2, T_3}^*$ and defaults when $V_{T_2} \leq K_2 + D_{T_2, T_3}^*$. As a result, the default point is defined as $V_{23}^* = K_2 + D_{T_2, T_3}^*$ which is random.

Another critical point is an economic condition where $K_2 < V_{T_2} < K_2 + D_{T_2, T_3}^*$. Under this circumstance, the firm defaults but K_2 is paid in full (i.e. 100% recovery). As a result, the owners of K_2 do not suffer any losses even if the firm defaults. The debt owners who suffer are those who own K_3 . For those debt owners the value of their debt is less than D_{T_2, T_3}^* which is the discounted expected value of $\min\{V_{T_3}, K_3\}$. For convenience, we label the second critical point $V_{22}^* = K_2$ which is constant.

In Table 1 at time T_2 , we can then derive the debt and equity values,

$$(16) \quad \begin{aligned} E_{T_2} &= \max\{V_{T_2} - V_{23}^*, 0\} \\ D_{T_2, T_3} &= \max\{V_{T_2} - K_2, 0\} - E_{T_2} \\ D_{T_2, T_2} &= \min\{V_{T_2}, K_2\} = V_{T_2} - \max\{V_{T_2} - V_{22}^*, 0\} \end{aligned}$$

At time T_1 , continuation values (ones with asterisks) are calculated as follows:

$$(17) \quad \begin{aligned} E_{T_1}^* &= e^{-r(T_2 - T_1)} \mathbb{E}_{T_1}[E_{T_2}] \\ D_{T_1, T_3}^* &= e^{-r(T_2 - T_1)} \mathbb{E}_{T_1}[D_{T_2, T_3}] \\ D_{T_1, T_2}^* &= e^{-r(T_2 - T_1)} \mathbb{E}_{T_1}[D_{T_2, T_2}] \end{aligned}$$

and then it can be seen from Table 1 that the following holds:

$$\begin{aligned}
(18) \quad E_{T_1} &= \max\{V_{T_1} - V_{13}^*, 0\} \\
D_{T_1, T_3} &= \max\{V_{T_1} - V_{12}^*, 0\} - \max\{V_{T_1} - V_{13}^*, 0\} \\
D_{T_1, T_2} &= \max\{V_{T_1} - V_{11}^*, 0\} - \max\{V_{T_1} - V_{12}^*, 0\} \\
D_{T_1, T_1} &= V_{T_1} - \max\{V_{T_1} - V_{11}^*, 0\}
\end{aligned}$$

where $V_{11}^* = K_1$, $V_{12}^* = K_1 + D_{T_1, T_2}^*$, and $V_{13}^* = K_1 + D_{T_1, T_2}^* + D_{T_1, T_3}^*$. Note that equations (16) and (18) are precisely how various tranches of a CDO (collateral debt obligation) are evaluated. In CDO pricing, such payoffs are convoluted with a complex loss distribution and here the simple log normal distribution is used for the asset value.

Finally at the current time t , we reach the current values of all the debts and equity as follows:

$$\begin{aligned}
(19) \quad E_t &= E_t^* = e^{-r(T_1-t)} \mathbb{E}_t[E_{T_1}] \\
D_{t, T_3} &= D_{t, T_3}^* = e^{-r(T_1-t)} \mathbb{E}_t[D_{T_1, T_3}] \\
D_{t, T_2} &= D_{t, T_2}^* = e^{-r(T_1-t)} \mathbb{E}_t[D_{T_1, T_2}] \\
D_{t, T_1} &= D_{t, T_1}^* = e^{-r(T_1-t)} \mathbb{E}_t[D_{T_1, T_1}]
\end{aligned}$$

It is now easy to induce that in an n -cash-flows model, we can obtain the following Corollary. While we shall not prove the Corollary, a numerical demonstration using $n = 3$ is provided to show that the random strike price model is identical to the constant strike model used in Geske (1979) and Geske and Johnson (1984).

[Corollary]

In the n -cash-flow case we can similarly replace all the constant default points $\bar{V}_{i,j}$ with the random values $V_{i,j}^$ and obtain the same result for the Geske model.*

In the n -period case, each debt can be expressed as a call spread as in equations (16) and (18) at each time step. Hence, the general formula for any debt at any time step is:

$$(20) \quad \begin{aligned} D_{T_i, T_j} &= \max\{V_{T_i} - V_{i,j-1}^*, 0\} - \max\{V_{T_i} - V_{i,j}^*, 0\} \\ E_{T_i} &= \max\{V_{T_i} - V_{i,n}^*, 0\} \end{aligned}$$

where $V_{i>j}^* = 0$, $V_{i=j}^* = K_i$, and $V_{i<j}^* = K_i + \sum_{k=i}^j D_{T_i, T_k}^*$ and D_{T_i, T_k}^* is the continuation value that needs to be computed recursively as $D_{T_i, T_k}^* = e^{-r(T_{i+1}-T_i)} \mathbb{E}_{T_i}[D_{T_{i+1}, T_k}]$.

Finally, as in equation (19), the value of equity and debts are given as follows:

$$(21) \quad \begin{aligned} E_t &= e^{-r(T_1-t)} \mathbb{E}_t[E_{T_1}] \\ D_{t, T_i} &= e^{-r(T_1-t)} \mathbb{E}_t[D_{T_1, T_i}] \quad i = 1, \dots, n \end{aligned}$$

Although we lose the closed-form solution as the “strike” price, $V_{i,j}^*$, is now random, the general form of the solution similar to (10) remains. To see that, we first recall that each $N_i[h_1^-(\bar{V}_{1,i}) \cdots h_i^-(\bar{V}_{i,i})]$ in equation (10) represents a risk-neutral survival probability. In Table 1, they are $N_1[h_1^-(\bar{V}_{1,1})]$, $N_2[h_1^-(\bar{V}_{1,2}), h_2^-(\bar{V}_{2,2})]$, and $N_3[h_1^-(\bar{V}_{1,3}) \cdots h_3^-(\bar{V}_{3,3})]$ in respectively. Under our approach where the strikes are random, the three survival probabilities for the three periods are $\Pr[\{V_{T_1} > V_{13}^*\}]$, $\Pr[\{V_{T_1} > V_{13}^*\} \cap \{V_{T_2} - V_{23}^*\}]$ and $\Pr[\{V_{T_1} - V_{13}^*\} \cap \{V_{T_2} - V_{23}^*\} \cap \{V_{T_3} - V_{33}^*\}]$ respectively. While there are no closed-form solutions to these probability functions, we can express them as Π_1^- , Π_2^- , and Π_3^- respectively. As a result, we can write the solution in the same form as (10) in a general n period model:

$$(22) \quad \sum_{i=1}^n D_{t, T_i} = V_t [1 - \Pi_n^+] + \sum_{i=1}^n e^{-r(T_i-t)} K_i \Pi_i^-$$

which is similar to equation (10) where Π_i^- is similar to the normal probability with the arguments h_i^- and Π_i^+ is similar to the normal probability with the arguments h_i^+ .

3.1 The Binomial Implementation

In this sub-section, we demonstrate how the model works in a binomial implementation. While not most computationally efficient, the binomial algorithm is the most intuitive and the easiest method to implement. To

demonstrate, we set up a six-period model with three cash-flows. Within two subsequent cash-flows, there is a time step in between. We also assume the following parameter values.

V_t	100	r	3%
σ	0.5	Δt	0.5
T_1	1	K_1	40
T_2	2	K_2	40
T_3	3	K_3	40

From these inputs, we can compute the following: $u = e^{\sigma\sqrt{\Delta t}} = 1.1421$, $d = 1/u = 0.7022$. The risk neutral probabilities are: $p = \frac{\exp(r\Delta t) - d}{u - d} = 0.4335$ and $1 - p = 0.5665$. The lattice is described in Table 2.

[Table 2 Here]

As the table shows, there are three cash-flows, K_1 , K_2 , and K_3 , paid at $T_1 = 1$, $T_2 = 2$, and $T_3 = 3$. For any two consecutive cash-flows, there is a partition in between and hence $\Delta t = 0.5$. We need to solve for three values of debts, D_{t,T_1} , D_{t,T_2} , and D_{t,T_3} , and the value of equity E_t .

Panel A of Table 2 is the standard binomial lattice of Cox, Rubinstein and Ross (1979) for the value of the firm. This is a 6-period binomial lattice with each period of 0.5 year. At the bottom of the panel, a time line and strikes are provided.

Panels B through E are the binomial lattices for various classes of liabilities where equity is regarded as the least priority in the order of claiming firm's assets. In each of the Panels B through E, the times at which the cash obligations are due (in the example, they are \$40 each at T_1 , T_2 and T_3 respectively) are further split into two separate columns – one immediately prior to exercise and one immediately post to exercise. For example, in Panel A, cash-flow paid is indicated in time T_1 , T_2 and T_3 which are in periods 2, 4 and 6

respectively. In Panel B, for example, periods 2 and 4 each is separated into 2 columns, each labeled as E_{T_1} and $E_{T_1}^*$ (for period 2) and E_{T_2} and $E_{T_2}^*$ (for period 4) respectively. The ones with asterisks (on the right) represent the continuation values and the ones without (on the left) represent the exercise values.

At time T_3 , there is only one cash obligation of \$40 (K_3) to be paid to the T_3 -maturity debt. Hence the payoffs for the equity and the debt are $\max\{V_{T_3} - K_3, 0\}$ and $\min\{V_{T_3}, K_3\}$ respectively. The numerical values are given in the last columns of Table 2's Panels B and C respectively.¹⁰

In between T_2 and T_3 , it is the usual expected discount cash-flow calculation and $E_{T_2}^*$ (shown in Panel B) and D_{T_2, T_3}^* (shown in Panel C) are continuation values of the equity and debt at time T_2 respectively. Then a cash-flow of K_2 is expected. Depending on various asset values at time T_2 , debt and equity holders will receive different payments. The usual Gscke rule for exercise, as explained earlier, is $E_{T_2}^* \stackrel{?}{=} K_2$ which is not sufficient to price the other two debts – T_2 -maturity and T_3 -maturity debts. The reason is clearly seen in Panels B, C, and D. In Panel B, the two bottom values of $E_{T_2}^*$ are less than K_2 and as a result, E_{T_2} is 0 in these two bottom states. Hence, it is clear that the default point for the firm value, \bar{V}_{23} , is between \$100 and \$49.31 which are the two corresponding values of the asset in Panel A. In other words, the payoff for the equity is positive when $V_{T_2} > 49.31$.

Under our approach, in place of \bar{V}_{23} , we define $V_{23}^* = K_2 + D_{T_2, T_3}^*$ which is random. The values of V_{23}^* are (from low to high) 62.6148, 73.9311, 78.8178, 78.8178, and 78.8178 for the asset value (V_{T_2}) at 24.31, 49.31, 100.00, 202.81, and

¹⁰ We shall note that effectively both payoffs can be regarded as a call spread on the firm value. In other words, the payoff of the equity can be written as $\max\{V_{T_3} - K_3, 0\} - \max\{V_{T_3} - \infty, 0\}$ and for the debt it is $\max\{V_{T_3} - 0, 0\} - \max\{V_{T_3} - K_3, 0\}$. As a result, all the liabilities (equity included) can be priced as call spreads. Note that these are precisely the tranche payoffs in a CDO structure.

411.33 respectively. As we can see, the option payoff is positive when $V_{T_2} > 49.31$, identical to using the constant strike \bar{V}_{23} . Consequently, it can be seen that the equity value (E_{T_2}) and the T_3 -maturity debt value (D_{T_2,T_3}) are unaffected using V_{23}^* in stead of \bar{V}_{23} even though when $V_{T_2} < 100$, V_{23}^* is decreasing in V_{T_2} . Finally, note that $V_{22}^* = K_2$.

As we progress backwards, at time T_1 , three strikes are determined as follows: $V_{13}^* = K_1 + D_{T_1,T_2}^* + D_{T_1,T_3}^*$ that replaces \bar{V}_{13} as the default point, $V_{12}^* = K_1 + D_{T_1,T_2}^*$, and $V_{11}^* = K_1$.

V_{T_1}	V_{13}^*	V_{12}^*	$V_{11}^*(= K_1)$
202.81	116.4884	78.8178	40
100.00	107.2961	78.8178	40
49.31	85.4447	73.9311	40

Such a process can easily be extended for more periods and ultimately converges to the continuous time result. As discussed earlier, we can solve for \bar{V}_{13} in the Geske model by letting $E_{T_1}^* = K_1$. In the Geske model this is a bi-variate search as $E_{T_1}^*$ involves bi-variate Gaussian probabilities. Here, by allowing \bar{V}_{13} to be replaced by V_{13}^* , we successfully avoid the complex process. Also as discussed earlier, this new method allows us to price other debts. As shown above, in this particular example, although the firm can default, T_1 -maturity debt is risk-free because it receives 100% recovery. T_2 -maturity debt defaults in one state (lowest) only, while the firm defaults in two lowest states.

In addition, we can also endogenously solve for the recovery value for each debt. T_2 -maturity debt recovers \$9.31 as a result of asset value being \$49.31. The seniority rule established by Geske and Johnson (1984) as a result is reserved in the binomial model.

The convergence of the binomial model is given in an appendix via a two-cash-flow example where the closed-form solution exists (1979 Geske, equation 4). For the given set of parameters, two-decimal convergence can be achieved with 400 steps.

As argued earlier, with this new methodology, we can easily extend the Geske model to include random interest rates that was previously impossible. Note that the original Geske model where the implied strike is a constant, \bar{V}_{jn} (where n is the total number of periods and $1 \leq j \leq n$), is not solvable under random interest rates.

3.2 Extension to Include Random Interest Rates

In this section, we extend the Geske/Geske-Johnson model to include random interest rates using the Vasicek model (1977). We use the Vasicek model only for the ease of exposition. More complex term structure models such as Heath-Jarrow-Morton (1992) and Hull-White (1990b) can be straightforwardly incorporated into our model.

Frey and Sommer (1998) and Geman, Karoui, and Rochet, (1995)) show that extending the Geske model to include random interest rates is not trivial. As we argue previously, this is because these efforts use the fixed strike formula. As we shall demonstrate in this section, once we convert the problem to random strikes, the extension is straightforward.

Let the joint dynamics (under the risk-neutral measure) be as follows:

$$(23) \quad \begin{aligned} \frac{dV_t}{V_t} &= r_t dt + \sigma dW_{1t} \\ df_{t,T} &= \left[v_{t,T} \int_t^T v_{t,u} du \right] dt + v_{t,T} dW_{2t} \end{aligned}$$

where $f_{t,T} = -\frac{\partial \ln P_{t,T}}{\partial T}$ is the instantaneous forward rate and $f_{t,t} = r_t$. Define $P_{t,T}$ as the price of a \$1 zero-coupon bond and $dW_{1t}dW_{2t} = \rho dt$. The interest rate process is an HJM which is general enough to include major interest rate models as special cases. In the Ho-Lee model (1986), $v_{t,T} = \gamma$ which is a constant and in this case the short rate process is $dr_t = \theta_t dt + \gamma dW_{2t}$. In the Hull-White model (which is an extension of the Vasicek model), $v_{t,T} = e^{-\alpha(T-t)}\gamma$ and in this case the short rate process is $dr_t = (\theta_t - \alpha r_t)dt + \gamma dW_{2t}$.¹¹ In the Cox-Ingersoll-Ross model, $v_{t,T} = \gamma\sqrt{r_t}\frac{\partial B_{t,T}}{\partial T}$ where $B_{t,T} = 2(e^{\zeta(T-t)} - 1)/[(\alpha + \zeta)(e^{\zeta(T-t)} - 1) + 2\zeta]$ and $\zeta^2 = \alpha^2 + 2\gamma^2$ and in this case the short rate process is $dr_t = (\theta_t - \alpha r_t)dt + \gamma\sqrt{r_t}dW_{2t}$.

While the lattice model is capable of incorporating any interest rate process described in equation (23), we shall present our numerical results using the Vasicek model in that we can demonstrate the convergence of our lattice algorithm by comparing to known closed-form solutions.

The lattice algorithm is explained in details in the Appendix. Basically, we follow the suggestion by Scott (1997) where the interest rate process is solved by the explicit finite difference method and the underlying asset process is solved by the implicit finite difference method.¹² We test the convergence of the algorithm (provided in the appendix) using the Rabinovitch model (1989) where the closed-form solution for the European call option exists. We find that convergence in two dimensions is achieved at 1000 steps.

Recall that our model replaces \bar{V}_{12} (fixed strike) with $V_{12}^* = K_1 + D_{T_1, T_2}^*$ (random strike) which is now a function of interest rates because D_{T_1, T_2}^* can be computed similar to (17) with a slight adjustment:

¹¹ If θ_t is constant, then usually it is expressed as $\theta_t = \alpha\mu$ where μ is regarded as the level of mean reversion. Then this degenerates to the Vasicek model.

¹² It is not feasible for both the processes to be solved by the explicit method which is more intuitive. The reason is that the random interest rates (similar to the random volatility in Scott (1997)) can blow up the stock price process. Detailed discussions can be found in an appendix.

$$(24) \quad \begin{aligned} E_{T_i}^* &= \mathbb{E}_{T_i} \left[\exp \left(- \int_{T_i}^{T_{i+1}} r_u du \right) E_{T_{i+1}} \right] \\ D_{T_i T_j}^* &= \mathbb{E}_{T_i} \left[\exp \left(- \int_{T_i}^{T_{i+1}} r_u du \right) D_{T_{i+1}, T_j} \right] \end{aligned}$$

The computation of (24) is straightforward in a bi-variate finite difference algorithm. Then, $V_{i < j}^* = K_i + \sum_{k=i}^j D_{T_i, T_k}^*$ and:

$$(25) \quad \begin{aligned} D_{T_i, T_j} &= \max\{V_{T_i} - V_{i, j-1}^*, 0\} - \max\{V_{T_i} - V_{i, j}^*, 0\} \\ E_{T_i} &= \max\{V_{T_i} - V_{i, n}^*, 0\} \end{aligned}$$

The process repeats until we reach the current prices of debts and equity:

$$(26) \quad \begin{aligned} E_t &= \mathbb{E}_t \left[\exp \left(- \int_t^{T_1} r_u du \right) E_{T_1} \right] \\ D_{t, T_j} &= \mathbb{E}_t \left[\exp \left(- \int_t^{T_1} r_u du \right) D_{T_1, T_j} \right] \end{aligned}$$

It is now apparent to see with the alternative formulation of the Geske model using random strikes, we can easily incorporate random interest rates. In the next section, numerical exemplifications will be provided to examine the impact of random interest rates on credit sensitive financial assets such as convertible bonds and credit default swaps.

4 Applications

Reduced-form models cannot be used to study endogenously how various risk factors interact with one another. Typically a reduced-form model uses an exogenous correlation matrix for the risk factors. As a result, reduced-form models cannot answer the question like how interest rates and credit spreads substitute or supplement each other; or how equity volatility can lead to lower or higher recovery.

Our model (i.e., Geske model with random interest rates) is a structural model that naturally endogenizes these risk factors. Hence, the interactions among these risk factors can be analyzed with full economic rigor. Our model is

particularly useful when all three risks – equity, credit, and interest rate, interact with one another. A perfect example for all three risks is convertible bonds where all three risks jointly determine the value of conversion. Moreover, it is common that convertible bonds embed call and put options that further convolute the relationships of these risks.

For the current section of the paper, the parameter values for the Vasicek model are set as follows:

α	0.4
θ (constant ¹³)	0.026
γ	0.06
r_0	0.03

The capital structure of the firm is given as follows:

t	0	V_0	300
T_1	1	K_1	30
T_2	2	K_2	30
T_3	3	K_3	30
T_4	4	K_4	30
T_5	5	K_5	30
T_6	6	K_6	30
T_7	7	K_7	30
T_8	8	K_8	30
T_9	9	K_9	30
T_{10}	10	K_{10}	30
Δt	1/10	σ	0.2
		ρ	-0.25

Unless otherwise specified, these are the parameter values used in the examples in this section. Given this set of parameter values, the equity value is \$55.55. The risky discount factor, $D_{t,T_i} / K_i$, and the risk-free discount factor, P_{t,T_i} , are given as follows:

¹³ In the Vasicek model, $\theta = \alpha\mu - \lambda\gamma$ where μ is the level of mean-reversion and λ is the market price of risk. In here we set $\mu = 0.05$ and $\lambda = -0.1$.

	$D_{t,T_i} / K_i$	P_{t,T_i} ¹⁴
T_1	0.9810	0.9810
T_2	0.9560	0.9561
T_3	0.9272	0.9273
T_4	0.8961	0.8962
T_5	0.8635	0.8638
T_6	0.8292	0.8307
T_7	0.7867	0.7975
T_8	0.7278	0.7646
T_9	0.6448	0.7322
T_{10}	0.5360	0.7005

We calculate the implied spread of the liabilities using the following equation:

$$(27) \quad s = \sum_{i=1}^{10} \ln[D_{t,T_i} / K_i / P_{t,T_i}]$$

Equation (27) represents an “aggregated” spread of all maturities. The result is 4.95% (or an average spread of 49.5 basis points). We shall use this quantity to measure credit risk in this section.

4.1 Interactions among Equity, Interest Rate, and Credit Risk

The first analysis of our model is convertible bonds in that convertible bonds are exposed to multiple risks. Convertible bonds are exposed to the interest rate risk due to their fixed coupons. They are also exposed to the equity risk due to conversion. Finally they are exposed to the credit risk in that many convertible bonds are high yields so default is a crucial concern.

Imagine a convertible bond with the coupon rate ξ and maturity T_m . It must have a corresponding straight bond with a coupon rate c and the same maturity T_m that is priced as:

¹⁴ Note that these risk-free discount factors are not direct results of the Vasicek model. Due to the discrete implementation of the binomial model, the Vasicek discount factors that are continuously compounded are converted into discrete time in order to be consistent with the binomial implementation. Details can be obtained on request.

$$(28) \quad B_{t,T_m} = c \sum_{i=1}^m \frac{D_{t,T_i}}{K_i} + \frac{D_{t,T_m}}{K_m}$$

where D_{t,T_i} for all i is defined in (26) and $D_{t,T_i}/K_i$ for $i = 1, \dots, n$ can be viewed as the i -th discount factor. We note that the coupons of the convertible bond share the same seniority structure as the corresponding zero coupon debts. In other words, if the firm defaults at time T_k for $k < m$, then the recoveries of the remaining coupons of the convertible bond must follow the same seniority structure of the zero coupon bonds.

It is interesting to note that (28) is similar to the formulation of the Duffie-Singleton model (1999). We note that similarities exist between our model and the Duffie-Singleton model in that each D_{t,T_i} term contains survival value as well as recovery value. Surprisingly interestingly, our model in (10) when we sum up all debt values is also parallel to the Jarrow-Turnbull model (1995) that separates the recovery value from the survival value. In the next section, we compare our model in details with the Duffie-Singleton model as well as the Jarrow-Turnbull model.

At any time t , the convertible bond is $\max\{N_B B_{t,T_m}, CS_t\}$ where N_B is the notional of the convertible bond, C is the conversion ratio, and S is the stock price which is the equity value divided by the number of outstanding shares N_S , i.e. $S_t = E_t / N_S$. In this sub-section, we set the conversion ration $C = 1$, number of stock shares $N_S = 1$, notional of the bond $N_B = 100$, the coupon rate $\xi = 0$, and finally $m = 4$. The convertible bond price is \$115. In other words, the conversion value is \$14.

We can now study the relationships among different risk factors. As an example, we choose to let vary the following two parameters:

- asset volatility

- current short rate¹⁵

The convertible bond prices are computed using equation (28). We simulate 100 interest rate scenarios (0.1% to 10% at an increment of 0.1%) and 150 volatility scenarios (5% to 20% at an increment of 0.1%) to obtain 15,000 combinations of asset volatility and interest rates. The outputs of these input changes are (1) convertible bond spread, (2) equity volatility, and (3) 10-year risk-free yield which is a common measure for interest rate levels. These outputs are the market observables and are jointly determined by our model. We present the interactions of the three key outputs that are relevant to convertible bond valuation in a series of contour plots in Figure 3. The regression result is given in the Appendix.¹⁶

[Figure 3 Here]

From these contour plots, we shall see that given CB prices, 10-year yield interacts with equity volatility but not with credit spread while equity volatility interacts with 10-year yield and spread. We also observe that the interaction between volatility and spread presents very different patterns at different CB prices. When CB prices are high, the relationship is non-monotonic. The same phenomenon is observed for the interaction between volatility and 10-year yield.

4.2 “Optimal” Capital Structure

The “optimal” capital structure here refers the concept of “rating chasing” that many companies adopt as their primary financing strategy. This is also known as mean-reversion of capital structure.¹⁷ There is ample evidence in the

¹⁵ The reverting level μ changes with the short rate in order to keep the “slope” of the yield curve fixed.

¹⁶ Regressions with higher powers and interaction terms are also run and the results are similar (available upon request).

¹⁷ See Kisgen (2006, 2007) for an excellent review on rating target.

literature on how credit ratings affect the capital structure of a company. For example, Kisingen (2006) argues that “firms near a credit rating upgrade or downgrade issue less debt relative to equity than firms not near a change in rating”. More recently, Hovakimian, Kayhan, and Titman (2009) investigate to see if firms have target credit ratings that are connected to various capital structure decisions. They find strong evidence that supports the idea that firms make corporate financial choices that offset shocks that move them away from their target capital structures.¹⁸

In this sub-section, we study how a firm can adjust its capital structure (via the amount of debt) to achieve the target credit risk (hence rating). As our model ties capital structure directly to credit quality (i.e. credit spreads) of the company, by assuming there is monotonic mapping between credit ratings and credit spreads, we can then show how managers of the company should change the capital structure in order to achieve the target rating for the company.

As a crude approximation, we continue to use equation (27) for the credit spread. To answer the question that what kind of capital structure can meet a certain rating target, we use the following linear equation for the term structure of debts: $K_i = a + bT_i$. When b is positive, the company adopts more long-term debts than short-term debts. Similarly, when b is negative, the company adopts more short-term debts than long-term debts. When $b = 0$, then the company adopts a flat debt structure. For any given credit spread produced by equation (27), we solve for a combination of a and b . In other words, there are infinite number of $\langle a, b \rangle$ combinations that can produce a certain credit spread.

¹⁸ The literature on dynamic capital structure (currently nicknamed “rating chase” by the industry) and how ratings affect capital structure decisions is voluminous. While our model provides a useful tool for making such decisions, this sub-section is not the main focus of our paper. Hence, we apologize that many classical papers in this area are omitted from the paper. Interested readers please refer to Kisingen (2007) for a review.

We then measure how each combination implies a certain capital structure. For example, at 1% spread, $a = 25.25$ and $b = -1$ are the resulting combination, which implies that the 1-year debt is \$25.25 and linearly drops to \$15.25 for the 10-year debt is the capital structure for 1% credit spread. To gain a more intuitive feel for such a capital structure, we represent the short-term and long-term debts as percentages of the asset value. For example \$25.25 is 8.42% (top left) of the asset value \$300 and \$15.25 is 5.42% of the asset value.

	1% credit spread		
1-yr debt	8.42%	6.15%	3.66%
10-yr debt	5.42%	6.15%	6.66%
	5% credit spread		
1-yr debt	11.11%	9.28%	7.01%
10-yr debt	8.11%	9.28%	10.01%
	10% credit spread		
1-yr debt	12.68%	10.83%	8.98%
10-yr debt	9.68%	10.83%	11.98%

The above table indicates that various term structures (i.e. combinations of a and b) can achieve a specified goal of credit spread (hence credit rating). For 1% credit spread, a short-term of 8.42% and a long-term debt of 5.42% will suffice as well as a short-term of 3.66% and a long-term debt of 6.66%; or even a flat term structure of 6.15%. However, the total leverage ratios (i.e. all 10 debts) are different – 69.17% for the case of decreasing term structure, 61.5% of flat term structure, and 51.65% for the case of increasing term structure.¹⁹

Same analysis can be carried out for other credit spreads. The model suggests that if a firm uses more short-term debts than long-term debts, then the overall leverage ratio must be higher, in order to achieve the same credit spread

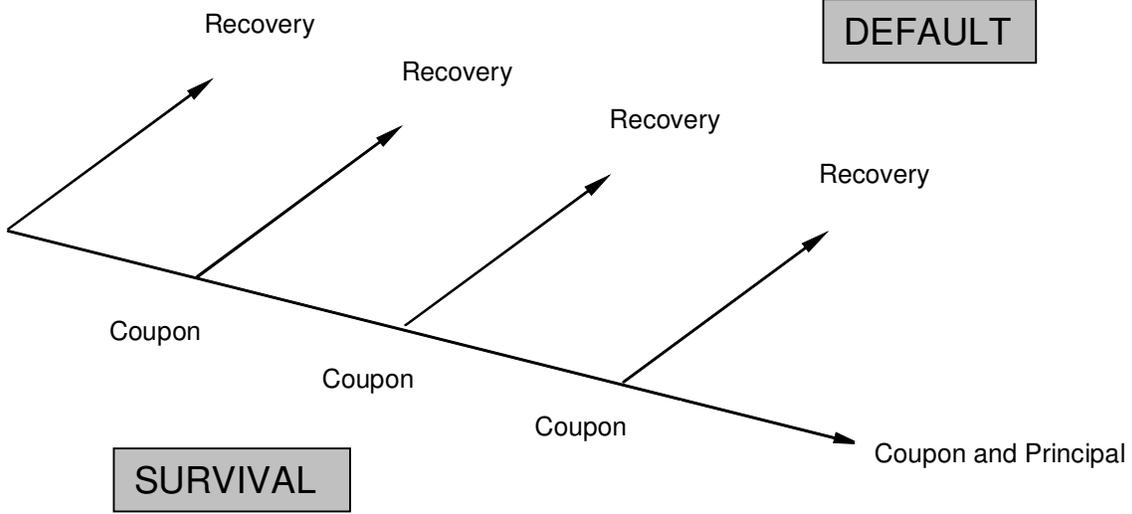
¹⁹ Note that each debt is just \$1/\$300 increment or decrement of its previous debt.

(hence credit rating). This substitution effect is more pronounced when the firm targets very high ratings. If the firm is targeting only mediocre ratings (e.g. 10% credit spread), then the substitution is not so obvious. In other words, our model indicates the term structure of liabilities matters substantially in a firm's target rating decisions.

Certainly, the term structure of liabilities does not need to be linear and it can further interact with investment decisions (i.e. asset volatility). To carry out a full analysis, we must generate multi-dimensional combinations of various parameters. Yet no matter how complicated the analysis could be, our model is a helpful tool in this important corporate finance decision.

5 Consistency with the Reduced-form Models

In this section, we provide an efficient comparison between the popular reduced-form models widely used in pricing credit derivatives and our model. The two popular reduced-form models are proposed by Jarrow and Turnbull (1995) who assume recovery of face value and Duffie and Singleton (1999) who assume recovery of market value. Regardless of their recovery assumptions, the default event is defined over the Poisson process, which can be graphically displayed by the following binomial diagram (for a coupon paying bond):



As a result, the price of a risky coupon bond under the Jarrow-Turnbull model can be written as:

$$\begin{aligned}
 (29) \quad B_{t,T_n}^{JT} &= \mathbb{E}_t \left[c \sum_{i=1}^n \exp\left(-\int_0^{T_i} r_u du\right) 1_{\{\tau > T_i\}} + \exp\left(-\int_0^{T_n} r_u du\right) 1_{\{\tau > T_n\}} \right. \\
 &\quad \left. + \exp\left(-\int_0^{\tau} r_u du\right) w_{\tau} 1_{\{\tau < T_n\}} \right] \\
 &= c \sum_{i=1}^n P_{t,T_i} Q_{t,T_i} + P_{t,T_n} Q_{t,T_n} + R_{t,T_n}
 \end{aligned}$$

where P_{t,T_i} is the risk-free discount factor for the future time T_i , c is percentage of the coupon payment over the face value, w is the recovery rate, and τ is default time. The first two terms of the last line represent the value of no default (under which the bond pays all coupons and the face value) and the last term represents the current value of recovery under default, which is an expected present value of the discounted recovery amount upon default.

The Duffie-Singleton model assumes recovery to be paid upon default and to equal a fraction of the value of the bond if it were not defaulted. Formally, we write the recovery as $w_t = \delta B_{t,T_n}$ where $0 < \delta < 1$ is constant. The Duffie-Singleton model is,

$$(30) \quad B_{t,T_n}^{DS} = c \sum_{i=1}^n P_{t,T_i} \hat{Q}_{t,T_i} + P_{t,T_n} \hat{Q}_{t,T_n}$$

where P_{t,T_i} is the risk-free discount factor for the future time T_i , c is percentage of the coupon payment over the face value, and $\hat{Q}_{t,T_i} = Q_{t,T_i}^{1-\delta}$. Note that since the recovery δ is incorporated in every \hat{Q}_{t,T_i} term, there is no need to have a separate term for recovery.

To show that our model is consistent with the Poisson process used by the reduced-form models, we first acknowledge that equation (28) is parallel to the Duffie-Singleton model, while equation (22) (or similarly equation (10)) is similar to the Jarrow-Turnbull model where the recovery is separate from the survival value.²⁰ In other words, if we evaluate the total debt value of a company when the true capital structure is considered, the Jarrow-Turnbull model can be appropriate. On the other hand, if we evaluate credit derivatives that are based upon a given capital structure, then Duffie-Singleton model is appropriate. From (22) and (28), we can find the consistency between the two popular reduced-form models.

In this section, we examine the multi-period behavior of the reduced-form models, namely Jarrow-Turnbull and Duffie-Singleton and our model (which is the Geske model with random interest rates). We incorporate stochastic interest rates in our analysis. The parameters for the Vasicek model are given in the previous section. To better reflect reality, we set $n = 30$.

5.1 Zero-coupon Bond

We first examine the case of extremely low coupons ($K_1 = \dots = K_{29} = c = 0$). The face value of debt (K_{30}) is 110 and the asset value of $V_0 = 184$. This is the case where we can see the fundamental difference between our model and the Jarrow-Turnbull model. We cannot solve the Duffie-

²⁰ This is the case where we assume the total liability of the firm is a gigantic coupon bond and in this case, our model.

Singleton model in that their model does separate recovery from the hazard rate. Note that the Jarrow-Turnbull model is (29) and our model is a binomial implementation of (28).²¹ We run our model with various volatility levels: 0.4, 0.6, 1.0, and 1.6. Once we obtain the expected recovery and the current debt value ($D_{t,T_{30}}$), we then solve for the intensity rate and the recovery rate in the Jarrow-Turnbull model. Two equations and two unknowns allow us to find the exact match between the Jarrow-Turnbull model and our model. Note that the last term of equation (29) is the Jarrow-Turnbull value of the expected recovery, which is easy to compute. The results are summarized as follows.

Our Model				
Volatility	0.4	0.6	1.0	1.6
equity value	169.13	177.97	183.60	183.99
debt value	14.87	6.03	0.40	0.01
Recovery	4.17	2.12	0.19	0.00
JT Model				
recovery rate	1.04%	0.21%	0.01%	0.00%
Intensity	1.48%	4.50%	13.55%	25.78%

As the volatility goes up, the equity value in our model goes up (i.e. call option value goes up.) Since the asset value is fixed at 184, the debt value goes down. The survival probability curves under various volatility scenarios of our model are plotted in Figure 4(a); and the default probability curves (unconditional, i.e. $Q_{t,T_{-1}} - Q_{t,T_i}$) are plotted in Figure 4(b).

[Figure 4 Here]

We observe several results. First, as the risk of default becomes eminent (i.e. high volatility and low debt value), the likelihood of default shifts from far

²¹ Note that with $c = 0$, our model is effectively the Rabinovitch model (1989). Yet, to obtain the entire default and survival curves, we must implement the binomial lattice.

terms (peak at year 30 for volatility = 0.4) to near terms (peak at year 5 for volatility = 1.6). Second, it is seen that the asset volatility has an influential impact on the shape of the survival probability curve. As volatility increases, the shape of the survival probability curve changes from strictly concave to an inverted S. Furthermore, our model is able to generate humped default probability curve (volatility = 1.6), often observed empirically. Third, these differently shaped probability curves are generated by one single debt, something not possible by reduced-form models. Both the Jarrow-Turnbull and Duffie-Single models cannot generate such probability curves with one single bond, due to the lack of information of intermediate cash-flows.

Corresponding to the recovery amounts under our model, we set the fixed recovery rate of the Jarrow-Turnbull model as shown in the above table. Given that one bond can only imply one intensity parameter value, we set it (under each scenario) so that the zero coupon bond price generated by our model is matched. As shown in the above table, the intensity rate goes from 1.48% per annum to 25.78% per annum. Note that flat intensity value is equivalent to a flat *conditional* default probability curve.

5.2 Coupon Bond

Our next analysis is to keep the bond value fixed, so that we can examine the default probability curve with the risk of the bond controlled. To do this, we assume that the company issues a par coupon bond at a coupon rate of 10%. At the volatility level of 0.4, the asset value is \$119, at 0.6, it is \$255, and at 1.0, it is \$9,091. The Vasicek model is assumed. Figure 5 demonstrates the results of our model for various volatility levels but keep the bond at par. We can see that for the same par bond, the default and survival probability curves are drastically

different as the asset/volatility combination changes. This is a feature not captured by either the Duffie-Singleton or the Jarrow-Turnbull model.²²

[Figure 5 Here]

To compare to the Jarrow-Turnbull and Duffie-Turnbull models, we keep the case where the volatility level is 0.6 and asset value is 255. The recovery amount is \$26.86 in this case. Our model solves for a survival probability curve and an expected recovery amount that evaluate a 10% par bond. We then try to use the Jarrow-Turnbull model and the Duffie-Singleton model to correctly price the par bond and match the recovery amount at the same time by solving for the fixed hazard rate and fixed recovery rate. As discussed in the literature, given that the Duffie-Singleton model does not separate recovery from hazard, it is not possible to solve separately for the hazard rate and the recovery rate in the Duffie-Singleton model, unlike the Jarrow-Turnbull model.

The hazard and the recovery rates for the Jarrow-Turnbull model are solved to be 8.38% and 43.09% respectively. Due to the under-identification problem suffered by the Duffie-Singleton model, we find that there is no perfect match by the Duffie-Singleton model and there are infinite number of recovery-hazard pairs for the minimum pricing error. We select the one pair where the hazard rate matches the closest to 8.38%. Under this criterion, we obtain 8% and 48.87% as the hazard rate and the recovery rate for the Duffie-Singleton model. Figure 6 shows the survival and default probability curves of the three models. From the survival probability curves, it is seen that our and the Jarrow-Turnbull models can be close.

[Figure 6 Here]

²² Although not shown here, it is noted that the default probability curve in the case of 100% volatility is exploding.

From Figure 6, we observe that the survival probability curve generated by our model is more convex than the two reduced-form models. It has two cross-over points with the curves generated by the two reduced-form models.²³ At near terms, the survival probabilities of our model are higher than the two reduced-form models, but then the probabilities decrease more rapidly than its two reduced-form counterparts. Finally, they recover to be higher than the two reduced-form counterparts. Given that the recovery amount is matched, it must be true that the areas of above and below are cancelled in order to generate the bond price (at par).

The coupon bond example in this sub-section is similar to the zero coupon bond example previously. The only difference is that we can now control for the target bond price (at par) and see how the probability curves react to various combinations of asset values and asset volatility levels. We witness very similar results as in the zero coupon bond case. The comparison to the reduce-form models sheds light on how prices can be matched and yet the fundamental default profiles can be drastically different.

5.3 Credit Default Swap

Using structural models for credit default swaps (CDS) has been difficult. Existing models all adopt short-cuts as approximations.²⁴ Here we provide an accurate and efficient solution to the price of the credit default swap. The reduced-form models can only evaluate CDS with both credit and interest rate risks and ignore the equity risk. Unfortunately, default risk is related to the

²³ Inferring from the survival probability curve, we can obtain the flat conditional forward default probability curves induced by the two reduced-form models and a humped curve by our model.

²⁴ For example, an exogenous flat barrier is often assumed while the true default barrier is stochastic. A constant recovery rate is also often assumed while the true recovery is stochastic as well.

firm's capital structure and hence must also be related to the equity risk. Indeed, high grade bonds that have low spreads are more sensitive to equity risk than credit or interest rate risk and high yields are more sensitive to interest rate and credit risks. Our model allows, for the first time, these risks to be jointly and accurately priced in CDS contracts.

Credit default swaps are the most widely traded credit derivative contract today. A default swap contract offers protection against default of a pre-specified corporate issue. In the event of default, a default swap will pay the principal (with or without accrued interest) in exchange for the defaulted bond.²⁵

Default swaps, like any other swap, have two legs. The premium leg contains a stream of payments, called spreads, paid by the buyer of the default swap to the seller till either default or maturity, whichever is earlier. The other leg, protection leg, contains a single payment from the seller to the buyer upon default if default occurs and 0 if default does not occur. Under some restrictive conditions, credit default swap spreads are substitutes for par floater spreads.²⁶ In many occasions, the traded spreads off credit default swaps are more representative of credit risk than corporate bonds. The valuation of a credit default swap is straightforward. For the default protection leg:

$$\begin{aligned}
 (31) \quad W_{t,T} &= \int_t^T (1 - w(V_t, r_t)) P_{t,s}[-dQ_{t,s}] \\
 &= \int_t^T P_{t,s}[-dQ_{t,s}] - \int_t^T w(V_t, r_t) P_{t,s}[-dQ_{t,s}] \\
 &= \int_t^T P_{t,s}[-dQ_{t,s}] - R_{t,T}
 \end{aligned}$$

²⁵ Default swaps can also be designed to protect a corporate name. These default swaps were used to be digital default swaps. Recently these default swaps have a collection of “representative” reference bonds issued by the corporate name. Any bond in the reference basket can be used for delivery.

²⁶ See, for example, Chen and Soprazzetti (2003) for a discussion of the relationship between the credit default swap spread and the par floater spread.

where $P_{t,s}$ is the risk-free discount factor between now t and future time s , and $Q_{t,s}$ is the survival probability between now t and future time s .

In a discrete time where spread payments are made at times $T_1 \dots T_n$, we can write (31) as:

$$(32) \quad W_{t,T_n}^{\text{prot}} = \sum_{i=1}^n P_{t,T_i} [Q_{t,T_{i-1}} - Q_{t,T_i}] - R_t$$

This is called the protection leg or the floating leg of the swap. For the premium leg, or the fixed leg, we can write the valuation equation as:

$$(33) \quad W_{t,T_n}^{\text{prem}} = s \sum_{i=1}^n P_{t,T_i} Q_{t,T_i}$$

and hence the default swap spread can be solved as follow by setting the values of the two legs equal:

$$(34) \quad s_{\text{CDS}} = \frac{\sum_{i=1}^n P_{t,T_i} [Q_{t,T_{i-1}} - Q_{t,T_i}] - R_t}{\sum_{i=1}^n P_{t,T_i} Q_{t,T_i}}$$

In our model, the survival probability $Q_{t,s}$ can be computed easily as:

$$(35) \quad \begin{aligned} Q_{t,T_i} &= \Pr[V_{T_1} > V_{1i}^*, V_{T_2} > V_{2i}^*, \dots, V_{T_i} > V_{ii}^*] \\ &= \mathbb{E}_t \left[\mathbf{1}_{V_{T_1} > V_{1i}^*, V_{T_2} > V_{2i}^*, \dots, V_{T_i} > V_{ii}^*} \right] \end{aligned}$$

where \mathbb{E}_t is the risk-neutral expectation. Equation (35) does not have a closed-form solution but can be easily computed via the binomial model.

Combining (32) and (33), we can use market credit default swap spreads to back out default probability curve. As the default swap market grows, more and more investors seek arbitrage trading opportunities between corporate bonds and default swaps. This suggests that we should use the calibrated corporate bond curves to compute default swap spreads. We use the results of Figure 6 to compute the 30-year CDS value.²⁷

²⁷ Note that CDS values for various tenors can be computed in the same way using equations (31) and (32).

Here, we demonstrate how such arbitrage trading strategies can be misleading. Arbitrage profits can be entirely due to model specification. To see that, we suppose that our model is the correct model. Continue to assume the Vasicek model with the given parameter values. Then, the probabilities that calculate the par bond are shown in Figure 7 for the case where asset value is \$255 and volatility is 0.6 (so that the coupon debt is at par). The 30-year default swap spread implied by our model is 499 basis points. This is done by implementing (31) to obtain the default swap protection value of \$35.47 and implementing (32) to solve for the spread.²⁸ Note that the credit default swap value and spread computed using our model are consistent with the recovery assumption of our model. To use the Jarrow-Turnbull model, we must fix the recovery rate. To get such value, we use that the probabilities generated by our model and a fixed recovery rate to compute the default swap value. This implied recovery rate is 0.4310. If we let the Jarrow-Turnbull model calibrate to the data (i.e. the 10% par bond), then as mentioned earlier the hazard rate and the recovery rate are 8.38% and 43.09% respectively. The CDS premium under the model would be 498 basis points, which is 1 basis point different from the correct value.

As the volatility of the asset rises, the error becomes larger. At 100% volatility (the corresponding asset value, as mentioned earlier, is \$9,091), the correct CDS premium is 506 basis points and the implied Jarrow-Turnbull recover rate is 20.89%.²⁹ If we let the Jarrow-Turnbull model calibrate to the data, then the hazard rate and the recovery rate are 6.07% and 21.80%

²⁸ The expected recovery value in (31) is calculated to be \$26.86 (which is 26.86% of the face value of the bond which is priced at par) under the volatility of 0.6 and asset value of \$255. The full default value (first term of (31)) is calculated to be 62.33% of the face value. As a result, the CDS protection value is 35.47% of the face value, or \$35.47. The risky annuity demonstrated in equation (32) is computed to be 7.1037 and hence the spread is 499 basis points.

²⁹ In this case, the expected recovery rate is \$11.69.

respectively. The resulting CDS spread is 489 basis points – a 17 basis point or 3.3% difference.

[Figure 7 Here]

The situation can be even more severe if we allow our model to generate more humped shaped default probability curve.³⁰

5.4 Cheapest-to-Deliver (CTD) Option

Finally we would like to make a note on how our model can be readily used for the valuation of the cheapest-to-deliver (CTD) option embedded in CDS contracts. The majority of CDS contracts require physical deliveries of the underlying bonds in the event of default. Being able to delivery any bond of the defaulted firm, a rational protection buyer of the CDS contract will surely deliver the cheapest bond. Such an option has been shown to have a non-trivial value (see, for example, Pan and Singleton (2008)). To evaluate the CTD option accurately, it must be the case that each and every bond of the defaulted firm is accurately evaluated. Reduced-form models, as a result, cannot accomplish such work as they provide no linkage between CDS and their underlying bonds. With our model, one can easily evaluate CDS contracts with the CTD option.

To see that, we improve equation (32) with the following changes:

$$(36) \quad W_{t,T_n}^{\text{prot}} = \sum_{i=1}^n P_{t,T_i} [Q_{t,T_{i-1}} - Q_{t,T_i}] - \min_i \{R_t^i\}$$

where $i = 1, \dots, n$ represents the i -th bond issued by the firm. As the cheapest bond is delivered in return for the full notional value, the protection provider of the CDS contract suffers the most loss. Given that this is a known behavior by

³⁰ We experiment a large number of various scenarios and discovered that the errors can be quite high in both overvalue and undervalue.

the protection buyer, the protection value is higher, as equation (36) shows, and hence the CDS premium is higher.

6 Conclusion

The capital structure model of credit risk began with Black and Scholes (1973)³¹ and Merton (1974) and then was extended by Geske (1977) and Leland (1994).³² However, none of these models has been widely used because the Geske model is computationally complex and expensive and the Leland model does not allow for flexible structures of debts.

The Geske model is computationally complex and expensive because of the necessity to compute multi-variate normal probabilities. Another even more bothersome complexity is the need to compute “implied strikes” (also known as the default barrier). This complexity prevents the Geske model from incorporating random interest rates. The Leland model, on the other hand, is a closed-form model which is inexpensive to compute and easy to incorporate random interest rates (see Huang, Ju, and Ou-Yang (2003)). However, the assumption of firms must continuously issue perpetual debts does not represent the reality well.

In this paper, we successfully resolve this issue. By transforming the Geske model from a fixed strike form to a random strike form, we can successfully incorporate random interest rates in the model. The binomial implementation of the model further facilitates the computation of the random strikes. As a result, we obtain a model that is as easy to implement as the

³¹ Their Section 6 “Common Stock and Bond Valuation”.

³² Also Leland-Toft (1996).

Leland model with random interest rates and keep the flexibility of the debt structure.

We then use our model for a number of applications. Using a convertible bond example, we illustrate how various risks (i.e. equity, interest rate, and credit) can interact with one another endogenously. There are substitution effects among these risk factors. Our model also makes possible to study the common phenomenon in the industry known as “rating chasing”. We show that with our model we can identify proper capital structure to achieve a particular rating objective. Lastly, we provide a consistency analysis of our model and popular reduced-form models in the industry. We find that our model is consistent with the Jarrow-Turnbull model from the whole company’s perspective and consistent with the Duffie-Singleton model from the individual debt’s perspective. When we apply our model to the valuation of credit default swaps, we identify some misleading results by the reduced-form models. In other words, reduced-form models could signal “false arbitrage opportunities” while the market is actually perfectly efficient.

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Table 1

t	T_1	T_2	T_3
$D_{01}^* = e^{-r\Delta t} \mathbb{E}[D_{11}] = e^{-r\Delta t} \mathbb{E}[\min V_1, K_1]$ $= e^{-r\Delta t} \left[\int_0^{K_1} V_1 + \int_{K_1}^{\infty} K_1 \right]$ $D_{02}^* = e^{-r\Delta t} \mathbb{E}[D_{12}]$ $= e^{-r\Delta t} \left[\int_{K_1}^{V_{12}^*} V_1 - K_1 + \int_{V_{12}^*}^{\infty} D_{12}^* \right]$ $D_{03}^* = e^{-r\Delta t} \mathbb{E}[D_{13}]$ $= e^{-r\Delta t} \left[\int_{V_{12}^*}^{V_{13}^*} V_1 - D_{12}^* - K_1 + \int_{V_{13}^*}^{\infty} D_{13}^* \right]$ $E_0^* = e^{-r\Delta t} \mathbb{E}[E_1] = e^{-r\Delta t} \mathbb{E}[\max E_1^* - K_1, 0]$ $= e^{-r\Delta t} \mathbb{E}[\max V_1 - D_{12}^* - D_{13}^* - K_1, 0]$ $= e^{-r\Delta t} \left[\int_{V_{13}^*}^{\infty} V_1 - D_{12}^* - D_{13}^* - K_1 \right]$	$\underline{K_1 + D_{12}^* + D_{13}^* < V_1 < \infty}$ $D_{11} = K_1$ $D_{12} = D_{12}^*$ $D_{13} = D_{13}^*$ $E_1 = V_1 - K_1 - D_{12}^* - D_{13}^* (= E_1^* - K_1)$ $\underline{K_1 + D_{12}^* < V_1 < K_1 + D_{12}^* + D_{13}^* (\equiv V_{13}^*)}$ $D_{11} = K_1$ $D_{12} = D_{12}^*$ $D_{13} = V_1 - D_{12}^* - K_1$ $E_1 = 0$ $\underline{K_1 < V_1 < K_1 + D_{12}^* (\equiv V_{12}^*)}$ $D_{11} = K_1$ $D_{12} = V_1 - K_1$ $D_{13} = 0$ $E_1 = 0$ $\underline{0 < V_1 < K_1 (\equiv V_{11}^*)}$ $D_{11} = V_1$ $D_{12} = 0$ $D_{13} = 0$ $E_1 = 0$	$\underline{K_2 + D_{23}^* < V_2 < \infty}$ $D_{22} = K_2$ $D_{23} = D_{23}^*$ $E_2 = V_2 - K_2 - D_{23}^* (= E_2^* - K_2)$ $\underline{K_2 < V_2 < K_2 + D_{23}^* (\equiv V_{23}^*)}$ $D_{22} = K_2$ $D_{23} = V_2 - K_2$ $E_2 = 0$ $\underline{0 < V_2 < K_2 (\equiv V_{22}^*)}$ $D_{22} = V_2$ $D_{23} = 0$ $E_2 = 0$ where	$\underline{K_3 < V_3 < \infty}$ $D_{33} = K_3$ $E_3 = V_3 - K_3$ $\underline{0 < V_3 < K_3 (\equiv V_{33}^*)}$ $D_{33} = V_3$ $E_3 = 0$

	<p>where</p> $D_{12}^* = e^{-r\Delta t} \mathbb{E}[D_{22}]$ $= e^{-r\Delta t} \mathbb{E}[\min V_2, K_2]$ $= e^{-r\Delta t} \left[\int_0^{K_2} V_2 + \int_{K_2}^{\infty} K_2 \right]$ $D_{13}^* = e^{-r\Delta t} \mathbb{E}[D_{23}]$ $= e^{-r\Delta t} \left[\int_{V_{23}^*}^{\infty} D_{23}^* + \int_{K_2}^{V_{23}^*} V_2 - K_2 \right]$ $E_1^* = e^{-r\Delta t} \mathbb{E}[E_2]$ $= e^{-r\Delta t} \mathbb{E}[\max V_2 - D_{23}^* - K_2, 0]$ $= e^{-r\Delta t} \left[\int_{V_{23}^*}^{\infty} V_2 - D_{23}^* - K_2 \right]$ $= V_1 - D_{12}^* - D_{13}^*$ <p>Note that:</p> $E_1 = \max\{V_1 - V_{13}^*, 0\}$ $D_{13} = \max\{V_1 - V_{12}^*, 0\} - \max\{V_1 - V_{13}^*, 0\}$ $D_{12} = \max\{V_1 - K_1, 0\} - \max\{V_1 - V_{12}^*, 0\}$ $D_{11} = V_1 - \max\{V_1 - K_1, 0\}$	$D_{23}^* = e^{-r\Delta t} \mathbb{E}[D_{33}]$ $= e^{-r\Delta t} \mathbb{E}[\min V_3, K_3]$ $E_2^* = e^{-r\Delta t} \mathbb{E}[E_3]$ $= e^{-r\Delta t} \mathbb{E}[\max V_3 - K_3, 0]$ $= e^{-r\Delta t} \mathbb{E}[V_3 - \min V_3, K_3]$ $= V_2 - D_{23}^*$ <p>Note that:</p> $E_2 = \max\{V_2 - V_{23}^*, 0\}$ $D_{23} = \max\{V_2 - K_2, 0\} - \max\{V_2 - V_{23}^*, 0\}$ $D_{22} = V_2 - \max\{V_2 - K_2, 0\}$
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Table 1 – In this table, in order to conserve space and present the table in the most intuitive way, we abbreviate the notation in the following way. Time index T_i is abbreviated as i and the current time t is abbreviated as 0. For example, D_{T_i, T_j} is abbreviated as D_{ij} , V_{T_i} is abbreviated as V_i , and E_t is abbreviated as E_0 .

The model is intuitive. For example, the first debt's default point is K_1 . As long as $V_{T_1} > K_1$ the first debt will be paid in full (although the firm's default point is $V_{13}^* > K_1$). The (risk-neutral) probability of $V_{T_1} > K_1$ is $N_1(h_1^-(K_1))$. And the probability of $V_{T_1} < K_1$ is $1 - N_1(h_1^-(K_1))$, which can be written as $1 - N_1(h_1^+(K_1))$ if we make the change of measure using the asset value as the numeraire (see Appendix for the proof). As a result, the expected recovery is $V_i[1 - N_1(h_1^+(K_1))]$.

Similarly, the second debt defaults when $V_{T_2} < \bar{V}_{12}$ where \bar{V}_{12} is the solution to $V_{T_1} = K_2 + D_{T_1, T_2}$. This is a uni-variate search of the normal probability. Note that the second debt survives if $V_{T_1} > K_1$ and $V_{T_2} > \bar{V}_{12}$, hence the survival probability is two-dimensional

$N_2(h_1^-(\bar{V}_{12}), h_2^-(K_2); \mathbb{C}_2)$. The default is either $V_{T_1} < K_1$ or $V_{T_1} > K_1 \cap V_{T_2} < \bar{V}_{12}$

Table 2 – The binomial lattice for the Geske Model

Basic inputs:

V_t	100	r	3%
σ	0.5	Δt	0.5
T_1	1	K_1	40
T_2	2	K_2	40
T_3	3	K_3	40

Lattice:

A - Firm Value						
						834.21
					585.78	
				411.33		411.33
			288.83		288.83	
		202.81		202.81		202.81
	142.41		142.41		142.41	
100.00		100.00		100.00		100.00
	70.22		70.22		70.22	
		49.31		49.31		49.31
			34.62		34.62	
				24.31		24.31
					17.07	
						11.99
		$T_1 = 1$		$T_2 = 2$		$T_3 = 3$
$t = 0$		$K_1 = 40$		$K_2 = 40$		$K_3 = 40$

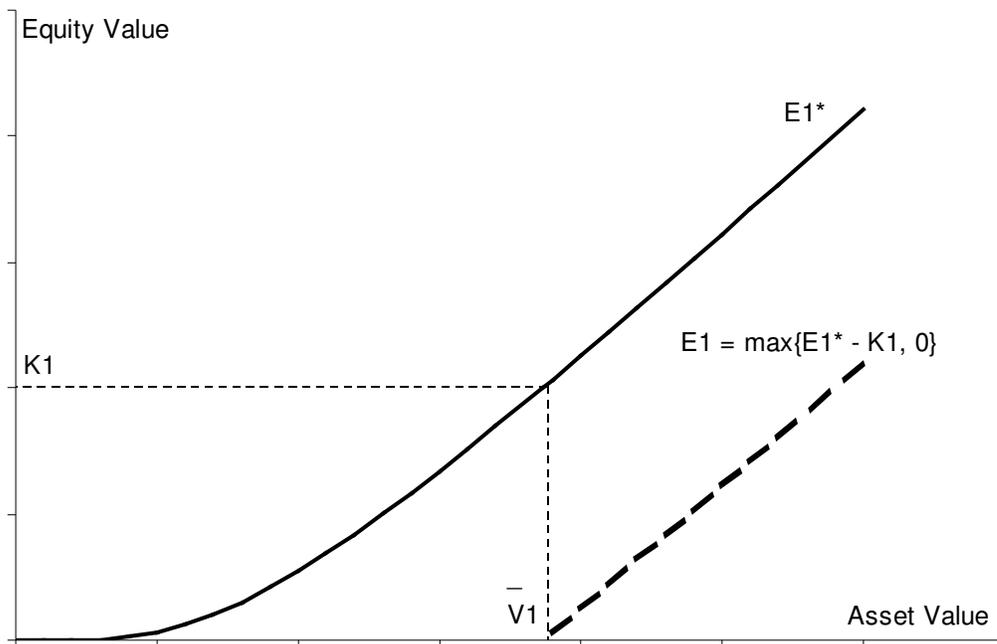
B - Equity							
E_{T_3}		E_{T_1}	$E_{T_1}^*$		E_{T_2}	$E_{T_2}^*$	E_{T_3}
							794.2145
						546.3713	
					332.5072	372.5072	371.3250
				211.1833		249.4232	
		86.3231	126.3231		123.9937	163.9937	162.8115
	36.8601			64.7675		103.0074	
15.7394		0.0000	32.7039		21.1822	61.1822	60.0000
	0.0000			9.0448		30.8144	
		0.0000	3.8622		0.0000	15.3758	9.3069
				0.0000		3.9741	
					0.0000	1.6969	0.0000
						0.0000	
							0.0000

C - T ₃ maturity Debt							
D_{t,T_3}	D_{T_1,T_3}	D_{T_1,T_3}^*	D_{T_2,T_3}	D_{T_2,T_3}^*	D_{T_3,T_3}		
							40.0000
						39.4045	
			38.8178	38.8178			40.0000
			38.2399			39.4045	
	37.6706	37.6706	38.8178	38.8178			40.0000
	27.9074		38.2399			39.4045	
16.9645		21.1822	28.4783	38.8178	38.8178		40.0000
	9.0448		21.7695			39.4045	
		0.0000	11.5136	9.3069	33.9311		40.0000
			3.9741			30.6487	
				0.0000	22.6148		24.3117
						17.0714	
							11.9873

D - T ₂ maturity Debt							
D_{t,T_2}	D_{T_1,T_2}	D_{T_1,T_2}^*	D_{T_2,T_2}				
			40.0000				
			39.4045				
	38.8178	38.8178	40.0000				
	38.2399		39.4045				
28.4783		38.8178	38.8178	40.0000			
	21.7695		39.4045				
		9.3069	33.9311	40.0000			
			30.6487				
			24.3117				

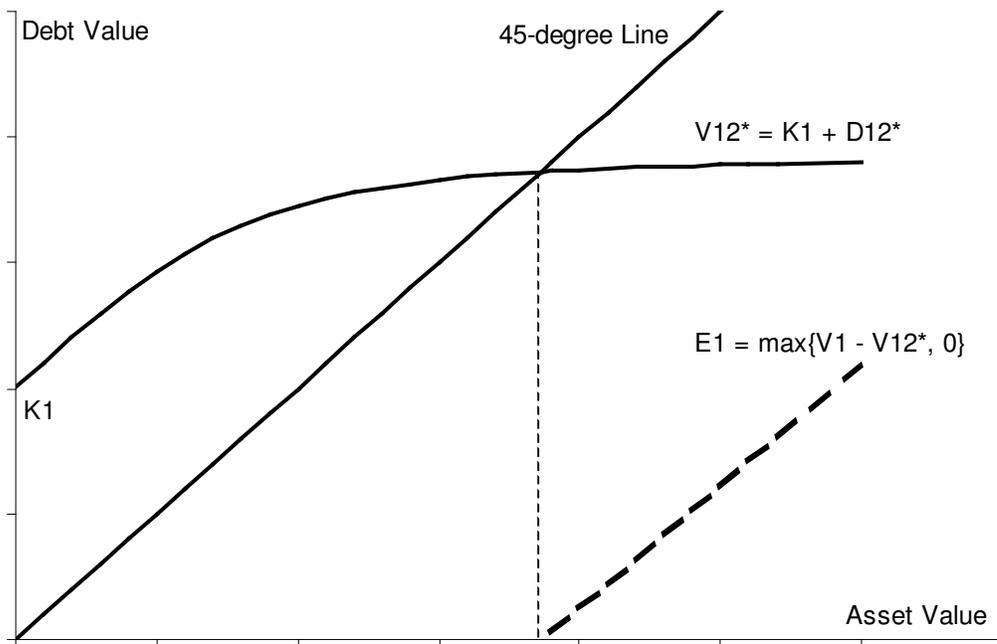
E - T ₁ maturity Debt							
D_{t,T_1}	D_{T_1,T_1}						
	40.0000						
	39.4045						
38.8178	40.0000						
	39.4045						
	40.0000						
$t = 0$	$T_1 = 1$		$T_2 = 2$		$T_3 = 3$		
	$K_1 = 40$		$K_2 = 40$		$K_3 = 40$		

Figure 1 – Constant Strike



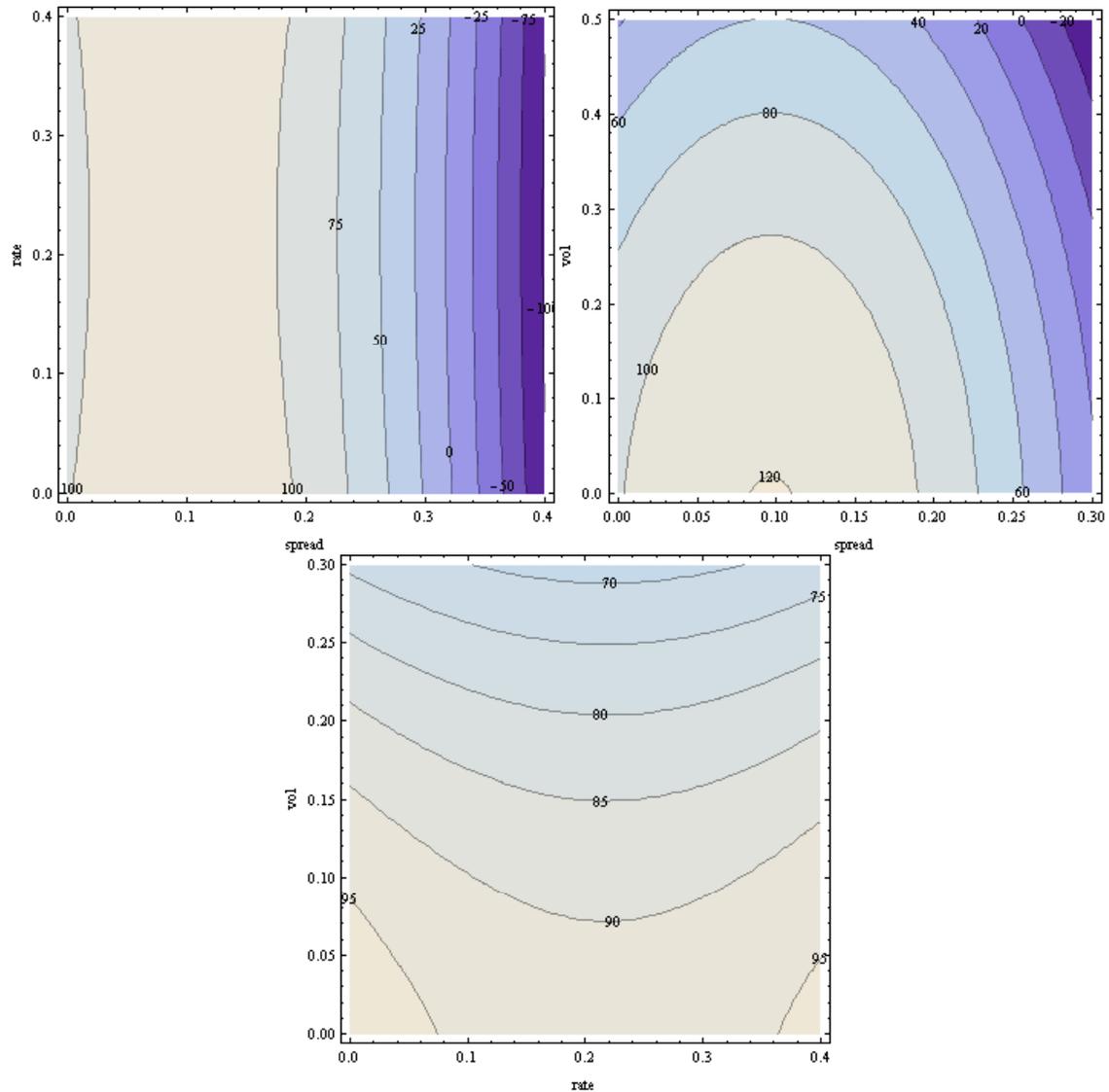
In the graph, K_1 (on the vertical axis) represents K_1 is the strike price at T_1 ; $E1^*$ (solid line) represents $E_{T_1}^*$ which is the continuation value equal to $e^{-r(T_2-T_1)}\mathbb{E}_{T_1}[\max\{V_{T_2} - K_2, 0\}]$ as seen in equation (4), $E1$ (long dotted line) represents E_{T_1} which is the current value of equity equal to $\max\{E_{T_1}^* - K_1, 0\}$, and finally $\bar{V}1$ represents \bar{V}_1 which is the implied strike price (or default barrier) as the solution to $E_{T_1}^* = K_1$ like the short-dotted lines in the graph demonstrates

Figure 2 – Random Strike



Same as Figure 1, except that $E_{T_1} = \max\{V_{T_1} - V_{12}^*, 0\}$ where, in the graph, $V_{12}^* = K_1 + D_{T_1, T_2}^*$ is labeled as $V12^*$ and D_{T_1, T_2}^* is labeled as $D12^*$ which is the T_2 -maturity debt at T_1 . Note that $E1$ in the graph is identical to $E1$ in Figure 1.

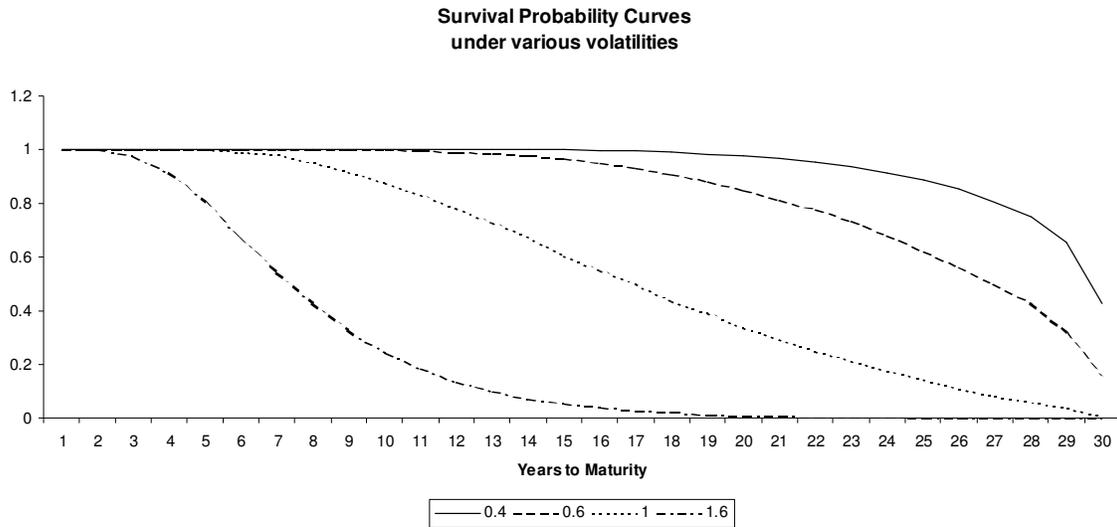
Figure 3 – Simulated Contour Plots of Key Inputs to CB Valuation



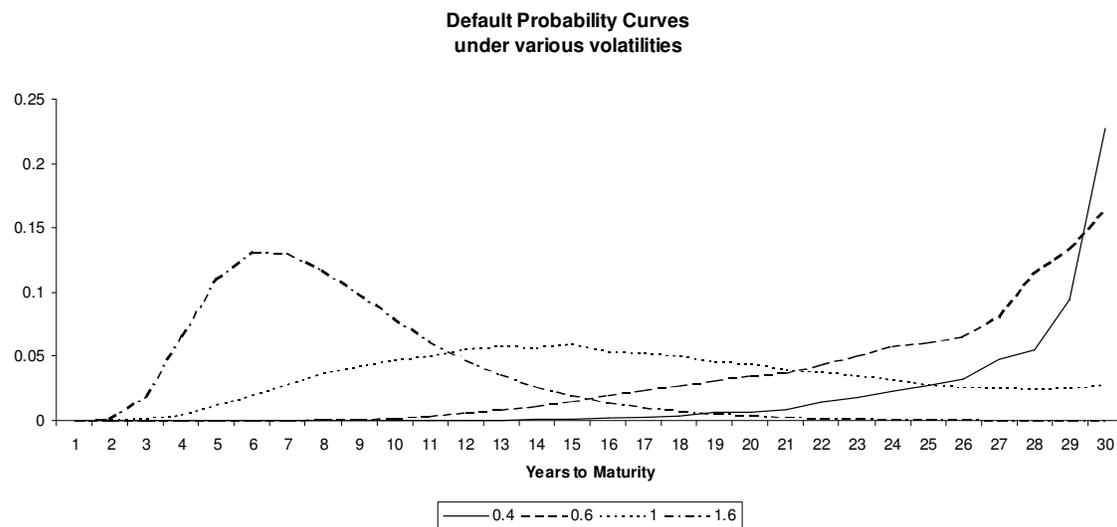
The convertible bond prices are computed using equation (28). We simulate 100 interest rate scenarios (0.1% to 10% at an increment of 0.1%) and 150 volatility scenarios (5% to 20% at an increment of 0.1%) to obtain 15,000 combinations of asset volatility and interest rates. The outputs of these input changes are (1) convertible bond spread, (2) equity volatility, and (3) 10-year risk-free yield which is a common measure for interest rate levels. These outputs are the market observables and are jointly determined by our model. We present the interactions of the three key outputs that are relevant to convertible bond valuation in a series of contour plots in Figure 3. The regression result is given in the Appendix.

Figure 4 – Example of Zero-coupon Bond

(a) Survival Probability Plot under Various Volatility Levels

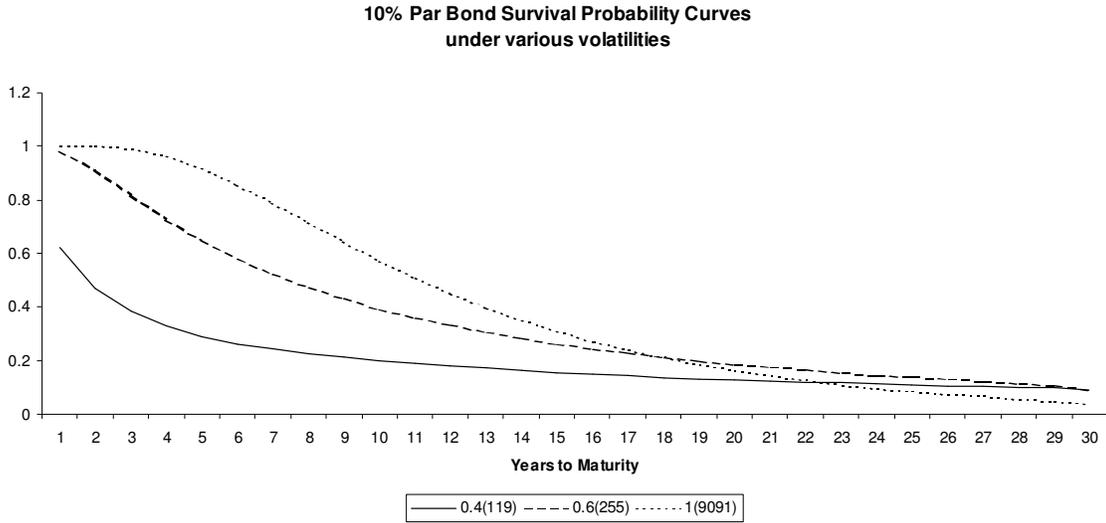


(b) Default Probability Plot under Various Volatility Levels



Note: This figure illustrates the survival probability curves under various asset volatility scenarios of the 30-year zero coupon bond under the Geske model. The Asset value is set to be \$184. The bond has no coupon and a face value of \$110. The yield curve is flat at 5%.

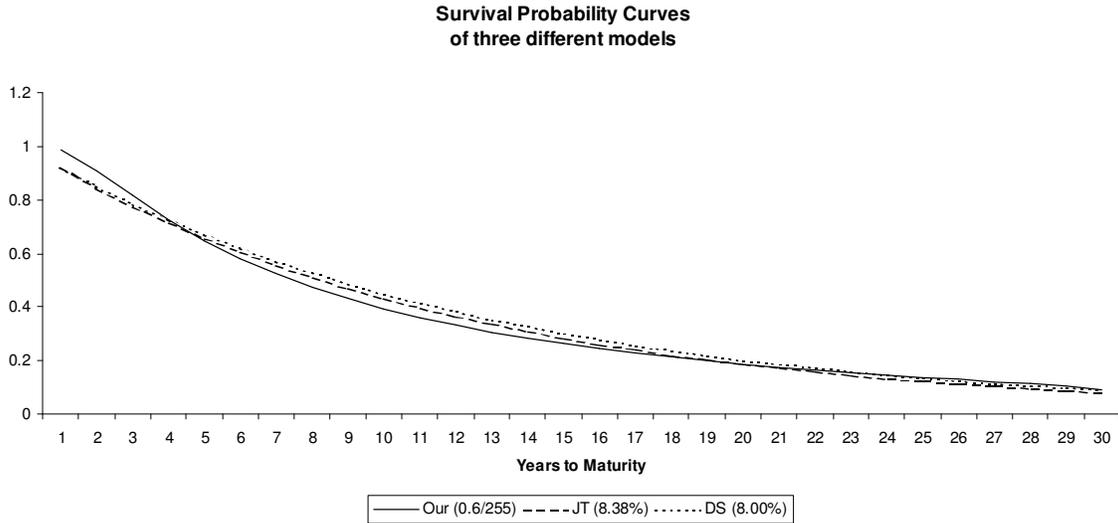
Figure 5 – 10% Par Coupon Bond Survival Probability Plot



Note: This figure illustrates the survival probability curves under various asset volatility scenarios of our model (i.e. Vasicek model is used). The asset values are \$119, \$255, \$9,091 for volatility levels of 0.4, 0.6, and 1.0 respectively. The bond has a \$10 coupon. The equity values are \$19, \$155, and \$8,991 respectively and hence the bond is price at par.

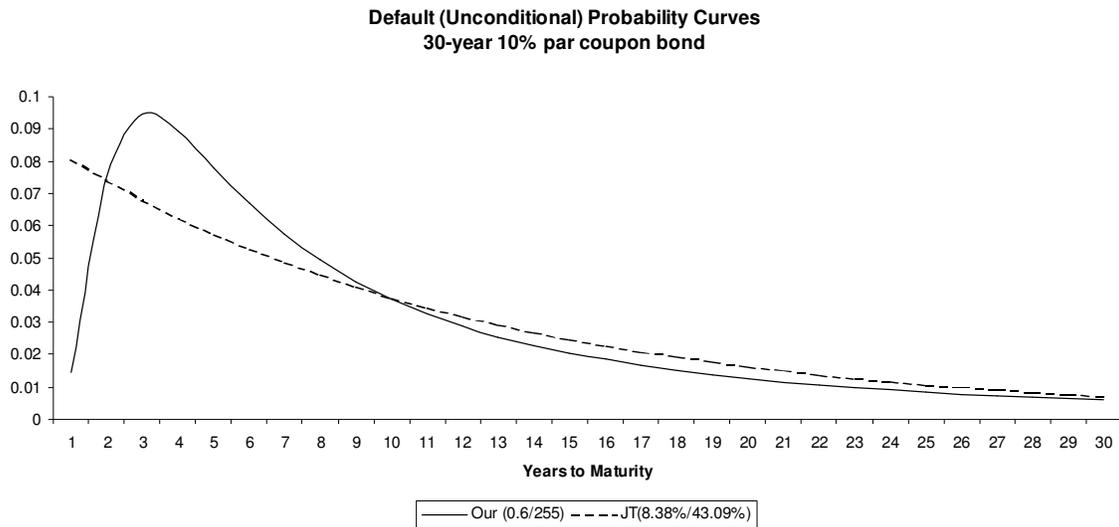
Figure 6 – Comparison of DS, JT and Our Models: Par Coupon Bond

(a) Survival Probability Plot



Note: This figure illustrates the survival probability curves under our model (with Vasicek parameters). The asset value is set to be \$255 and volatility 0.6 so that the bond is priced at par. The bond has \$10 coupon and a face value of \$110. The hazard rates of the Jarrow-Turnbull and Duffie-Singleton models are 8.38% and 8.00% respectively. The recovery rates are 43.09% and 48.87% respectively.

Figure 7: (Unconditional) Default Probability Curve for JT and Our models



Note: In our model (where Vasicek model is assumed), asset = \$255, volatility = 0.6, coupon = 10, face = 100 (so that debt is at par). The credit default swap protection value is \$35.20. The hazard rate and recover rate for the Jarrow-Turnbull model are 8.38% and 0.4309 respectively.

8 Appendix

8.1 Proof of the Theorem

[Lemma]

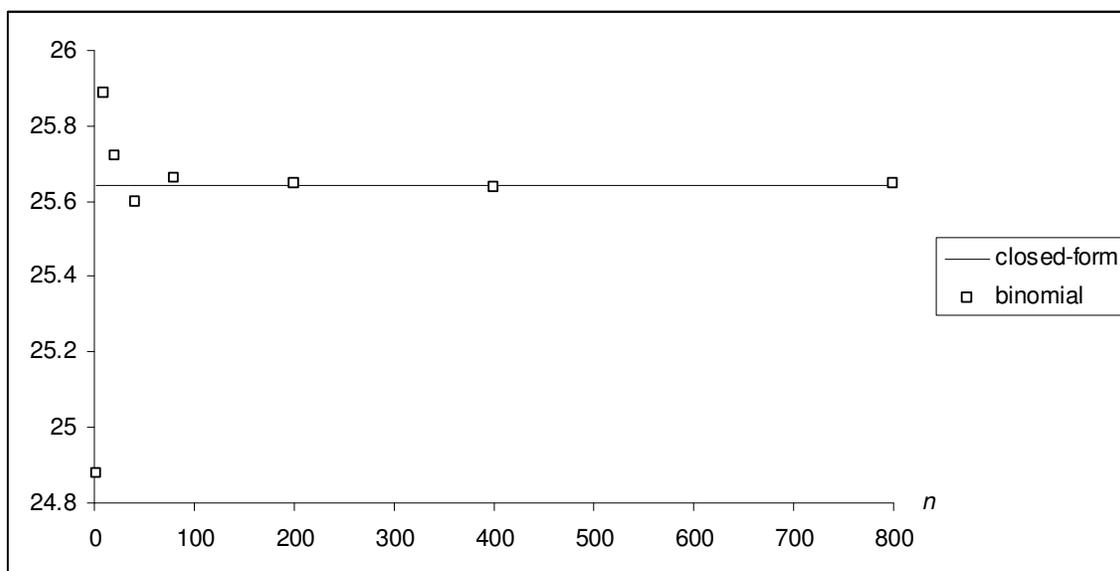
$$(37) \quad E_t^V \left[\frac{V_{12}^*}{V_{T_1}} \right] = \frac{e^{-r(T_1-t)} E[V_{12}^*]}{V_t}$$

[Q.E.D.]

8.2 Convergence of the Binomial Model – Geske

Figure 8 demonstrates the speed of convergence of the binomial model in a two-cash-flow model where the closed-form solution exists. The solid line in Figure 8 is computed with the closed form solution of equation (7) and the dotted line is generated with the algorithm described in equation (19).

Figure 8 – Convergence of the Binomial Model



8.3 Two-Dimensional Lattice

In this Appendix, we demonstrate how to construct the two-dimensional lattice. The Vasicek model (by setting $\beta = 0$ in (23)) can be described as follows:

$$(38) \quad dr = (\theta - \alpha r)dt + \gamma dW_2$$

For any contingent claim written on r , it must follow the following PDE:

$$(39) \quad \frac{1}{2}\gamma^2 P_{rr} + (\theta - \alpha r)P_r + P_t = rP$$

and the partial derivatives are approximated as follows:

$$(40) \quad \begin{cases} \frac{\partial P}{\partial r} = \frac{P_{i+1,j+1} - P_{i+1,j-1}}{2\Delta r} \\ \frac{\partial^2 P}{\partial r^2} = \frac{P_{i+1,j+1} - 2P_{i+1,j} + P_{i+1,j-1}}{\Delta r^2} \\ \frac{\partial P}{\partial t} = \frac{P_{i+1,j} - P_{i,j}}{\Delta t} \end{cases}$$

where i represents the time dimension and j represents various economic states.

Substitute the above discrete approximations back to the PDE and get:

$$(41) \quad \frac{1}{2}\gamma^2 \frac{P_{i+1,j+1} - 2P_{i+1,j} + P_{i+1,j-1}}{\Delta r^2} + (\theta - \alpha r) \frac{P_{i+1,j+1} - P_{i+1,j-1}}{2\Delta r} + \frac{P_{i+1,j} - P_{i,j}}{\Delta t} = rP_{i,j}$$

Re-arranging the terms (and letting r at various states as r_j), we get:

(42)

$$\begin{aligned} & \left(\frac{\gamma^2}{2\Delta r^2} \Delta t + \frac{(\theta - \alpha r_j)}{2\Delta r} \Delta t \right) P_{i+1,j+1} + \left(1 - \frac{\gamma^2}{\Delta r^2} \Delta t \right) P_{i+1,j} + \left(\frac{\gamma^2}{2\Delta r^2} \Delta t - \frac{(\theta - \alpha r_j)}{2\Delta r} \Delta t \right) P_{i+1,j-1} \\ & = (1 + r_j \Delta t) P_{i,j} \end{aligned}$$

The condition for the explicit method to converge (see Hull-White (1990a)) is all probabilities must be between 0 and 1. This condition translates into the following inequalities:

$$(43) \quad \begin{cases} \frac{\gamma^2}{2\Delta r^2} \Delta t \geq \frac{(\theta - \alpha r_j)}{2\Delta r} \Delta t \\ \frac{\gamma^2}{2\Delta r^2} \Delta t + \frac{(\theta - \alpha r_j)}{2\Delta r} \Delta t \leq 1 \\ \frac{\gamma^2}{\Delta r^2} \Delta t \leq 1 \end{cases}$$

The easiest way to set up the lattice is to set $\Delta t = \Delta r^2 / \gamma^2$ (last line in (43)). As a result, the middle probability (i.e. middle term in (42)) is 0 and the up probability can be solved for (by matching the mean of the interest rate process) as follows:

$$(44) \quad \frac{1}{2} \left(1 + \frac{(\theta - \alpha r_j) \Delta t}{\gamma \sqrt{\Delta t}} \right)$$

Given that the probability must be between 0 and 1, it must hold that:

$$(45) \quad -1 < \frac{(\theta - \alpha r_j) \Delta t}{\gamma \sqrt{\Delta t}} < 1$$

which defines the upper and lower bounds for the interest rate:

$-\frac{\theta + \gamma / \sqrt{\Delta t}}{\alpha} < r_j < \frac{\theta + \gamma / \sqrt{\Delta t}}{\alpha}$. Note that, as Hull and White (1990a) point out, this constraint is caused by mean reversion. Without mean reversion (i.e. $\alpha = 0$), then the interest rate is unbounded.

Hence, effectively the finite difference model is degenerated into a binomial model with equal up and down probabilities.

Note that the interest rate process and the firm value process are correlated. Hence, it is impossible to build an explicit finite difference lattice in two dimensions. As a result, we follow Scott (1997) where the interest rate process is built with an explicit method and the firm value process is built with an implicit method.³³ To do so, we first must orthogonalize the two processes.

³³ In other words, when either the drift or the diffusion of the underlying asset process is stochastic, the explicit method will fail.

Define $dR = dr - (\rho\gamma/\sigma)d\ln V$. By doing so, $dRd\ln V = 0$. The dR process is similar to the dr as a Gaussian mean-reverting process except for the mean and variance as follows:

$$\begin{aligned}
 \mathbb{E}[dR] &= \mathbb{E}[dr] - \frac{\rho\gamma}{\sigma} \mathbb{E}[d\ln V] \\
 (46) \quad &= (\theta - \alpha r)dt - \frac{\rho\gamma}{\sigma} \left(r - \frac{1}{2}\sigma^2 \right) dt \\
 &= \left[\left(\theta + \frac{1}{2}\rho\gamma\sigma \right) - \left(\alpha + \frac{\rho\gamma}{\sigma} \right) r \right] dt
 \end{aligned}$$

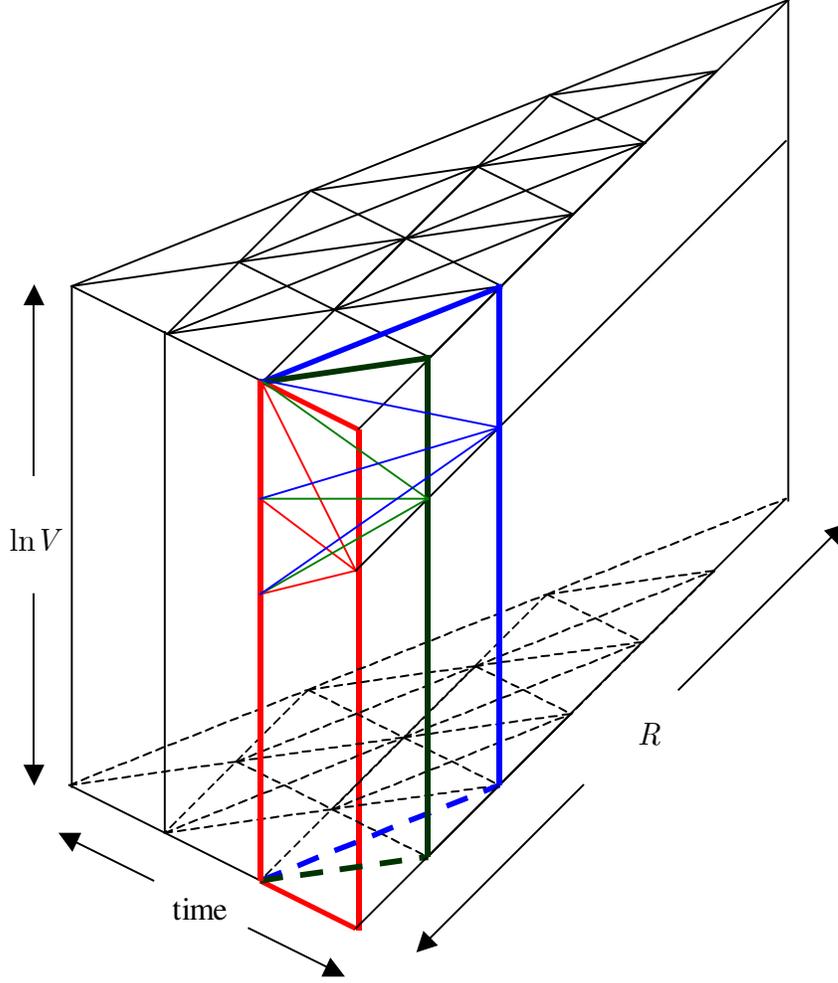
and

$$\begin{aligned}
 \mathbb{V}[dR] &= \mathbb{V}[dr] + \mathbb{V} \left[-\frac{\rho\gamma}{\sigma} d\ln V \right] - 2\mathbb{E} \left[dr \frac{\rho\gamma}{\sigma} d\ln V \right] \\
 (47) \quad &= (\gamma^2 + \rho^2\gamma^2 - 2\rho^2\gamma^2) dt \\
 &= \gamma^2 (1 - \rho^2) dt
 \end{aligned}$$

As a result, the explicit finite difference algorithm for the dR process is similar to the dr process with the simple substitution that $\Delta t = \frac{\Delta R^2}{\gamma^2(1-\rho^2)}$.

The binomial lattice for R and $\ln V$ can be visualized in the following diagram. The vertical dimension is the log of the firm value; the bottom dimension is time, and the dimension of the right is the transformed interest rate. Given that R and $\ln V$ are orthogonal, we can simply set up the lattice for R first. Once the lattice of R is determined, we can then use the implicit method described in Hull (2000), as the red, blue and green lines demonstrate, to compute the firm asset values backwards.

The implicit method is described clearly in Hull (2000). The inversion of the matrix is an easy recursive calculation (tridiag) that can be found in various numerical libraries, such as Numerical Recipe.



The convergence is demonstrated below. The call option value in this setting has a closed-form solution, derived by Rabinovitch (1989), as follows:

$$(48) \quad C_{t,T} = V_t N(d_{t,T}^+) - P_{t,T} KN(d_{t,T}^-)$$

where

$$d_{t,T}^{\pm} = \frac{\ln V_t - \ln P_{t,T} \pm \frac{1}{2}\xi_{t,T}}{\sqrt{\xi_{t,T}}}$$

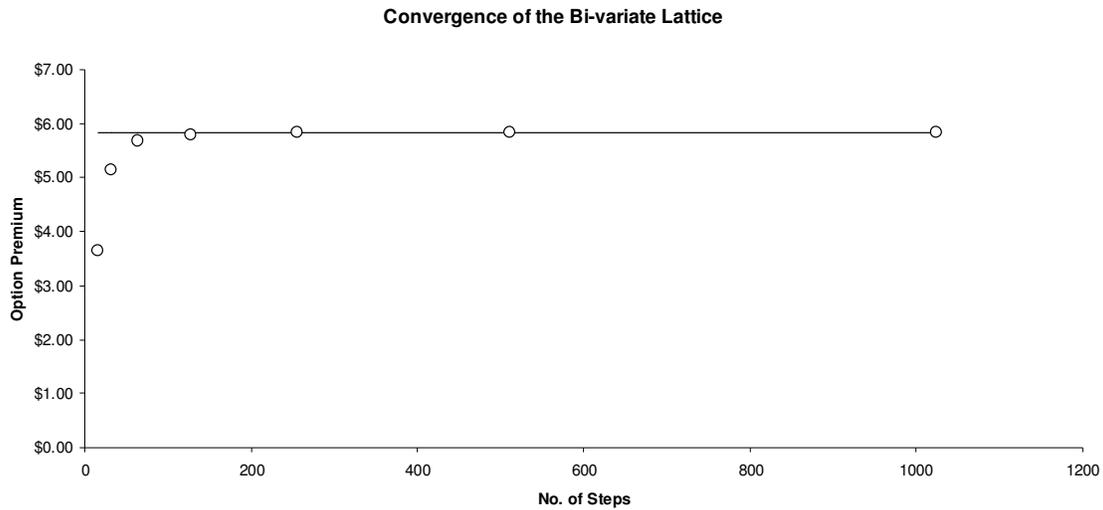
and

$$\xi_{t,T} = \frac{\gamma^2}{\alpha^2} \left[T - t - \frac{2}{\alpha} (1 - e^{-\alpha(T-t)}) + \frac{1}{2\alpha} (1 - e^{-2\alpha(T-t)}) \right] + 2 \frac{\rho\sigma\gamma}{\alpha} \left[T - t - \frac{1}{\alpha} (1 - e^{-\alpha(T-t)}) \right] + \sigma^2 (T - t)$$

We present the convergence of the model in the case of a European call option when the interest rate follows the Vasicek model using the following parameter values:³⁴

α	0.25
θ	0.005
γ	0.03
r_0	0.02
T	0.5
V_0	50
K	50
ρ	0.5

The Rabinovitch value of the call option is \$5.85 and in 1000 steps, the bi-variate lattice converges to this value.³⁵



8.4 Regression Results

$$y = a + b1 \text{ spread} + b2 \text{ spread}^2 + c1 \text{ rate} + c2 \text{ rate}^2 + d1 \text{ vol} + d2 \text{ vol}^2$$

where

Coefficients	Standard Error	t Stat
--------------	----------------	--------

³⁴ In the Vasicek model, $\theta = \alpha\mu - \lambda\gamma$ where μ is the level of mean-reversion and λ is the market price of risk. In here we set $\mu = 0.02$ and $\lambda = 0$.

³⁵ We have also tested other correlation values and uniformly at 1000 steps the bi-variate lattice converges at the second decimal place.

a	98.285	0.048	2039.037
b1	457.329	0.705	648.927
c1	-53.096	0.412	-128.902
d1	-20.418	0.771	-26.490
b2	-2363.420	13.631	-173.390
c2	121.361	2.451	49.515
d2	-199.905	3.125	-63.966
adj R2	0.9859		

$$y = a + b1 \text{ spread} + b2 \text{ spread}^2 + b3 \text{ spread}^3 + c1 \text{ rate} + c2 \text{ rate}^2 + c3 \text{ rate}^3 + d1 \text{ vol} + d2 \text{ vol}^2 + d3 \text{ vol}^3$$

where

	Coefficients	Standard Error	t Stat
a	88.732	0.061	1455.441
b1	382.088	0.521	733.792
c1	-136.623	0.539	-253.707
d1	272.802	1.657	164.648
b2	-3152.383	7.421	-424.769
c2	1231.374	6.406	192.211
d2	-2743.940	13.950	-196.692
b3	48739.439	301.602	161.602
c3	-3575.876	19.443	-183.916
d3	6973.638	36.378	191.702
adj R2	0.9968		